CBPF-NF-013/95

Dynamics of Two Interacting Particles in a Magnetic Field in Two Dimensions

Sergio Curilef and *Francisco Claro**

Centro Brasileiro de Pesquisas Físicas - CBPF
Rua Dr. Xavier Sigaud, 150
22290-180 – Rio de Janeiro, RJ – Brasil

*Pontificia Universidad Católica de Chile, Facultad de Física
Vicuña Mackenna 4860, Casilla 306, Santiago 22, Chile

Abstract

The classical dynamics of two interacting particles of equal mass and equal or opposite charge, moving in a plane and a perpendicular magnetic field, is discussed. The simplest trajectories are similar to those of a single particle in the presence of crossed electric and magnetic fields (Hall configuration). Such motion occurs over a ribbon that may be straight (for opposite charges), or bent into a circle (for identical particles).

Key-words: Classical dynamics; Two-body problem; Two dimensions.

I Introduction

Since the discovery of the Quantum Hall Effect there has been much interest in understanding the dynamics of electrons confined to move in two dimensions in the presence of a magnetic field perpendicular to the plane of motion[1]. The confinement is possible at the interface between two materials, typically a semiconductor and an insulator such as GaAs and AlGaAs, where a quantum well that traps the particles is formed, forbidding their motion in the direction perpendicular to the plane of the interface at low energies. The integral Quantum Hall Effect has been explained using a free electron model while a proper treatment of the fractional effect requires that the electron-electron interaction be included[2]. The interacting quantum problem has been treated in the Hartree-Fock[3, 4] and variational[5, 6] approximations, as well as with numerical methods[7]. It is a difficult many body problem for which further understanding than that provided by the approximate treatments is needed. The simplest case, that of just two interacting particles in an additional confining parabolic potential has been treated by Taut[8]. He found exact analytical solutions for a selected values of the magnetic field. Why other values do not lend themselves for such solutions is unclear.

In this paper we present a complete solution of the classical two-body problem ignoring radiation and relativistic effects. Our purpose is to provide information on the trajectories in order to guide further efforts in the understanding of the quantum effects. Also, there have been recent experiments involving interesting effects such as the Weiss oscillations[9], in which electrons behave semiclassically. Although these effects may be explained using non interacting electrons it is possible that the interaction becomes relevant in the limit of very dilute electron systems, as is the case in the fractional Quantum Hall effect (low filling fraction).

In Sec. II we study the case of two identical particles. The problem is separable in center of mass and relative coordinates. The center of mass moves as a free particle in the magnetic field, of twice the charge and mass as each constituent of the pair. The Coulomb repulsion affects the relative motion. We find that this motion is similar to that of a single particle in crossed electric and magnetic fields (Hall configuration), only that the rectilinear strip in which this latter motion takes place is bent into a circle (Corbino geometry). In Section III we discuss the case of two particles of the same mass and opposite charge. The problem is non-separable, yet becomes one dimensional in a special case for which we find solutions also similar to those in the Hall configuration over a rectilinear strip.

II Identical particles

We consider two identical particles of mass m and charge e in a uniform magnetic field $\vec{B}$. We are interested in the motion over a plane perpendicular to the magnetic field. The particles interact with the field, and with each other through the Coulomb repulsion. The dynamics is derived from the lagrangian

$$L(\vec{r}_1, \vec{r}_2, \dot{\vec{r}}_1, \dot{\vec{r}}_2) = \frac{1}{2}m\dot{\vec{r}}_1^2 + \frac{1}{2}m\dot{\vec{r}}_2^2 + \frac{e}{c}\vec{A}_1(\vec{r}_1)\cdot\dot{\vec{r}}_1 + \frac{e}{c}\vec{A}_2(\vec{r}_2)\cdot\dot{\vec{r}}_2 - V(|\vec{r}_1 - \vec{r}_2|). \quad (1)$$

Here $\vec{r}_1$ ($\vec{r}_2$) is the position vector of particle 1 (2), $\vec{A}(\vec{r})$ is the vector potential, and $V(r) = e^2/\epsilon_m r$, where ϵ_m is the dielectric constant of the medium in which the particles move. The problem is separable if center of mass (CM) and relative coordinates are used. Let $\vec{R} = (\vec{r}_1 + \vec{r}_2)/2$ denote the position of the CM $\vec{R} = (\vec{r}_1 + \vec{r}_2)/2$, and $\vec{r} = \vec{r}_2 - \vec{r}_1$ denote the relative position vector. In the symmetric gauge $\vec{A}(\vec{r}) = \frac{1}{2}\vec{B}\times\vec{r}$ we obtain $L = L_{cm} + L_{rel}$, where

$$L_{cm}(R, \dot{R}; \theta_{cm}, \dot{\theta}_{cm}) = \frac{1}{2}M(\dot{R}^2 + R^2\dot{\theta}_{cm}^2) + \frac{1}{2}\frac{Q}{c}BR^2\dot{\theta}_{cm} \quad (2)$$

describes the dynamics of the CM, and

$$L_{rel}(r, \dot{r}; \theta_{rel}, \dot{\theta}_{rel}) = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}_{rel}^2) + \frac{1}{2}\frac{q}{c}Br^2\dot{\theta}_{rel} - qQ/\epsilon_m r \quad (3)$$

describes the relative motion, all in polar coordinates. From these lagrangians we see that the CM motion is that of a single particle with charge $Q = 2e$ and mass $M = 2m$ in the presence of a magnetic field $\vec{B}$. It describes a circle with an angular frequency $\dot{\theta} = -\omega_c$, with $\omega_c = eB/mc$ the cyclotron frequency. The relative motion is in turn that of a particle with charge $q = e/2$ and mass $\mu = m/2$ in the presence of an external magnetic field, and the electric field produced by a charge $Q = 2e$ fixed at the origin.

We will use the dimensionless notation $\xi = R/\ell_B$, $\dot{\xi} = (d\xi/dt)/\omega_c$, $\rho = r/\ell_B$, $\dot{\rho} = (d\rho/dt)/\omega_c$, where $\ell_B = (mc^2/B^2)^{1/3}$ is the the natural classical length scale and $\omega_c = eB/mc$ is the cyclotron frequency. As units of energy and angular momentum we use $E_o = e^2/\ell_B$ and $P_o = m\omega_c\ell_B^2$, respectively. The lagrangians (2) and (3) do not contain the azimutal angle. The conjugate momenta

$$p_\theta^{cm} = 2\xi^2(\frac{\dot{\theta}_{cm}}{\omega_c} + \frac{1}{2}) \quad (4)$$

$$p_\theta^{rel} = \frac{1}{2}\rho^2(\frac{\dot{\theta}_{rel}}{\omega_c} + \frac{1}{2}) \quad (5)$$

are therefore constants of motion. The energies associated with the classical motion are

$$\epsilon_{cm} = \dot{\xi}^2 + \frac{1}{4}(\frac{p_\theta^{cm}}{\xi} - \xi)^2 \quad (6)$$

$$\epsilon_{rel} = \frac{1}{4}\dot{\rho}^2 + (\frac{p_\theta^{rel}}{\rho} - \frac{1}{4}\rho)^2 + \frac{1}{\epsilon_m\rho}, \quad (7)$$

The form of ϵ_{cm} and ϵ_{rel} differ by the presence of a Coulomb term in ϵ_{rel}, making the relative and CM motion very different.

