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A Path-Integral Approach for Bosonic Effective Theories for Fermion Fields in Four and Three Dimensions

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Abstract

We study Four dimensional Effective Bosonic Field Theories for A) Massive Fermion Field in the infrared region and B) Massive Fermion in Ultraviolet region by using an appropriate Fermion Path Integral Chiral variable change and C) The Polyakov's Fermi-Bose Transmutation in the 3D-Abelian Thirring Model.

Key-words: Bosonization in higher dimensions.

I Introduction

Analysis of Fermionic Quantum Models in Four-Dimensional Space-Time always have been a very difficult mathematical problem ([1]). Fortunatelly, non perturbative Effective actions have shown its usefulness to analyse new phenomena in these theories. It is the purpose of this paper to propose a new technique to arrive at an Effective Bosonic Action suitable adapted from similar exact obtained results on two-dimensions. This main result of our study is the content of section II, section III. In the section IV we present our study of Polyakov's Fermi-Bose Transmutation in the Abelian Thirring model in detailies ([3]).

Finally in section V we comment some papers in the literature related to the topic of higher-dimensional Bosonization.

II The Bosonic High-Energy Effective Theory

We start this section by considering the Generating Functional for the correlations functions generated by vectorial and axial currents in a Theory of Euclidean Abelian Massive Fermions in a Euclidean Four-Dimensional Space-Time R^4.

$$
\begin{aligned}
& Z\left[V_\mu, A_\mu\right](m) = \frac{1}{Z(0,0)} \int D^F\left[\psi(x)\right] D^F(\bar{\psi}(x)) \\
& \delta^{(F)}\left(\partial'_\mu(\bar{\psi}\gamma^\mu\psi)(x)\right)\delta^{(F)}\left(\left[\partial_\mu(\bar{\psi}\gamma^\mu\gamma^5\psi) - 2im\bar{\psi}\psi\right](x)\right) \\
& \exp\left(-\int d^4\left[\bar{\psi}\left(i\gamma^\mu\partial + m + \gamma_\mu\gamma_5 A_\mu + \gamma_\mu V_\mu\right)\psi\right](x)\right)
\end{aligned}
\tag{1}
$$

where we have taken into account in a explicity way in the Functional Domain of Integration of eq. (1), the current-charge law for theory, response to *phase* local variable field change

$$
\begin{aligned}
\psi(x) &\rightarrow e^{ig_V\ \theta(x)} e^{ig_A\gamma_5\omega(x)}\psi(x) \\
\psi(x) &\rightarrow \bar{\psi}(x) e^{-ig_V\ \theta(x)} e^{ig_A\gamma_5\omega(x)}
\end{aligned}
\tag{2}
$$

It is worth point out that our Fermionic Functional Measures are defined in terms of the spectral set (eigenfunctions and eingenvalues) associated to the Free Massless Dirac Operator $\partial\!\!\!/ \equiv i\gamma^\mu\partial_\mu$ instead of the Full Massive Dirac Operator $\partial\!\!\!/(A,V) + m \equiv i\gamma^\mu(\partial_\mu + V_\mu + \gamma_5 A_V) + m$ since the external sourcer (A_μ, V_μ) are not Dynamical and, leading to the absence of the axial-anomaly piece in the chiral current law associated to these fields. Besides the mass term is defined as a perturbation of the massless case as in 2D-models ([4]). We now write the Generating Functional eq. (1) in a local way by expressing the functional Delta constraints in Fourier Functional Domain

$$
\begin{aligned}
& Z[V_\mu, A_\mu](m) = \frac{1}{Z(0,0)} \int D^F[\psi(x)] D^F[\bar{\psi}(x)] \int D^F[\theta(x)] D^F[\omega(x)] \\
& \exp\left[-\int\{d^4x\ ig_A(\bar{\psi}\gamma^\mu\gamma^5\psi)(x)\partial_\mu\omega(x) - 2m \int d^4x(\bar{\psi}\psi)(x)\omega(x)\right] \\
& \exp\left[-i\int d^4x\ g_V(\bar{\psi}\gamma^\mu\psi)\partial_\mu\theta(x)\right] \exp\left[-\int d^4x\bar{\psi}[i\partial\!\!\!/(A,V) + m]\psi\right]
\end{aligned}
\tag{3}
$$

At this point of our study, we implement the Phase Variable Change eq. (2) into eq. (3) by taking into consideration the non-unity Jacobian associated to the Chiral rotation ([5]-eq. 9).

$$D^F[\bar{\psi}(x)]D^F[\psi(x)] = D^F\left[\left(\bar{\psi}(e^{-ig_A\gamma_5\omega}e^{-ig_V\theta})\right)(x)\right]$$
$$\times D^F\left[\left((e^{ig_A\gamma_5\omega}e^{ig_V\omega})\psi\right)(x)\right]\frac{\det_F[e^{ig_A\gamma_5\omega}(i\partial\!\!\!/)e^{ig_A\gamma_5\omega}]}{\det_F[i\partial\!\!\!/]} \tag{4}$$

The ratio of the functional Dirac Determinants were evaluated in ref. [5] - eq. (17) - eq. (18) and yielding the Functional Weigth for the Chiral Dynamical Phase $\omega(x)$ (with a U.V. cut-off Λ).

$$\det_F\left[e^{ig_A\gamma_5\omega}(i\partial\!\!\!/)e^{ig_A\gamma_5\omega}\right]/\det_F(i\partial\!\!\!/)]$$
$$\exp\left[\left(\frac{g_A}{\Lambda}\right)^2\int d^4x\omega(-\partial^2)\omega\right]\exp\left[-\frac{(g_A)^2}{4\pi^2}\int d^4x[(-\partial^2\omega)(-\partial^2\omega)(x)\right]$$
$$\exp\left[\frac{(g_A)^4}{12\pi^2}\int d^4\left((\omega(\partial_\mu\omega)^2(-\partial^2\omega)\right)\right](x) \tag{5}$$