Integrating Eq.(6) with the aid of Eq.(4) we obtain for the radial coordinate of the CM, the equation

$$\xi^2 - 2\xi\sqrt{\epsilon_{cm} + p_\theta^{cm}}\cos(\theta_{cm} - \theta_0) + p_\theta^{cm} = 0, \tag{8}$$

where θ_o is a constant of integration. Equation (8) represents a circle of radius $\sqrt{\epsilon_{cm}}$ centered at $\xi_o = \sqrt{\epsilon_{cm} + p_\theta^{cm}}$, so that ϵ_{cm} defines the orbit radius while p_θ^{cm} fixes the position of its center.

The integral of motion for the relative coordinate is obtained from Eqs.(5) and (7). We get,

$$\Delta\theta_{rel} = 4\int_{\rho_<}^{\rho} \frac{d\rho'(\frac{p_\theta^{rel}}{\rho'} - \frac{1}{4}\rho')}{\sqrt{-\rho'^4 + (16\epsilon + 8p_\theta^{rel})\rho'^2 - 16\rho'/\epsilon_m - 16{p_\theta^{rel}}^2}}. \tag{9}$$

Here $\rho_< < \rho < \rho_>$, where $\rho_>$ and $\rho_<$ are the extreme values of the relative coordinate of the orbit. They are determined by the two real and non-negative solutions of Eq.(7) under the condition $\dot{\rho} = 0$. The other two solutions are $c_\pm$ (see Appendix A1). In terms of these constants we can rewrite Eq.(9) in the form

$$\begin{aligned}\Delta\theta_{rel}(\rho) = \; & 2\frac{c_+ - \rho_<}{\sqrt{(\rho_> - c_+)(\rho_< - c_-)}}\left(2p_\theta^{rel}\Pi(\lambda(\rho), \eta\frac{c_+}{\rho_<}, \sigma) + \Pi(\lambda(\rho), \eta, \sigma)\right) \\ & + 2\frac{1}{\sqrt{(\rho_> - c_+)(\rho_< - c_-)}}(2\frac{p_\theta^{rel}}{c_+} - c_+)F(\lambda(\rho), \sigma),\end{aligned} \tag{10}$$

where $F(a, b)$ and $\Pi(a, b, c)$ are elliptic integrals of the first and third kind, respectively, and

$$\lambda(\rho) = \arcsin\sqrt{\frac{(\rho_> - c_+)(\rho - \rho_<)}{(\rho_> - \rho_<)(\rho - c_+)}}, \tag{11}$$

$$\eta = \frac{\rho_> - \rho_<}{\rho_> - c_+}, \tag{12}$$

$$\sigma = \sqrt{\frac{(\rho_> - \rho_<)(c_+ - c_-)}{(\rho_> - c_+)(\rho_< - c_-)}}. \tag{13}$$

The classical motion of the pair is the composition of the circular CM motion and the relative motion. The latter is in general non-circular and is not necessarily periodic. A simplifying property common to pairs of particles of equal mass is however that in the CM system both particles describe identical orbits with a phase difference of π.

The simplest possible classical orbit one can obtain for this system is the circle. It corresponds to a situation in which the Lorentz force and Coulomb repulsion combine to exactly produce the centripetal acceleration necessary to maintain a circular motion. The condition is

$$2\frac{v^2}{\rho} = v - \frac{1}{\epsilon_m\rho^2}, \tag{14}$$

where $v = \rho|\dot{\theta}_{rel}|/2\omega_c$ is the dimensionless constant speed of each particle. If the CM is at rest the motion in truly circular in the laboratory frame, while only the relative motion is circular if the CM moves. The constant relative distance may be obtained from our formalism by noting that Eq. (7) is the sum of a kinetic energy term and the effective potential

$$V_{eff}(\rho) = \left(\frac{p_\theta^{rel}}{\rho} - \frac{1}{4}\rho\right)^2 + \frac{1}{\epsilon_m \rho}. \tag{15}$$

This potential has a minimum at which the relative motion is circular. Setting the derivative to zero one then obtains Eq. (14) with the aid of Eq. (5). One also obtains for the distance between the particles,

$$\rho_m = v + \sqrt{v^2 + \frac{1}{\epsilon_m v}}. \tag{16}$$

In ordinary units the radius of the circle is then $R_m = \rho_m \ell_B/2$. With the aid of Eqs. (5) and (16) one obtains for the frequency of the circular motion in terms of the parameter ρ_m,

$$\frac{\dot{\theta}_{rel}}{\omega_c} = \begin{cases} -\frac{1}{2} + \frac{1}{2}\sqrt{1 - 8\frac{1}{\epsilon_m \rho_m^3}} & p_\theta^{rel} > 0 \\ -\frac{1}{2} & p_\theta^{rel} = 0 \\ -\frac{1}{2} - \frac{1}{2}\sqrt{1 - 8\frac{1}{\epsilon_m \rho_m^3}} & p_\theta^{rel} < 0. \end{cases} \tag{17}$$

The role of p_θ^{rel} is clear in the non-interacting limit for which there is relative motion between the particles if $p_\theta^{rel} < 0$ only. Then the center of mass and the particles themselves describe concentric circles with the same angular frequency $-\omega_c$ and radii $\sqrt{\epsilon_{cm}}$ and $\sqrt{\epsilon_{cm}} \pm \rho_m/2$, respectively. This is shown in Fig. 1(a). When $p_\theta^{rel} > 0$ in the same limit there is no relative motion and the particles and the center of mass describe circles of the same radius, if they move at all, their centers aligned and a distance $\rho_m/2$ appart, as shown in Fig. 1(b). When the interaction is turned on the motion is more complex. An example is shown in Fig. 1(c), drawn for $p_\theta^{rel} = 1$. The frequency of the small oscillations about the circular motion is easily obtained and we get $\omega_\rho = 2\sqrt{1/\epsilon_m \rho_m^3}$.

The equations of motion are greatly simplified when $p_\theta^{rel} = 0$. The trajectory in the CM frame is a circular orbit of radius $\rho_m = 2/\epsilon_m^{1/3}$ and frequency $\dot{\theta}_{rel} = -\omega_c/2$. Since the CM moves with angular frequency $-\omega_c$ then the frequencies are commensurate and the orbits are closed. Figure 1(c) shows the case in which the CM is at rest, and is just a circle with the particles moving in diametrally opposite points. If the CM is moving then one has a situation as shown in Fig. 1(d), obtained by giving the same initial velocity to both particles.

When the motion is such that the distance between the particles is not constant one has, always for $p_\theta^{rel} = 0$, that the general integral of motion Eq. (9) is reduced to

$$\Delta\theta_{rel} = -\frac{2\rho_<}{\sqrt{\rho_>(2\rho_> + \rho_<)}}\Pi\left(\arcsin\sqrt{\frac{\rho_>(\rho - \rho_<)}{\rho(\rho_> - \rho_<)}}, \frac{\rho_> - \rho_<}{\rho_>}, \sqrt{\frac{{\rho_>}^2 - {\rho_<}^2}{\rho_>(2\rho_> + \rho_<)}}\right), \tag{18}$$

where $\Pi(a, b, c)$ is the elliptic integral of the third kind,

$$\rho_< = 4\sqrt{\tfrac{2\epsilon_{rel}}{3}}(\sqrt{3}\sin\tfrac{\alpha}{3} - \cos\tfrac{\alpha}{3}), \tag{19}$$

$$\rho_> = 8\sqrt{\tfrac{2\epsilon_{rel}}{3}}\cos\tfrac{\alpha}{3}, \tag{20}$$

and $\cos\alpha = -(2\sqrt{3/2\epsilon_{rel}})/\epsilon_m$, with $\pi/2 < \alpha \leq \pi$.