By substituting eq. (4) into eq. (3) and by noting the validity of the equation

$$\int D^F\left[\bar{\psi}(x)e^{i(g_A\gamma_5\omega-ig_V\omega)(x)}\right]D^F\left[e^{i(g_A\gamma_5\omega+ig_V\theta)(x)}\psi(x)\right]e^{i(g_A\gamma_5\omega-ig_V\theta)(x)}$$
$$\exp\left[-\int d^4x\{\left(\bar{\psi}e^{ig_A\gamma_5\omega-ig_V\theta}\right)\left[i\partial\!\!\!/(A,V)+me^{-2(ig_A\gamma_5\omega)}\times(1+2\omega)\right]\left(e^{ig_A\gamma_5\omega+ig_V\theta}\psi\right)\}(x)\right]$$
$$=\det_F\left[i\partial\!\!\!/(A,V)+m(1+2\omega)\exp g_A\gamma_5\omega)\right)] \tag{6}$$

we finally obtain the searched result at the leading limit of high ultra-violet region $m\to 0$, which improves somewhat those models studied in the second reference of [1].

$$\tilde{Z}[V_\mu,A_\mu](m)=\frac{1}{\tilde{Z}(0,0)}\int D^F[\theta(x)]D^F[\omega(x)]$$
$$\exp\left(\int d^4x\omega(x)\left\{-\frac{\Lambda_F^2}{(g_A)^4\pi^2}\left[\frac{1}{-(\partial^2)+\left(\frac{2\pi}{\Lambda_F}\right)^2}-\frac{1}{-\partial^2}\right]\right\}^{-1}(x,y)\omega(y)\right)$$
$$\exp\left(\frac{(g_A)^4}{12\pi^2}\int d^4\left(\omega(\partial_\mu\omega)^2(-\partial^2\omega)\right)(x)\right)$$
$$\exp\left\{-2\int d^4xd^4y[(m(1+2\omega)e^{-2i(g_A\gamma_5\omega)}+(x)(i\partial\!\!\!/)^{-1}(x,y)\right.$$
$$\left.(V_\mu+\gamma_5A_\mu)(y)\,(i\partial\!\!\!/)^{-1}(y,x)]+O(m^2)\right\}) \tag{7}$$

Comments related to this Effective High-Energy Bosonic Field Theory for the current algebra of observables are made in section IV of this paper.

III The Bosonic Low Energy Effective Theory

Let us start our analysis in this section by writing the Generating Functional for the correlations generated by vectorial and axial currents in a Theory of Free Massive Euclidean Fermion Fields in R^4

$$\tilde{Z}[V_\mu, A_\mu] = \frac{1}{Z(0,0)} \int D^F[\psi(x)] D^F[\bar{\psi}(x)]$$
$$\exp\left[- \int d^4x \left(\bar{\psi}((i\partial\!\!\!/(A,V,m)\psi\right)(x) \right] \tag{8}$$

The main point of our approximate bozonization procedure to eq. (8) is to introduce a Massive Fermion Field Theory invariant under the Field rotation eq. (2) by elevating the involved local $(\omega(x), \theta(x))$ to be Dynamical and Functionally integrating them out. As a consequence, we propose to approximate eq. (8) in the Infrared region by means of the Chirial Invariant Functional Integral with a mass parameter term

$$\tilde{Z}[V_\mu, A_\mu]_{IR} = \lim_{m\to\infty} \int D^F[\omega(x)] D^F[\theta(x)]$$
$$\int D^F[\bar{\psi}^{\theta,\omega}(x)] D^F[\psi^{\theta,\omega)}(x)]$$
$$\exp\left\{ - \int d^4x \bar{\psi}^{\theta,\omega)}(i\partial\!\!\!/(A,V) + m)\psi^{(\theta,\omega)}(x) \right\} \tag{9}$$

where the fields rotated in eq. (9) are given by eq. (2)

$$\begin{aligned} \psi^{(\theta,\omega)}(x) &= e^{ig_V\theta(x)} e^{ig_A\gamma_5\omega(x)}\psi(x) \\ \bar{\psi}^{(\theta,\omega)}(x) &= \bar{\psi}(x) e^{-ig_V\theta(x)} e^{ig_A\gamma_5\omega(x)} \end{aligned} \tag{10}$$

We, thus, proceed in the inverse path of that followed in section II by using the inverse field variable change eq. (4)

$$\tilde{Z}[V_\mu, A_\mu]_{IR} = \lim_{m\to\infty} \int D^F[\omega(x)] D^F[\theta(x)]$$
$$\exp\left(\int d^4x \omega(x) \left\{ \frac{\Lambda_F}{(g_A)^4 4\pi^2} \left[\frac{1}{(-\partial^2) + \left(\frac{2\pi}{\Lambda_F}\right)^2} - \frac{1}{(-\partial^2)} \right]^2 \right\} (x,y)\omega(y) \right)$$
$$\exp\left(-\frac{(g_{\Lambda_F})^4}{12\pi^2} \int d^4x \left(\omega(\partial_\mu\omega)^2(-\partial^2\omega)(x) \right) \right)$$
$$\det{}_F\left[i\partial\!\!\!/(V_\mu + ig_V\partial_\mu\theta, A_\mu + ig_A\partial_\mu\omega) + m\exp(2ig_A\gamma_5\omega) \right] \tag{11}$$

where Λ_F denotes the intrinsic cult-off from the original Fermion Field Theory which by its turn, determines the effective energy scale where our Effective Bosonic Theory is expected to be working. Another point to remark it that the Effective Bosonic Action in eq. (11) is exactly the inverse of that used in the ultra-violet region eq. (5).