A general statement about the motion is that it is confined in spite of the Coulomb repulsion. This may be seen by noting that the effective potential (15) diverges both at the origin and in the limit $\rho \to \infty$, at the separation (16). Confinement is provided by the magnetic field.

A special point in the relative motion is the separation at which the parenthesis in Eq.(15) vanishes, that is, at $\rho_o = 2\sqrt{p_\theta^{rel}}$. Then, the potential is just the Coulomb repulsion. Figure (2) shows the possible orbits in the relative coordinates, which we shall describe assuming the CM to be at rest. Figure 2(a) is for $\epsilon_{rel} = 1/\epsilon_m\rho_o$ and is obtained when the particles are released from rest at an initial separation ρ_0. They move instantaneously appart and then the Lorentz force curves the trajectories. The orbit is closed or open depending on whether Eq.(9) is an integral multiple of 2π or not. Figure 2(b) is for $\epsilon_{rel} > 1/\epsilon_m\rho_o$ and corresponds to equal and opposite initial velocities in the direction perpendicular to the line joining the particles and directed such that the initial motion is counterclockwise. Note that the sense of rotation is changed whenever $\rho = \rho_o$. Finally, Fig.2(c) is for $\epsilon_{rel} < 1/\epsilon_m\rho_o$ and is obtained, always with the CM at rest, when the initial motion is as before, but the initial sense of rotation is clockwise. In this case $\dot\theta_{rel}$ does not change sign. When the CM is moving the above orbits are superposed to the uniform rotation of the CM. Also, while the the three kinds are possible for $p_\theta^{rel} > 0$, in the case $p_\theta^{rel} < 0$ they are all, in their relative motion, of the type shown in Fig. 2(b).

The curves in Fig.2 correspond qualitatively to the possible trajectories of a particle in crossed magnetic and electric fields in the usual Hall configuration [10]. While in the standard Hall effect the field sources are fixed and the motion is over a straight strip, in our case they move over a circular strip (Corbino geometry). The observed curvature is produced by the motion of the field sources.

III Particles with opposite charge

Another simple case is that of two particles that differ only in the sign of the charge, such as a particle and its antiparticle. Consider two particles of mass m and charge e and $-e$ respectively. As before the motion is limited to a planar surface and a magnetic field $\vec{B}$ perpendicular to this plane is present. The particles interact with the magnetic field and with each other through the Coulomb attraction $V(|\vec{r}_1 - \vec{r}_2|) = -e^2/\epsilon_m|\vec{r}_1 - \vec{r}_2|$. As before we use CM and relative coordinates. In the symmetric gauge the lagrangean is

$$L(\vec{R},\vec{r};\dot{\vec{R}},\dot{\vec{r}}) = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 + \frac{1}{2}\frac{e}{c}(\vec{B}\times\vec{R}\cdot\dot{\vec{r}} + \vec{B}\times\vec{r}\cdot\dot{\vec{R}}) + e^2/\epsilon_m r. \tag{21}$$

Using the dimensionless units defined in Sec. 1 a first constant of motion is

$$-2\hat{z}\times\dot{\vec{\xi}} + \vec{\rho} = \vec{\rho}_o. \tag{22}$$

The CM position $\vec{\xi}$ and the relative position vector $\vec{\rho}$ are coupled in this equation. In fact, in contrast to the case of identical particles, now the lagrangean (21) is not separable and the CM and relative motions are coupled.

A second constant of motion is the energy, which may be written in terms of relative coordinates only in the form,

$$\epsilon = \frac{1}{4}\dot{\vec{\rho}}^2 + \frac{1}{4}(\vec{\rho} - \vec{\rho}_o)^2 - \frac{1}{\epsilon_m \rho}, \tag{23}$$

Note that information about motion of the CM is contained in this expression through the constant $\vec{\rho}_o$.

Motion is in general quite complex in this case. Simple trajectories are however obtained when the motion of the CM is uniform. Then, according to Eq. (22) the relative position vector $\vec{\rho}$ is also constant and the particles move in parallel straight lines. This situation corresponds to a Lorentz force that just cancels the Coulomb attraction between the particles. Defining the effective potential

$$v_{eff}(\rho) = \frac{1}{4}(\rho - \rho_o)^2 - \frac{1}{\epsilon_m \rho}. \tag{24}$$

it is easy to show that parallel motion occurs at the minimum of this function. Motion is perpendicular to the line joining the particles, which move separated by the constant distance

$$\rho_m = \frac{\rho_o}{3}(1 + 2\cos\frac{\alpha}{3}) \tag{25}$$

with the CM speed $\dot{\xi} = (\rho_m - \rho_o)/2$. Here α is given by the relation $\cos\alpha = 1 - 9/\epsilon_m \rho_o^3$. This parallel motion is not always possible. In fact, for Eq.(24) to have extrema the condition $\rho_o > 3/(2\epsilon_m)^{1/3}$ must be fulfilled. Figure 3(a) shows v_{eff} when this condition is not met, while relative minimum and maximum are present, while Fig. 3(b) is for the case when there is a relative minimum and a relative maximum. The distance to the origin in this latter case is given by

$$\rho_M = (1 + \sqrt{3}\sin\frac{\alpha}{3} - \cos\frac{\alpha}{3})\frac{\rho_o}{3}, \tag{26}$$

The effective potential (24) is a usefull quantity in general when $p_\theta^{rel} = 0$, in which case the problem becomes one-dimensional. In polar coordinates this means $\dot{\theta}_{rel} = 0$, so the relative position vector does not change direction in time, though in general its length changes. The integral of motion is given by

$$\Delta t(\rho) = \frac{2}{\omega_c}\frac{\rho_>}{\sqrt{(\rho_> - c_-)\rho_<}}\Pi(\lambda(\rho), 1 - \frac{\rho_>}{\rho_<}, \rho), \tag{27}$$

where $\rho_< \leq \rho \leq \rho_>$. Here $\rho_{\lessgtr}$ are the extremes of the orbit in the relative coordinate, c_- is a constant (see Apendix A2) and $\lambda(\rho)$ is given by Eq.(11) with $c_+ = 0$. $\Pi(a, b, c)$ is the elliptic integral of the third kind.

In Fig. 4 we show trajectories when the effective potential is of the form shown in Fig. 3(a), that is, when $\rho_o \leq 3/(2\epsilon_m)^{1/3}$ and there are no extrema in the potential. In this case only $\rho_>$ is real and positive, while the other extreme of the orbit is $\rho_< = 0$. In the figure the trajectory of only one particle is exhibited, since the other particle performs a motion that is the mirror image about the axis $y = 0$. The magnetic field is assumed coming out of the paper towards the reader. The particles start their motion placed over the y-axis at a distance $\rho_>$, and with the CM at the origin. All motion has a periodic oscillation with an average drift in the x-direction. We have the following cases:

(i) for $\epsilon = -1/\epsilon_m \rho_o$ the motion starts from rest at an initial maximum separation $\rho_> = \rho_o$. The initial radial motion due to the Coulomb attraction is deflected by the magnetic field (Fig.4a).

(ii) for $\epsilon < -1/\epsilon_m \rho_o$, the maximum separation is always greater than ρ_o. The particles have equal initial velocities in the $-x$ direction (fig. 4b);

(iii) for $\epsilon > -1/\epsilon_m \rho_o$, one has $\rho_> < \rho_o$. It corresponds to particles with equal initial velocities in the positive sense of the x axis (Fig. 4c).

Neither of these cases allow parallel motion at constant speed. When $\rho_o > 3/(2\epsilon_m)^{1/3}$ and the potential has a minimum (and a maximum) as in Fig. 3(b), the motion is qualitatively as described above, only that in case i the derivative of the trajectory is at all points continuous and in all cases the particles move in non-overlapping parallel strips (Fig. 5). Also, as discussed above, parallel uniform motion is possible when the separation is the minimum ρ_m of the potential.

IV Conclusions

In summary, we have shown that the planar motion of two interacting charged particles in a perpendicular magnetic field is bounded even when the Coulomb force is repulsive. We have ignored radiation and relativistic effects, as is usual in the low energy limit that applies in current day experiments with two dimensional motion at the interface between a semiconductor and an insulator. When the particles are identical the center of mass describes a circle with the single particle cyclotron frequency, while in the relative coordinates the motion is in a bounded circular ribbon. The simplest trajectory is a circle with the particles always in diametrally opposite points. When the particles have opposite charges the simplest trajectories are straight parallel lines with uniform motion. More complex trajectories include a family in which there is constant average parallel drift with periodic oscillations about this average. It is hoped that the insight that our classification of orbits provides will be helpfull in the search for a complete solution of the quantized problem.

Acknowledgments

This work was supported in part by Fundación Andes/Vitae/Antorchas, grant 12021-10, and by Fondo Nacional de Ciencias, grants 1930553 and 1940062.

Appendix: Roots of the orbit equation

A1 Identical particles

The equation for the extremes of the orbit is

$$f = \rho^4 - 8(2\epsilon + p_\theta^{rel})\rho^2 + 16\frac{1}{\epsilon_m}\rho + 16 {p_\theta^{rel}}^2. \tag{28}$$

The roots are [11]

$$\rho_> = \frac{1}{2}\sqrt{8(2\epsilon + p_\theta^{rel}) + u_1} \pm \frac{1}{2}\sqrt{8(2\epsilon + p_\theta^{rel}) - u_1 + 4\sqrt{\frac{u_1^2}{4} - 16 {p_\theta^{rel}}^2}}, \tag{29}$$

and

$$c_\pm = -\frac{1}{2}\sqrt{8(2\epsilon + p_\theta^{rel}) + u_1} \pm \frac{1}{2}\sqrt{8(2\epsilon + p_\theta^{rel}) - u_1 - 4\sqrt{\frac{u_1^2}{4} - 16 {p_\theta^{rel}}^2}}. \tag{30}$$

Here

$$u_1 = \left(r + \sqrt{q^3 + r^2}\right)^{1/3} + \left(r - \sqrt{q^3 + r^2}\right)^{1/3} + \frac{8}{3}(2\epsilon + p_\theta^{rel}), \tag{31}$$

where

$$r = \frac{2048}{9}\left({p_\theta^{rel}}^2 \epsilon + \frac{2}{3}{p_\theta^{rel}}^3 - \frac{2}{3}\epsilon^3 - \epsilon^2 p_\theta^{rel}\right) + 128/\epsilon_m^2, \tag{32}$$

$$q = -\frac{256}{9}\left({p_\theta^{rel}}^2 + \epsilon^2 + \epsilon p_\theta^{rel}\right). \tag{33}$$

A2. Particles of opposite charge

The equation for the extremes of the orbit is

$$\rho^3 - 2\rho_o\rho^2 + (\rho_o^2 - 4\epsilon)\rho - 4\epsilon_c = 0. \tag{34}$$

The roots are [11]

$$\rho_{>} = f_{+} + f_{-} + \frac{2}{3}\rho_o, \tag{35}$$

$$\rho_{<} = -\frac{1}{2}(f_{+} + f_{-}) + \frac{2}{3}\rho_o + \frac{\sqrt{3}i}{2}(f_{+} - f_{-}), \tag{36}$$

$$c_{-} = -\frac{1}{2}(f_{+} + f_{-}) + \frac{2}{3}\rho_o - \frac{\sqrt{3}i}{2}(f_{+} - f_{-}), \tag{37}$$

where f_{+} and f_{-} are given by

$$f_{\pm} = \left(\frac{2}{\epsilon_m} + \frac{4}{3}\rho_o\epsilon - \frac{1}{27}\rho_o^3 \pm \sqrt{\left(\frac{2}{\epsilon_m} + \frac{4}{3}\rho_o\epsilon - \frac{1}{27}\rho_o^3\right)^2 - \left(\frac{4}{3}\epsilon + \frac{1}{9}\rho_o^2\right)^3} \right)^{1/3}. \tag{38}$$

Figures

Fig.1 Orbits for $\epsilon^{rel} = 1/\epsilon_m \rho_m$. Both particles describe the same orbit. For details see text.

Fig.2 Orbit for energies (a) $\epsilon^{rel} = 1/\epsilon_m \rho_o$, (b) $\epsilon^{rel} = 1/\epsilon_m \rho_o$ and (c) $\epsilon^{rel} = 1/\epsilon_m \rho_o$. The center of mass is at rest.

Fig.3 Effective potential for (a) $\rho_o \leq 3(1/2\epsilon_m)^{1/3}$ and (b) $\rho_o > 3(1/2\epsilon_m)^{1/3}$.

Fig.4 Trajectories for the potential of Fig.3a for (a) $\epsilon^{rel} = -1/\epsilon_m \rho_o$, (b) $\epsilon^{rel} < -1/\epsilon_m \rho_o$ and (c) $\epsilon^{rel} > -1/\epsilon_m \rho_o$. In each case $p_\theta^{rel} = 0$.

Fig.5 Trajectories for the potential of Fig.3b and (a) $\epsilon^{rel} = -1/\epsilon_m \rho_o$, (b) $-1/\epsilon_m \rho_o < \epsilon^{rel} < v_{eff}(\rho_{\rho_M})$, (c) $v_{eff}(\rho_m) < \epsilon^{rel} < 1/\epsilon_m \rho_o$.