Let us now analyse the Fermion Functional Determinant involving the sources in this Low Energy Limit $m \to \infty$. At this limit, we can easily improve the asymptotic expansion in terms of the inverse power of the mass parameter m of ref. [6] and approximating the term $m\exp(2g_A\gamma_5\omega)$ by the simple mass term m (this procedure being correct only at this limit of $m \to \infty$).

We, thus, consider the following differential equation for this Functional Determinant, where parameter s ranges in the interpolating $0 \leq s \leq 1$.

$$\begin{aligned}
&\lim_{m\to\infty} \frac{d}{ds}\left\{\det{}_F\left[i\partial\!\!\!/(s(V_\mu + ig_V\partial_\mu\theta); s(A_\mu + ig_A\partial_\mu\omega)) + m\right]\right\} \\
&\sim \int_0^\infty dt e^{-tm^2} \times Tr_F\left[(\gamma^\mu V_\mu + \gamma_5\gamma_\mu A_\mu)\right]\cdot\left[i\partial\!\!\!/(s(V_\mu + ig_V\partial_\mu\theta); s(A_\mu + g_A\partial_\mu\omega)) + m\right] \\
&\exp\left\{-t\left[i\partial\left(s(V_\mu + ig_V\partial_\mu\theta); s(A_\mu + ig_A\partial_\mu)\right)\right]^2\right\} \qquad (12)
\end{aligned}$$

By applying the saddle point technique to evaluate the Laplace Transform (see Appendix 1), we obtain the leading effective infrared effective source dependent action (the well-know Hesenberg Effective Lagrangean)

$$\begin{aligned}
&S_{eff}(A_\mu, V_\mu)_{IR} = \exp\{+\frac{m/\Lambda_F)^2}{4\pi} \int d^4x([V_\mu + ig_V\partial_\mu\theta)^2 + \\
&(A_\mu + ig_A\partial_\mu\omega)^2(x)] + [c_1F^2_{\mu V}(A_\mu) + c_2F^2_{\mu V}(A_\mu) + c_3F_{\mu\nu}(V_\mu)F^{\mu\nu}(A_\mu)(x)] \\
&+0((m/\Lambda_F)^{-2})\}\ , \qquad (13)
\end{aligned}$$

Hence c_1 and c_2 are positive constants whose value depend on the regularization scheme used and the Dirac Matrices representation. By substituting the Massive Abelian Gauge Field (Source) action above into the Functional Integral eq. (11), we get our propose IR Effective Bosonic Theory for the Algebra generated by vectorial and axial currents of a Massive Free Fermion Field Theory. At this point the reader should compare the UV-Effective action eq. (13) with IR-Effective Action given by eq. (7).

It is instructive point out that in the important use of $D \equiv 2$, all Functional Integrals are of Gaussian Type and leading to the following result in the IR-region.

$$\begin{aligned}
&\tilde{Z}[A_\mu, V_\mu] = \int D^F[\omega(x)]D^F[\theta(x)] \\
&\exp\left[-\frac{1}{2\pi}\int d^2x\left[(\partial\theta)^2 + (\partial\omega)^2(x)\right]\right. \\
&\exp\left[\frac{m^2}{2\pi}\int d^3\left[(V_\mu + ig_V\partial_\mu\theta)^2 + (A_\mu + ig_A\partial_\mu\omega)^2\right](x)\right] \\
&= \exp\left\{\int d^2xd^2yV_\mu(x)\left[m^2\delta_{\mu\nu} - 4\,\frac{g_V^2}{\left(\frac{1+m^2g_V^2}{\pi}\right)}\,\frac{\partial_\mu\partial_V}{(-\partial^2)}\right]V_\nu(y)\right\} \\
&\times\exp\left\{\int d^2xd^2yA_\mu(x)\left[m^2\delta_{\mu\nu} - \frac{4g_A^2}{\left(1+\frac{m^2g_A^2}{\pi}\right)}\frac{\partial_\mu\partial_\nu}{(-\partial^2)}\right](x,y)A_\nu(y)\right\} \qquad (14)
\end{aligned}$$

By analyzing the Two-Dimensional Effective Bosonic Theory we conclude that the result is clearly not Gauge Invariant on the Source Gauge Fields as the Gauge Symmetry is dynamically broken in two-dimensional space-time.

In the important case of the presence of a Quantized Electromagnetic Field $G_\mu(x)$, we can follow our previous of the section. The main difference is the introduction of the "Topological Charge" of the Electromagnetic Field in the delta function of eq. (1)

$$\begin{aligned} \delta^{(F)}([\partial_\mu(\bar{\psi}\gamma^\mu\gamma^5\psi) &- 2im\bar{\psi}\psi]) \to \\ \delta^{(F)}([\partial_\mu(\bar{\psi}\gamma^\mu\gamma^5\psi) &- 2im\bar{\psi}\psi - \frac{1}{32\pi^2} \int d^4x(*F_{\mu\nu}F^{\mu\nu})(G_\mu)]) \end{aligned} \quad (15)$$

and the replacing of the Full Dirac Operator below in eq. (12)

$$\partial\!\!\!/(A,V) + m \to \partial\!\!\!/(A,V+G) + m$$

It is worth point the natural appereance of an "Axion Like" interaction between the Chiral Phase Neutral Field $\omega(x)$ and the Electromagnetic Field $G_\mu(x)$, namely

$$S_{axion}[\omega, G_\mu] = \exp\left\{ i \int d^4x\omega(x)(*F_{\mu\nu}F^{\mu\nu})(G)(x)\right\} \quad (16)$$

The generalization of our study for the non-Abelian case is straightforward and leading to the non-Abelian generalization of our previous study (see ref. [12]) where the non-Abelian Evaluation of the chiriality rotated Jacobian eq. (4) is presented in full details).