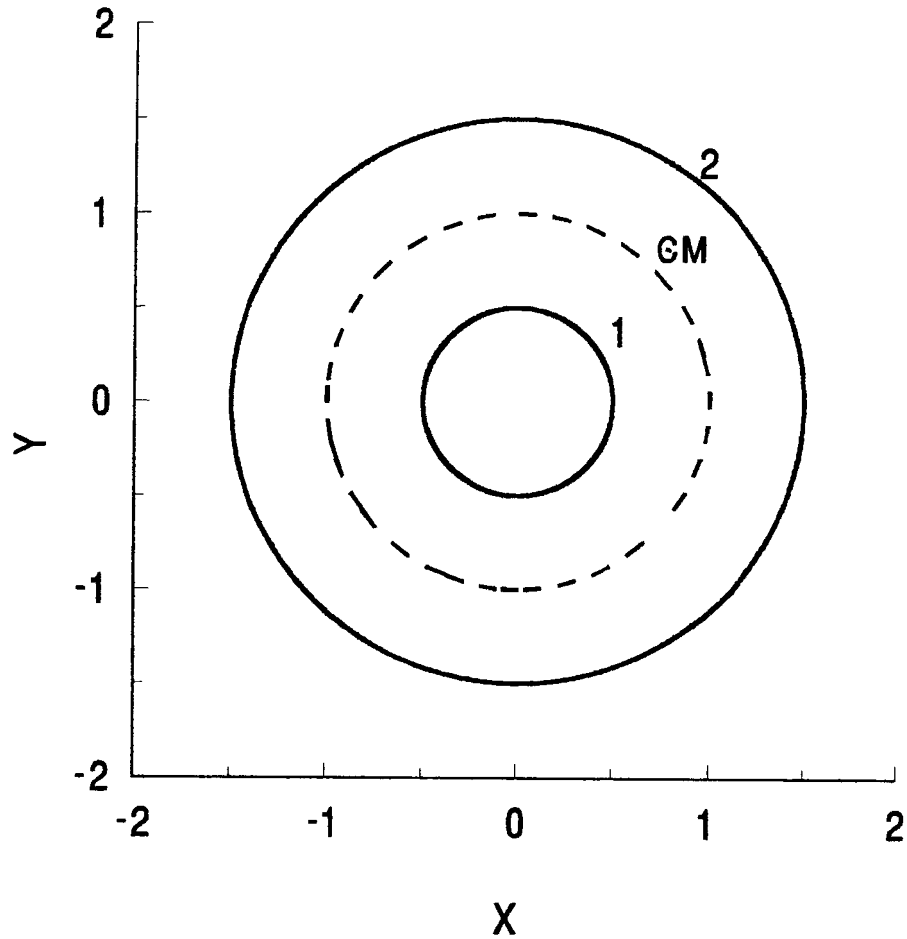

Fig. 1a

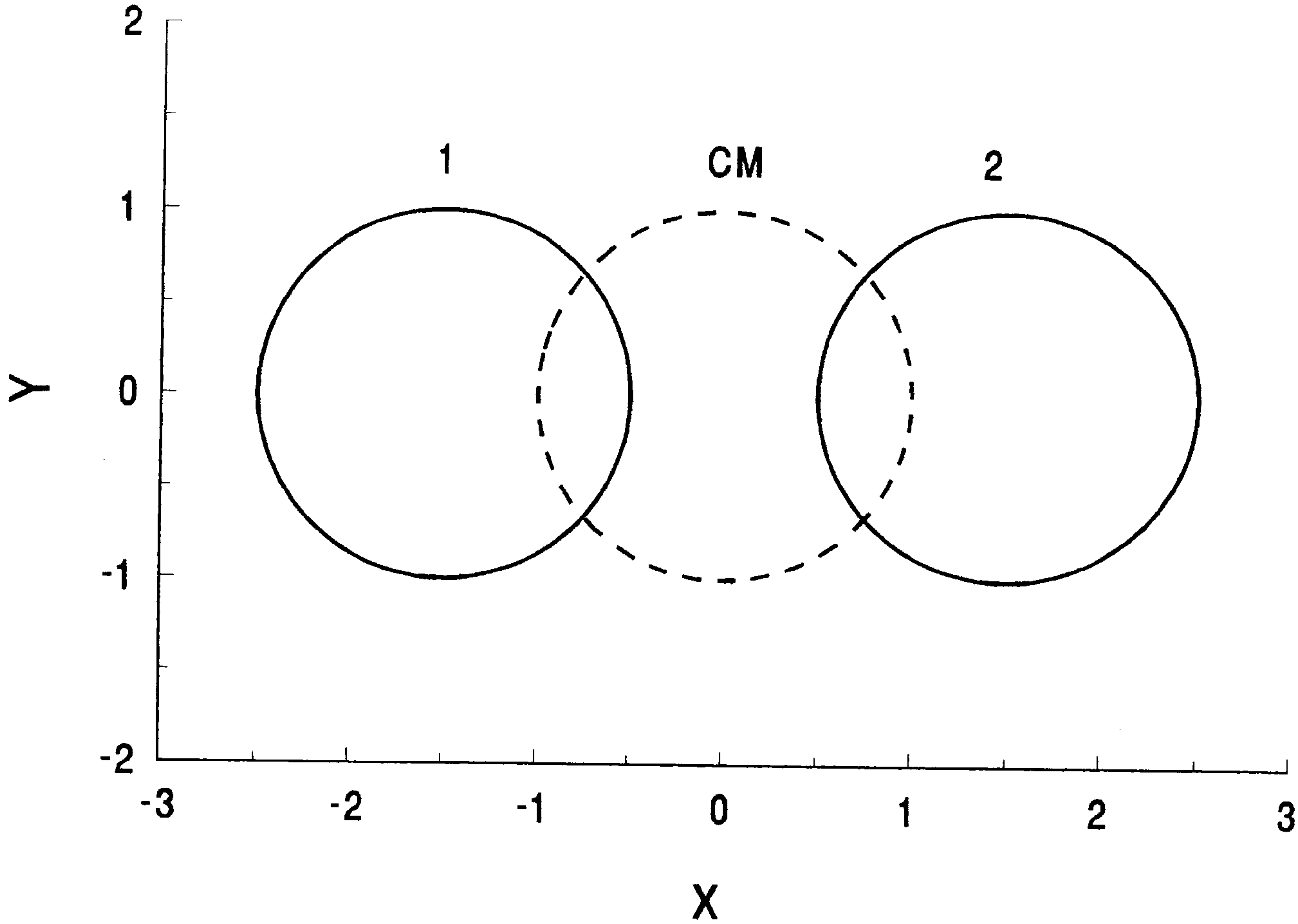

Fig. 1b

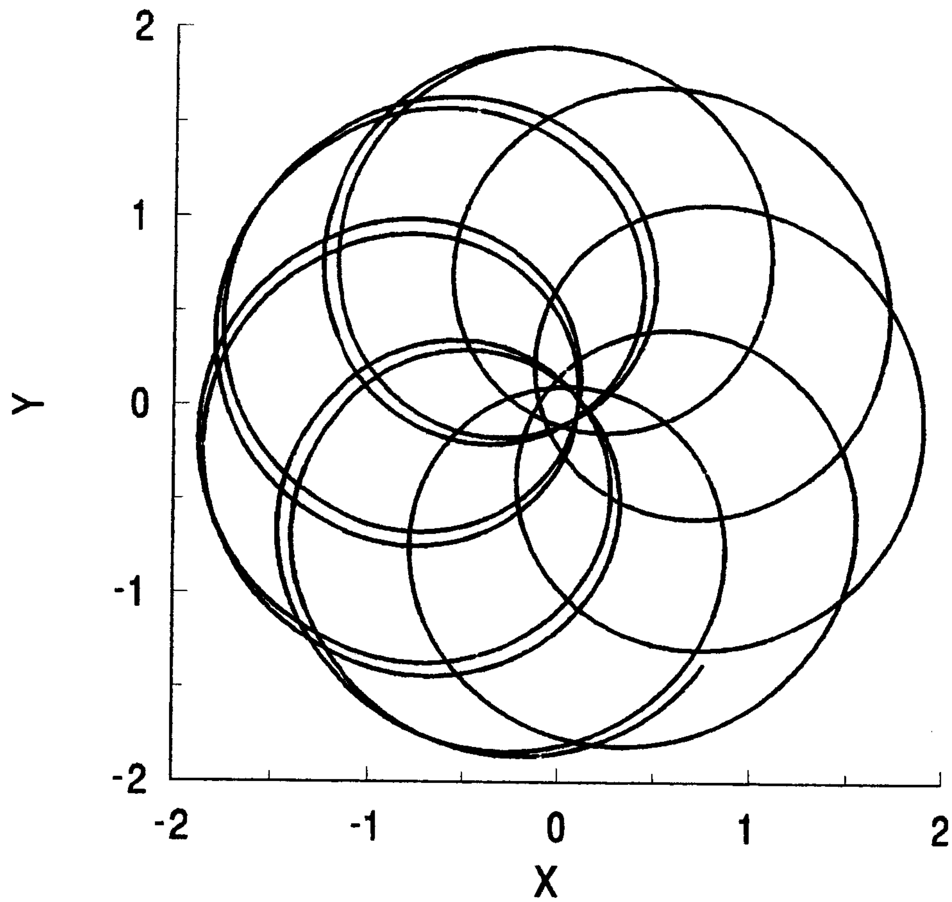

Fig. 1c

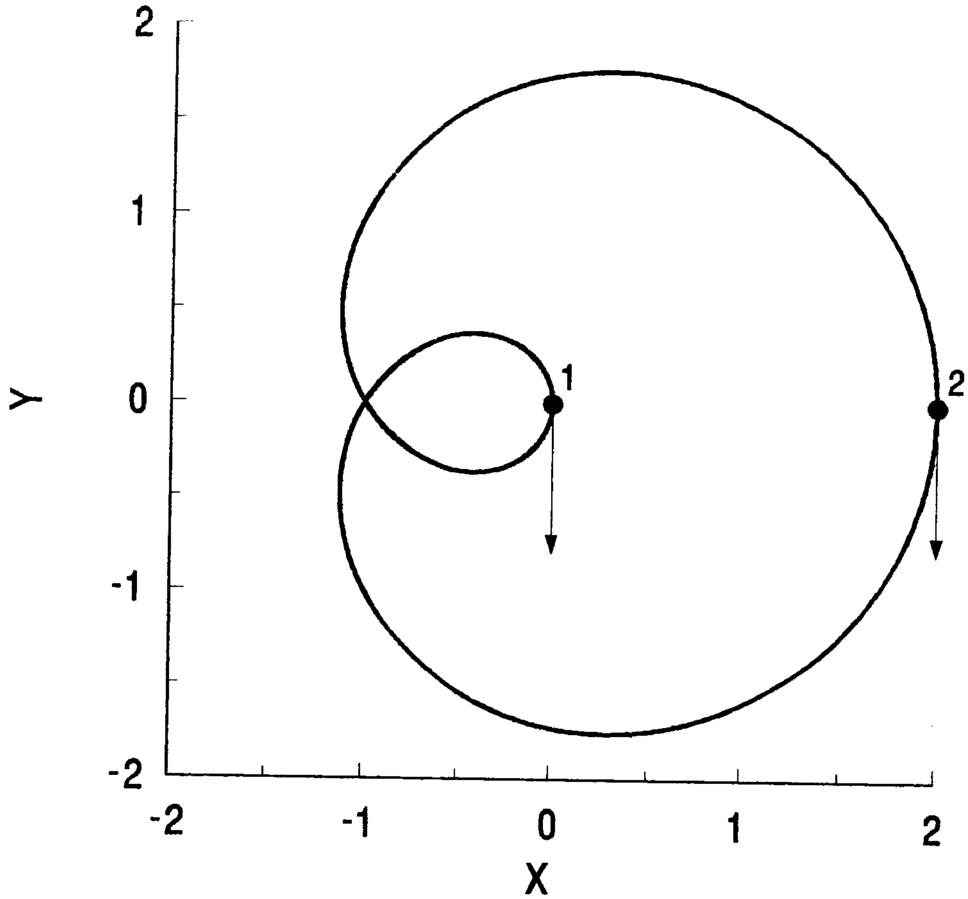

Fig. 1d

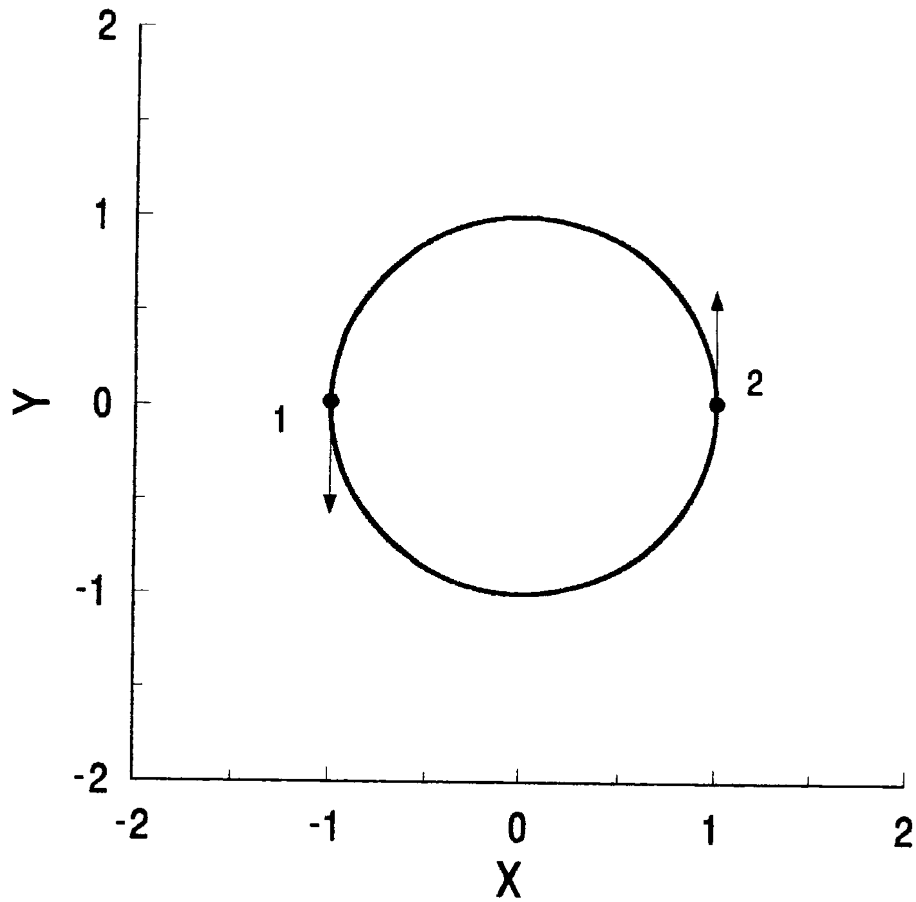

Fig. 1e