Finally, it is instructive point out that one should explicit the "Euclideanicity" of our approach by considering the *non-unitary* (Euclidean) variable change below

$$\begin{aligned} \psi(x) &\to e^{g_V\theta(x)}e^{g_V\gamma_5\omega(x)}\psi(x) \\ \bar{\psi}(x) &\to \bar{\psi}(x)e^{g_A\gamma_5\omega(x)}e^{-g_V\theta(x)} \end{aligned} \quad (16')$$

instead of the Classical Unitary eq. (2), the Jacobian will be a now a Full Functional involving the non-unitary phases ($\theta(x)$, $\omega(x)$). Note that eq. (16') is allowed in Euclidean Space-time since the energy density $\bar{\psi}\psi$, $\bar{\psi}\gamma^5\psi$, $\bar{\psi}\gamma^5 A_\mu$ are not real as $\bar{\psi}$ and ψ are independent, anticummuniting Euclidean Fields and, thus, living in different Functional Spaces.

IV Polyakov's Fermi-Bose Transmutation in 3D-Abelian Thirring Model

The Polyakov's Fermi-Bose transmutation in the infrared regime of the Cp^1 model has became a basic phenomenon for understanding approximate bosonization in Fermion Field Theory in Three-Dimensional Space-Time. In this section we present in details the above cited phenomenon in the Thirring Model. This study is based on our unpublished research ((7)).

Let us start our study in this section by considering the Massive Three-Dimensional Thirring Lagrangian in the Euclidean Space Time with a repulsive interaction.

$$\mathcal{L}(\psi,\bar{\psi}) = \bar{\psi}(i\gamma\partial)\psi + m\bar{\psi}\psi - \frac{g^2}{2}\,(\bar{\psi}\gamma^\mu\psi)^2 \tag{17}$$

The 3D-Euclidean Hermitean γ^μ matrices which we are using obey the relationship

$$\{\gamma^\mu,\gamma^\nu\} = \delta^{\mu\nu} \quad ; \quad [\gamma^\mu,\gamma^\nu] = \frac{1}{2}\,\varepsilon^{\mu\nu\rho}\gamma_\rho \tag{18}$$

The independent Euclidean fields $\psi^{(\alpha)}(x)$ and $\bar{\psi}^{(\beta)}(x)$ satisfy the Euclidean anticommuting relation $(\alpha,\beta = 1,2,3)$

$$\left\{\psi^{(\alpha)}(x),\bar{\psi}^{(\beta)}(y)\right\} = \delta^{\alpha\beta}\delta^{(3)}(x-y) \tag{19}$$

The Lagrangian (17) is invariant under the *global Abelian group* $\psi \to \exp(i\Omega)\psi_1$; $\bar{\psi} \to \exp(-i\Omega)\bar{\psi}$ with the Noetherian conserved current

$$\partial_\mu(\psi\gamma_\mu\bar{\psi}) \equiv 0 \tag{20}$$

In order to analyse the Polyakov's Boson-Fermion Transmutation, we consider the generating function

$$\begin{aligned} Z[n,\bar{n}] = \frac{1}{Z(0,0)} \times \bigg\{ \int D^F[\psi(x)]D^F[\bar{\psi}(x)] \\ \times \exp\left[\int d^3x\left(\mathcal{L}(\psi,\bar{\psi}) + \eta\bar{\psi} + \psi\bar{\eta}\right)(x)\right]\bigg\} \end{aligned} \tag{21}$$

By making use of the Hubbard-Stratonovich field reparametrization, we rewrite eq. (21) in a form useful for our Bosonization purpose

$$\begin{aligned} Z[\eta,\bar{\eta}] &= \frac{1}{Z(0,)} \times \bigg\{ \int D^F[\psi(x)]D^F[\bar{\psi}(x)]D^F[A_\mu(x)] \\ &\times \exp + \left(-\frac{1}{2}\int d^3x A_\mu^2(x)\right)\delta^{(F)}[(\partial_\mu A_\mu)] \\ &\times \exp\left(-\int d^3x\left[\bar{\psi}(i\gamma\partial + g\gamma A + m)\psi + \eta\bar{\psi} + \psi\bar{\eta}\right](x)\right) \end{aligned} \tag{22}$$

where $A_\mu(x)$ is an auxiliary Euclidean Abelian real vector field satisfying the Landau gauge as consequence of Eq. (20), since it should concides with the vectorial current at the operator level.

At this point, it becomes important to remark that the fermionic measures $D^F[\bar{\psi}_1(x)]D^F[\psi(x)]$ in Eq. (22) are defined in terms of the normalized eigenvectors of the self-adjoint Euclidean Dirac operator $i\gamma_\mu(\partial - igA_\mu)$ since we want to keep the model's physical local gauge invariance in the pure fermion sector of the theory

$$\begin{aligned} \psi(x) &\to \psi(x)\exp(ig\Omega(x)) \\ \bar{\psi}(x) &\to \bar{\psi}(x)\exp(-ig\Omega(x)) \\ A_\mu(x) &\to A_\mu(x) \end{aligned} \tag{23}$$