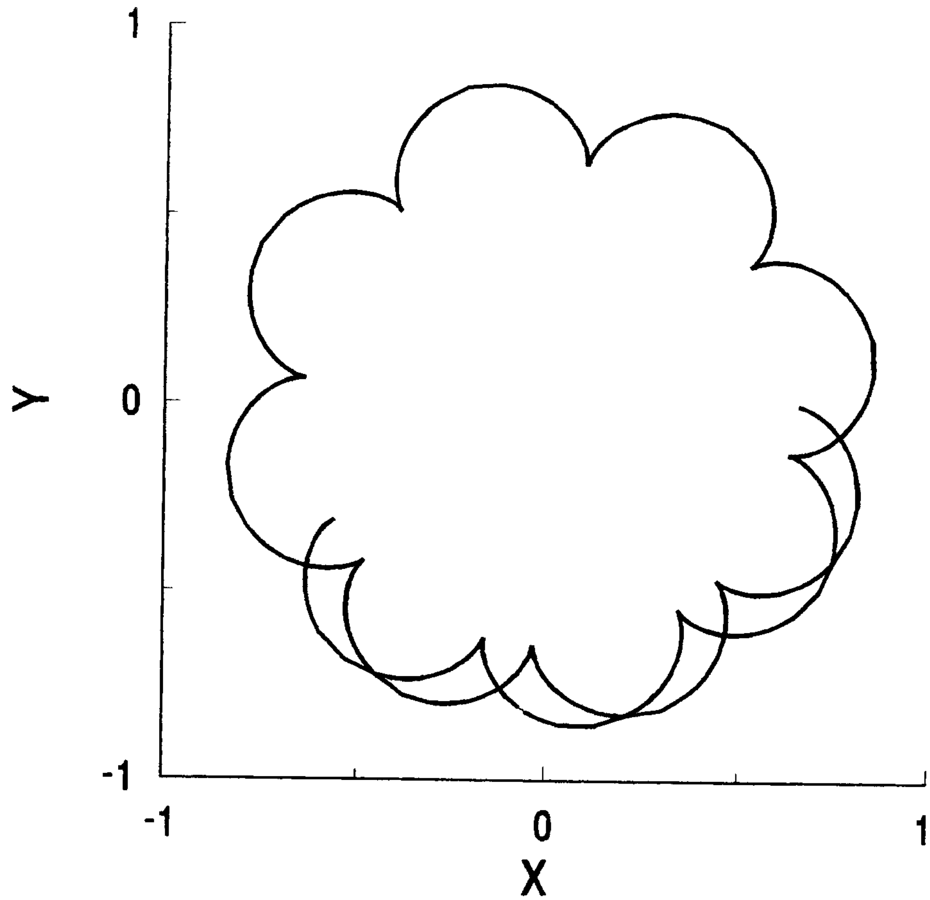

Fig. 2a

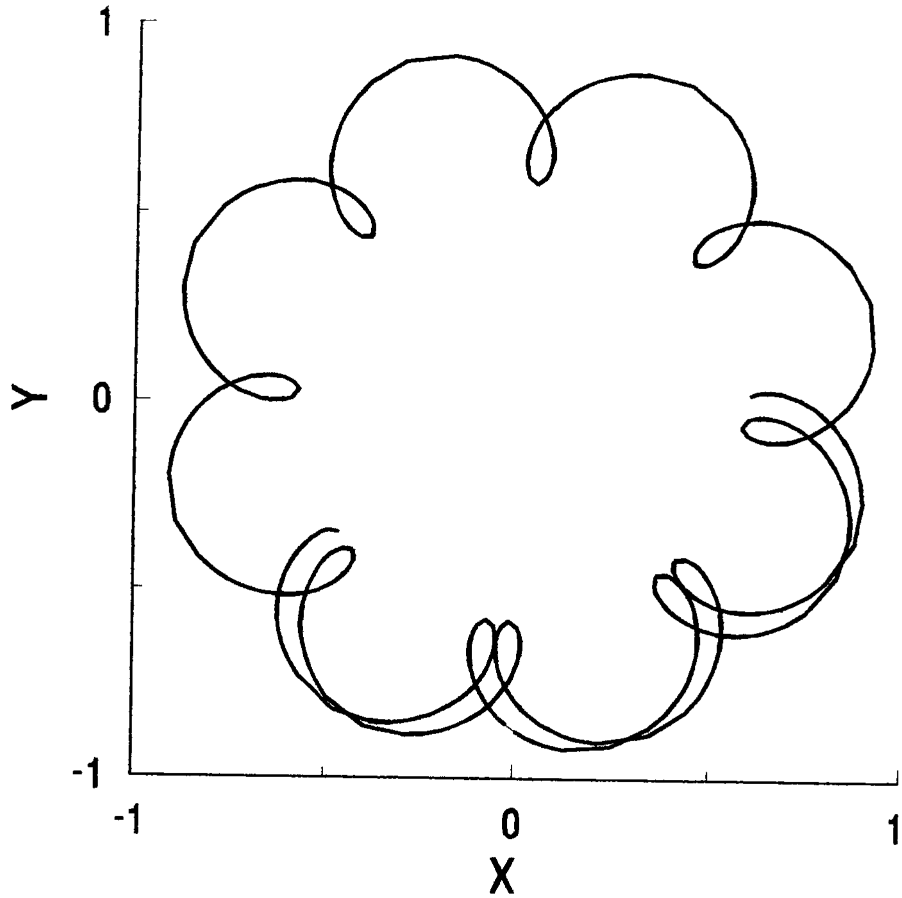

Fig. 2b

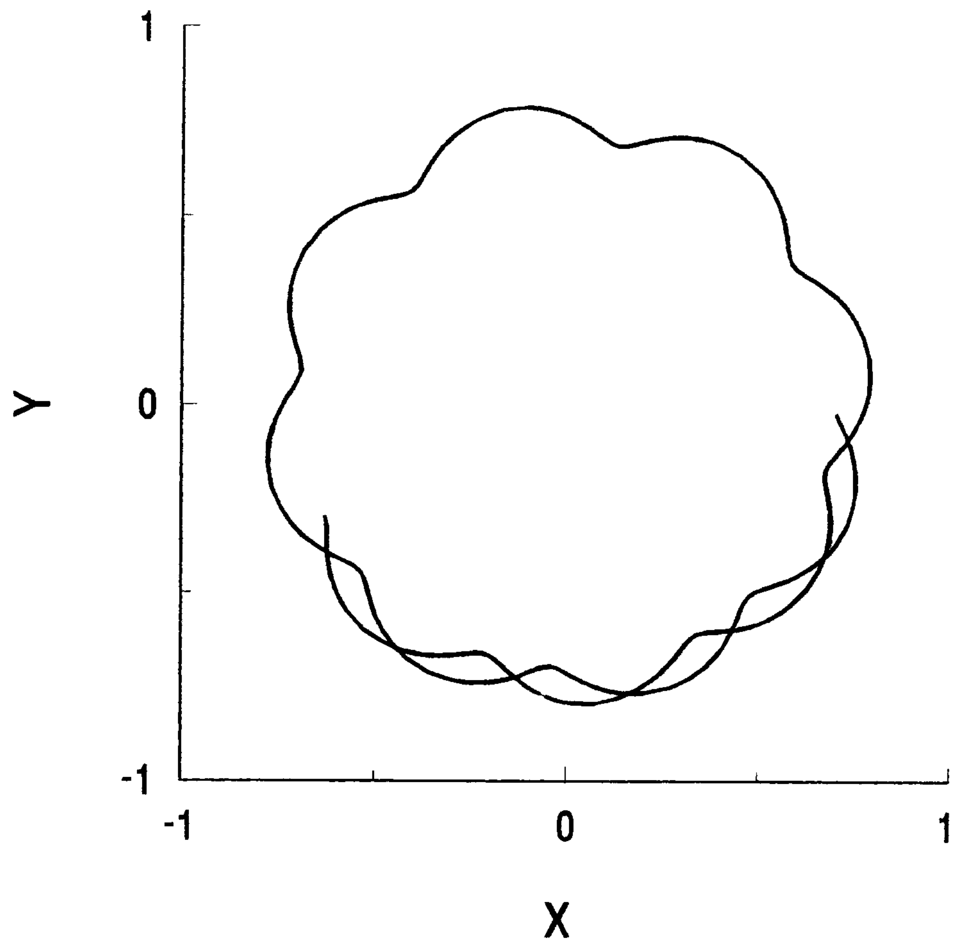