Note that this local Abelian gauge invariance in the fermionic parametrization Eq. (17) is a consequence of the current conservation Eq. (20) at the quantum level of the generating functional Eq. (21) and differs from the usual local gauge invariance of the gauge models involving the shift $A_\mu \to A_\mu + g\partial_\mu\Omega$. The local invariance Eq. (23) is a consequence of the following path integral identify

$$\begin{aligned} & \int D^F\left[\psi(x)e^{ig\Omega(x)}\right] D^F\left[\bar\psi(x)e^{-ig\Omega(x)}\right] \\ & \exp\left\{-\int d^3\mathcal{L}\left(\psi(x)e^{ig\Omega}(x),\ \bar\psi(x)e^{-ig\Omega}(x)\right)\right\} \\ = & \int D^F[\psi(x)]D^F[\bar\psi(x)]\exp\left\{-\int d^3x\mathcal{L}\left(\psi(x),\ \bar\psi(x)\right)\right\} \\ \times & \exp\left[-i\int d^3\Omega(x)\left(\partial_\mu(\bar\psi\gamma_\mu\psi)\right)(x)\right] \end{aligned} \tag{24}$$

In this quantum field path integral framework, the infrared Polyakov's Fermi-Bose transmutation (3) may be understood as the large fermion of the otherwise trivial 3D-Abelian Quantum Field Thirring model ([8]).

Explicitly, we first, introduce an ultra-violet cutt-off in Eq. (22) and integrate out the Euclidean Fermi Fields. Let us, thus, consider the Effective Path Integral.

$$\begin{aligned} Z[\eta,\bar\eta] &= \frac{1}{Z(0,0)}\times\int D^F[A_\mu(x)]\times \\ &\times\ \exp\left(-\frac{1}{2}\int d^3xA_\mu^2(x)\right)\times\delta^{(F)}[(\partial_\mu A_\mu)] \\ &\times\ \det[i\gamma\partial+g\gamma A+m]\times \\ &\times\ \exp\left\{+\frac{1}{2}\int d^3xd^3y\left[\bar\eta(x)(i\gamma\partial+g\gamma A+m)^{-1}(x)\cdot\eta(y)\right]\right\} \end{aligned} \tag{25}$$

The fermion vacuum loops associated to the fermion functional determinant may be easily evaluated at the limit of large mass by using the proper-time definition for this functional determinant, see appendix 1

$$\begin{aligned} & \log\det(i\gamma\partial+g\gamma A+m) = \frac{1}{2}\times\lim_{\varepsilon\to 0^+}\int_\varepsilon^\infty\frac{dt}{t}\times \\ & Tr_{(F)}\left[\exp\left(-t[i\gamma\partial+g\gamma A+m]^2\right)\right] \end{aligned} \tag{26}$$

where $Tr_{(F)}$ denote the functional trace.

We have, thus the following result for the family of interpolating Dirac operator $i\gamma\partial+sg\gamma A+m(0\le s\le)$

$$\begin{aligned} & \frac{d}{ds}\left(log\det[i\gamma\partial+sg\gamma A+m]\right)\times \\ & \lim_{\varepsilon\to 0^+}\int_\varepsilon^\infty dt\ e^{-tm^2}Tr_{(F)}\{(A\gamma A) \\ & (i\gamma\partial+sg\gamma A+m)\cdot\times \\ & \exp\left(-t[i\gamma\partial+sg\gamma A+m]^2\right)\} \end{aligned} \tag{27}$$

By taking the limit of large fermion mass as in ref. [6] and appendix 1, we get the result below, after integrating the interpolating parameter in the range $0 \le s \le l$

$$\begin{aligned}&\log\left[\det(i\gamma\partial + g\gamma A + m)/\det(i\gamma\partial + m)\right]_{(\varepsilon)} \\ &= \frac{g^2 m}{(4\pi)^{\frac{3}{2}}}\cdot\left(\frac{1}{\varepsilon}\right)\cdot\int d^3x\left(\frac{1}{2}A^2_\mu(x)\right) \\ &-g^2\,\frac{\sqrt{\pi}}{2}\,\frac{m}{|m|}\int d^3x(A_\mu\varepsilon^{\mu\nu\rho}F_{\nu\rho}(A))(x) + O\left(\frac{1}{m}\right)\end{aligned} \tag{28}$$

It is worth pointing out the existence (in principle) of an induced (cutt-off dependent) mass term for the auxiliary vector field (this auxiliary vector at the quantum level coincides with the Noetherian $U(1)$ global current; $A_\mu(x) = (\bar\psi\gamma_\mu\psi)(x)$).

Note that this mass term signals the dynamics breaking of the usual gauge invariance in the pure fermionic sector of Eq. (25) which involves the gauge change $A_\mu(x) \to A_\mu(x) + \frac{1}{g}\,\partial_\mu\Omega(x)$ as in 2D-models (see Eq. (23)).

The physical consequence of this term is a formal renormaliztion of the bare fermion mass at one loop, as similar phenomenon happened in The Jacobian Evaluation of Eq. (4).

$$m_R = \lim_{\varepsilon\to 0^+}\; m_{Bare}/\varepsilon \tag{29}$$

The second term in the right-hand side of Eq. (28) is the Chern-Simons lagrangian. Substituting Eq. (28) – Eq. (29) in Eq. (25) we get the result with fermions loops integrated out at large mass

$$\begin{aligned}&Z[\eta,\bar\eta] \sim \frac{1}{Z(0,0)}\times\left\{D^F[A_\mu(x)]\times\right. \\ &\exp\left\{-\frac{1}{2}\left(1-\frac{g^2}{(4\pi)^{\frac{3}{2}}}\,m_R\right)\cdot\int d^3x A^2_\mu(x)\right\} \\ &\exp\left\{-\frac{g^2\sqrt{\pi}}{2}\;\int d^3x\,(A_\mu\varepsilon^{\mu\nu\rho}F_{\gamma\rho}(A))\,(x)\right\} \\ &\delta^{(F)}[(\partial_\mu A_\mu]\times\exp\left\{+\frac{1}{2}\;\int d^3x d^3y\;\bar\eta(x)\right. \\ &\left.(i\gamma\partial + g\gamma A + m_R)^{-1}(x,y)\eta(y)\right\}\end{aligned} \tag{30}$$

Following closely ref. [3] now we analyse the large m_R limit of the external fermion sources by considering the Feynman path integral representation for the Feynman Green function of the Dirac operator in the presence of $A_\mu(x)$.

$$\begin{aligned}&(i\gamma\partial + g\gamma A + m_R)^{-1}_{\alpha\beta}(x,g) = \\ &\int_0^\infty dt\; e^{-m_R t}\times\left\{\int_{\substack{X^\mu(0)=x^\mu \\ X^\mu(t)=y^\mu}} D^F[X^\mu(\sigma)]\times\right. \\ &\Phi_{\alpha\beta}(x,y)\cdot\times\exp\left(ig\int_0^t d\sigma A_\mu(X(\sigma))\dot X^\mu(\sigma)\right)\end{aligned} \tag{31}$$

where the spin-factor is explicitly given by

$$\begin{aligned}\Phi_{\alpha\beta}(x,y) &= \int D^F[\pi^\mu(\sigma)]\exp\left(i\int_0^t d\sigma(\pi^\mu(\sigma)\cdot\dot{X}^\mu(\sigma))\right)\\ &\times \quad I\!P\left\{\exp i\int_0^t d\sigma(\gamma^\mu\cdot\pi_\mu\sigma))\right\} \qquad (32)\end{aligned}$$

Here $I\!P$ means the path ordenation of the 3D-γ^μ matrices along Feynman trajectory $X_\mu(\sigma)$; $(0\le\sigma\le t)$.

At the limit of large m_R, only the classical straight-line trajectory entering in the Path Integral is leading to Eq. (31) – Eq. (32) and producing the result

$$\begin{gathered}(i\gamma\partial+g\gamma R+m_R)^{-1}_{x\beta}(x,y)\sim\\ \left(U^{(1)}_\alpha U^{(2)}_\beta\right)\exp\left(iy\int_x^y A_\mu(x)dX^\mu\right) \qquad (33)\end{gathered}$$

where $U^{(1),(2)}_\alpha$ are usual Euclidean spinorial base associated to the free massive fermion fields $\{\bar\psi(x),\psi_\alpha(x)\}$

By grouping Eq. (33) and Eq. (34), we finally obtain our Polyakov's infrared Bosonic Theory for the 3D-Thirring model

$$\begin{aligned}&Z[\eta,\bar\eta,(m\to\infty)]=\int D^F[A_\mu(x)]\times\\ &\exp\left\{-\frac{1}{2}\left(1-\frac{g^2m_R}{(\eta\hbar)^{\frac{3}{2}}}\cdot\int d^3xA^2_\mu(x)\right\}\times\right.\\ &\exp\left\{-\frac{g^2\sqrt{\pi}}{2}\int d^3x(A_\mu\varepsilon^{\mu\nu\rho}F_{\nu\rho}()(x)\right\}\\ &\times\delta^{(F)}[(\partial_\mu A_\mu)]\exp\left\{+\frac{1}{2}\int d^3xd^3y\right.\\ &(\bar\eta(x)\eta_\beta(y))\cdot(U^\alpha_1U^\beta_2\times\\ &\exp\left(ig\int_x^y A_\mu(x)\dot{X}^\mu\right) \qquad (34)\end{aligned}$$

Now it is a straightforward consequence of Eq. (34) *the infrared (large mass) Bosonization formulae of the 3D-Abelian Thirring model* analogous to those associated to 2D-Thirring model

$$\begin{aligned}\psi^1_\alpha(x) &\overset{m\to\infty}{\sim} \psi^1_{\alpha,free}(x)\times\exp\left(ig\int_{-\infty}^x A_\mu\dot{X}^\mu\right)\\ \psi^2_\alpha(x) &\overset{m\to\infty}{\sim} \psi^2_{\alpha,free}(x)\times\exp\left(ig\int_{-\infty}^x A_\mu\dot{X}^\mu\right) \qquad (35)\end{aligned}$$

Here $A_\mu(x)$ is the quantum field associated to the "massive" Chern-Simon theory

$$\begin{aligned}\bar{\mathcal{L}}(A_\mu) &= \frac{1}{2}\left(1-\frac{g^2m_R}{(\eta\pi)^{\frac{3}{3}}}\right)\cdot\int d^3dxA^2_\mu(x)\\ &- \frac{g^2\sqrt{\pi}}{2}\frac{m_R}{|m_R|}\cdot\int d^3x(A_\mu\varepsilon^{\mu\nu\rho}F_{\nu\rho}(A))(x) \qquad (36)\end{aligned}$$

Equations (28) and (36) are our main result in this section about approximate Bosonization for the Thirring Model in the large mass limit.

In the important case for high T_c-superconductivity, modeled by by the Thirring Model coupled to an external divergence free current

$$W(\bar{\psi}, \psi, J_\mu) = \mathcal{L}(\psi, \psi) + \int d^3x J_\mu(x)(\bar{\psi}\gamma_\mu\psi)(x) \tag{37}$$

we can proceed as exposed above and obtain the associated Polyakov's full bosonized generating functional for correlations functions involving vectorial currents from the 3D-Thirring Model Ewq. (17).

$$\begin{aligned} W_{eFF}(J_\mu) &= \int D^F[A_\mu(x)] \cdot \delta^{(F)}[(\partial_\mu A_\mu)] \\ & \exp\left\{-\frac{1}{2}\int d^3x(A_\mu + J_\mu)^2(x)\right\} \\ & \exp\left\{\frac{g^2 m_R}{Z\cdot(4\pi)^{\frac{3}{2}}}\cdot d^3x(A_\mu^2(x))\right. \\ &- \left.\frac{g^2\sqrt{\pi}}{2}\,\frac{m_R}{|m_R|}\cdot\int d^3x(A_\mu\varepsilon^{\mu\nu\rho}F_{\gamma\rho}(A))(x)\right\} \end{aligned} \tag{38}$$

Finally, we point out that we have neglected in Eq. (25) the zero modes of the 3D-Dirac Operator which will be a subject of a particular paper on this problem.

V Effectives Four-Dimensional Bosonic Actions

The effective Bosonic Action obtained in section II, III are higher-order Four Dimensional Bosonic Field Theories and, this, should be considered only as an approximate and effective action as it shares all drawbacks and usefulness as all Effective Action proposed in the Literature ([9], [12]). However, there are some hints that Theories of the kind obtained in this my be given a meaning by a non-perturbative procedure as proposed in ref. [10] and this point may be advantageous for implement realizable approximate calculations usefull for realistic 4D-Field Theories.

In Three-Dimensions, we disagree from similar studies presented in ref. ([11]), since in this reference it was used the Deser-Jackiw interpolating Field to rewrite the Effective Action in terms of on Maxwell-Chern-Simon Field Theory *which do not hold true when one is analysing observables* and leads to a cumbersome theory in the Non-Abelian case (a theory in the strong limit $g_{phy}^2 \to \infty$). Finally, the *Wilson Loops of ref. ([11]) are unclear since the Non-Abelian Stokes Theorem was proved only in* R^2*, namelly for* $R^\rho(\rho > 2)$ *it was not unambigously proved that*

$$Tr_s P(\exp \oint_c, A_\mu a' X^\mu) = Tr_S\left[\left(\exp\int_\Sigma d\Sigma^{\mu\nu} Tr_t\left(\frac{\delta}{\delta\Sigma_{\mu\nu}}\cdot W[\tilde{C}^S W_t]\right)\right)\right] \tag{39}$$

where $\tilde{C}_t^S$ *are closed trajectories in the surface* Σ *(see ref. [13] for the notation) in* R^3.

As an alternative for the study of ref. [11] one should write Loop Wave Equation for the Wilson Loops eq. (39) and solve them by means of effective Theory of Chern-Simons String exposed in ref. ([14]).

We start this final part of our paper by considering the fermionic determinant of the self-adjoint Dirac Operator in $L^2(R^3)$

$$\begin{aligned} &\log\det\,(\not{D}(A)m) = S_{eff}(A', s) \\ = \;& -\frac{1}{2}\lim_{\varepsilon\to 0^+}\int_\varepsilon^\infty \frac{dt}{t}\; Tr_{(F)}\left(e^{[-t(\not{D}_D(A)+m)^2}\right) \end{aligned} \tag{40}$$

where we have introduced a non-parameter family of Dirac Operators interpolating the free Operator and that one in presence of a external gauge field.

$$\not{D}_S(A) + m = i\gamma^\mu(\partial_\mu - sigA_\mu) + m \tag{41}$$

We have regulated the fermion determinant by the proper-time method. At this we remark that $S_{eff}(A;s)$ satisfies the differential equation

$$\begin{aligned} &\frac{d}{ds}\; S_{eff}(A;s) = \\ &\lim_{\varepsilon\to 0^+}\int_\varepsilon^\infty dt\; Tr_{(F)}[g(\gamma^\mu A_\mu)\cdot(\not{D}_S(A)+m) \\ &\times\left(-t\not{D}_S^2(A) + m^2 + 2m\not{D}_S(A)\right))] \end{aligned} \tag{42}$$

Since we are interested at the large fermion mass limit $m\to\infty$, we neglect the term $\exp(-2m\not{D}_S(A)) \sim 1$ inside the trace operation of Eq. (3-A). We have, thus the result at large m;

$$\begin{aligned} &\lim_{m\to}\frac{d}{ds}\; S_{eff}(A,s) \\ &\overset{m\to\infty}{\sim} \lim_{\varepsilon\to 0^+}\int_\varepsilon^\infty dt\; e^{-tm^2} Tr_{(F)}[+(g(\gamma A)\cdot \\ &\times\not{D}_S(A)+m)\cdot\exp(-t)\not{D}_S(A)))] \\ &\overset{m\to\infty}{\sim} -\frac{g}{(4\pi)^{\frac{3}{2}}}\sum_{\ell 0}^{\infty}\left(\int_0^\infty dte^{-tm^2}\cdot t^{\ell-\frac{3}{2}}\right) \\ &\times\int d^3 Tr_{(F)}\left((\gamma A)\left[\not{D}_s(A)+m)b_\ell(x,x,A;s)\right]\right) \end{aligned} \tag{43}$$

where we have introduced a one-parameter family of Dirac Operators interpolating the free Operator and that on in presence of a external gauge field.

$$\not{D}(A) + m = \gamma^\mu(\partial_\mu - sigA_\mu) + m \tag{44}$$

We have regulated the fermion determinant by the proper-time method. At ths point we remark that $S_{eff}(A;s)$ satisfies the differential

$$\frac{d}{ds}\; S_{self}(A;s) =$$

$$\lim_{\varepsilon 0^+} \int_{\varepsilon}^{\infty} dt\ Tr_{(F)} \left[(g(\gamma^{\mu} A_{\mu}) \cdot (\not{D}(A) + m) \right.$$
$$\left. \times \exp\left(-t(\not{D}_S^2(A) + m^2 + 2m\not{D}_S(A)) \right) \right] \tag{45}$$

Since we are interested at the large fermion mass limit $m \to \infty$, we neglected the term $\exp(-2m\not{D}(A)) \sim 1$ inside the trace operation of Eq. (3-A). We have, thus at large m;

$$\lim_{m\to\infty} \frac{d}{ds} S_{eff}(A,s)$$
$$\overset{m-\infty}{\sim} \lim_{\varepsilon\to 0^+} \int_{\varepsilon}^{\infty} dt\ e^{-tm^2} Tr_{(F)}[+g(\gamma A) \cdot$$
$$\times \not{D}(A) + m) \cdot \exp(-t)\not{D}_S(A))]$$
$$\overset{m-\infty}{\sim} -\frac{g}{(4\pi)^{\frac{3}{2}}} \sum_{\ell=0}^{\infty} \left(\int_0^{\infty} dt e^{-tm^2} \cdot t^{\ell-\frac{3}{2}} \right)$$
$$\times \int d_x^3 Tr_{(F)} ((\gamma_A) [\not{D}_S(A) + m \times b_\ell(x,x,A,s])$$
$$\tag{46}$$

where $b_\ell(x,x,A,s)$ are the Seeley-De Witt coefficients associated to the asymptotic short-time $\tau \to 0^+$ of eq. (2.A) since we are considering the asymptotic limit of $m \to \infty$ means of the Laplace method for handling Saddle-Points of integrals (ref. [6]). Explicitly expressions for these coefficients are easily are easily calculated ([5]). At large fermion mass limit, in R^3 only the 2 first's. Seeley De Witt coefficients will be needed.

$$b_0(x,x,A,s) = 1\!\!1_{idem} \tag{47}$$

and

$$b_1(x,x,A,s) = -\frac{g_s}{2}[\gamma^{\mu},\gamma^{\nu}]F_{\mu\nu}(A) + g^2 A_{\mu}^2 + ig(\partial_{\mu} A_{\mu}) \tag{48}$$

After substituting eq. (5-1) and Eq. (6-A) into Eq. (4-A) and solving the S-differential equation we get Eq. (28) displayed in the text.

We point out that similar procedure may be used to evaluate the Fermion Propagator at large mass limit. However this evaluation is of no help in deducing infrared bosonization formulae of the kind of Eq. (25).

Finally we remark that the same procedure now involving the Seeley De Witt coefficient $b_2(x,xA,s)$ was used to deduce eq. ((13)).

Appendix A

Let us write a Formal Path Integral for Dirac Particles by using only Bosonic trajectories $X^{\mu}(\sigma)$ instead od super symmetric trajectories of Ref. [8].

By using the usual Plane Wave Euclidean spinor basis

$$\begin{aligned} |x,\alpha> &= e^{ipx} U_{\alpha}^{(1)}(p) \\ <g,\beta| &= U_{\beta}^{(2)}(p) e^{ips} \end{aligned} \tag{A.1}$$

where the spinors $\left\{U_\alpha^{(1)}(p), U_\beta^{(2)}(p)\right\}$ satisfy the free Dirac equation and the completeness relation

$$U_\alpha^{(1)}(p) \cdot U_\beta^{(2)}(p) = \delta_{\alpha\beta} \tag{A.2}$$

one can write the Fermion Propagator in the present of a external field in the following form (see Ref. [3]).

$$\begin{aligned}
&S_{\alpha\beta}(x-y) = \int_0^\infty \; dt < x, \alpha | \exp(-T(-i\gamma\partial + g\gamma A + m))|y, \beta > \\
&\int_0^\infty dT \; e^{-mT} \; \int_{\substack{X_\mu(0)=x \\ X_\mu(t)=y}} D^F \left[X_\mu(\sigma)\right] \int \left[p_\mu(\sigma)\right] \\
&\exp\left(i\int_0^T d\sigma p_u \cdot \dot{X}^\mu(\sigma)\right) \\
&{}_{Dirac}\left\{\exp\left(i\int_0^T \gamma^\mu(p_\mu(\sigma) + gA_\mu(x(\sigma)d\sigma\right)\right\}
\end{aligned} \tag{A.3}$$

Where$_{Dirac}$ means the ordenation along the bosonic trajectorie of the Dirac indexes coming the γ^μ-exponential involving the external Gauge Fields $A_\mu(X)$. Note that the $P_\mu(\sigma)$ Path Integral is free at this ends points.

Let us now consider the formal variable change in the Path Integral in Eq. (4-A)

$$P_\mu(\sigma) + gA_\mu(X(\sigma)) = \pi_\mu(\sigma) \tag{A.4}$$

As a consequence of Eq. (5-A), we get the more Transport expression used in Eq. (23) of the text.

$$\begin{aligned}
&S_{\alpha\beta}(x-y) = \int_0^\infty dT \; e^{-mT} \int_{\substack{X_\mu(p)=x \\ X_\mu(t)=y}} D^F[X_\mu(\sigma)] \\
&\times \int D^F[\pi_\mu(\sigma)] \times \exp\left(i\int_0^T d\sigma\pi_\mu(\sigma) \cdot \dot{X}^\sigma\right) \\
&\times \exp\left(-ig\int_0^T d\sigma A_\mu(X(\sigma)\dot{X}(\sigma))\right) \\
&\times_{Dirac}\left\{\exp i\int_0^T d\sigma(\gamma\Pi_\mu)(\sigma)\right\}
\end{aligned} \tag{A.5}$$

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