Fig. 2c

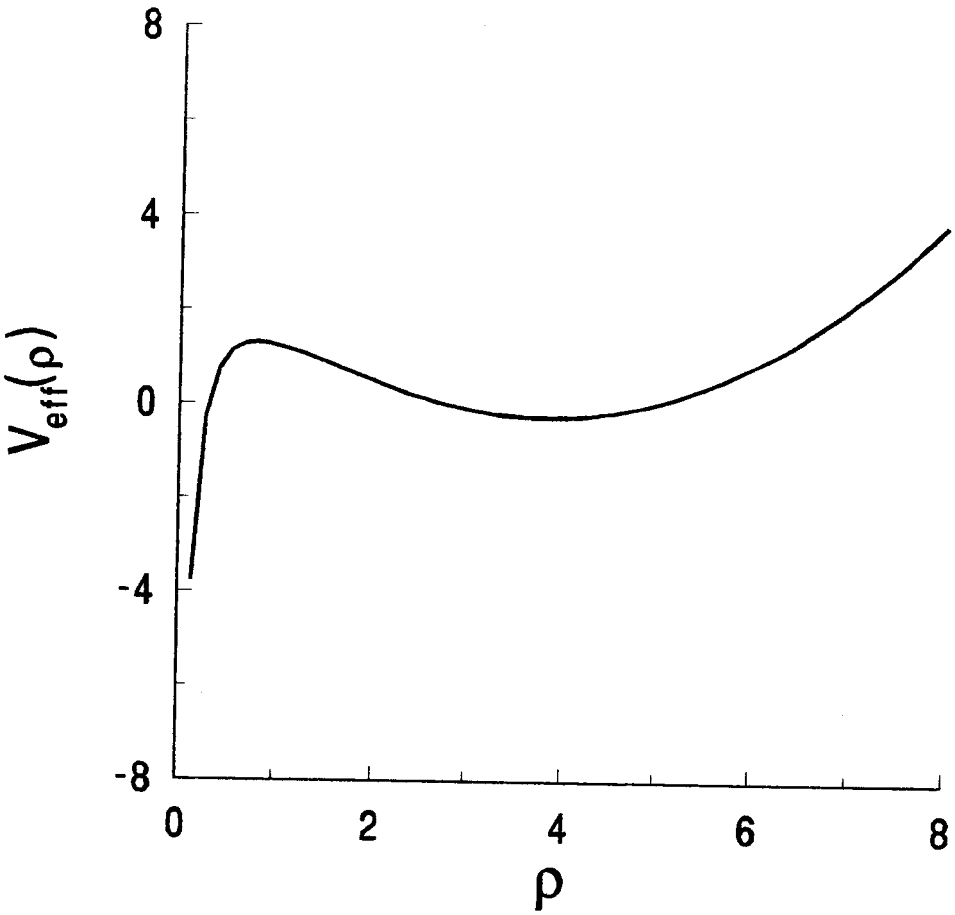

Fig. 3a

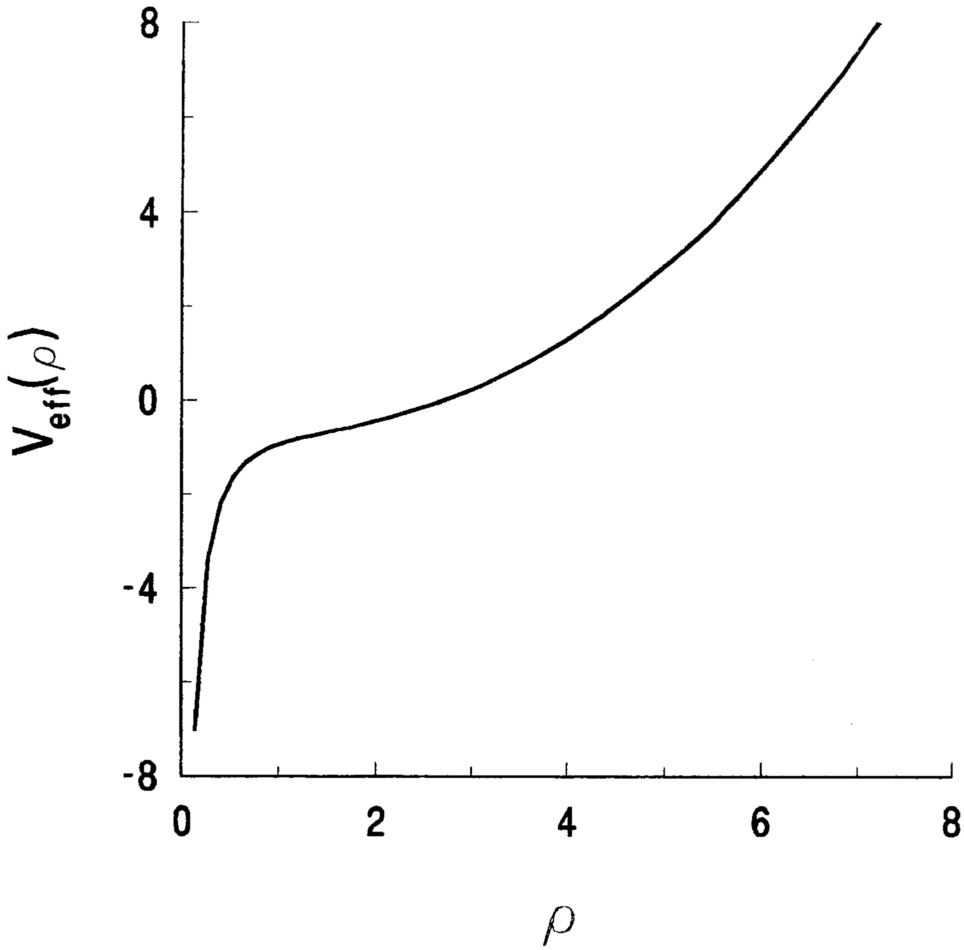

Fig. 3b

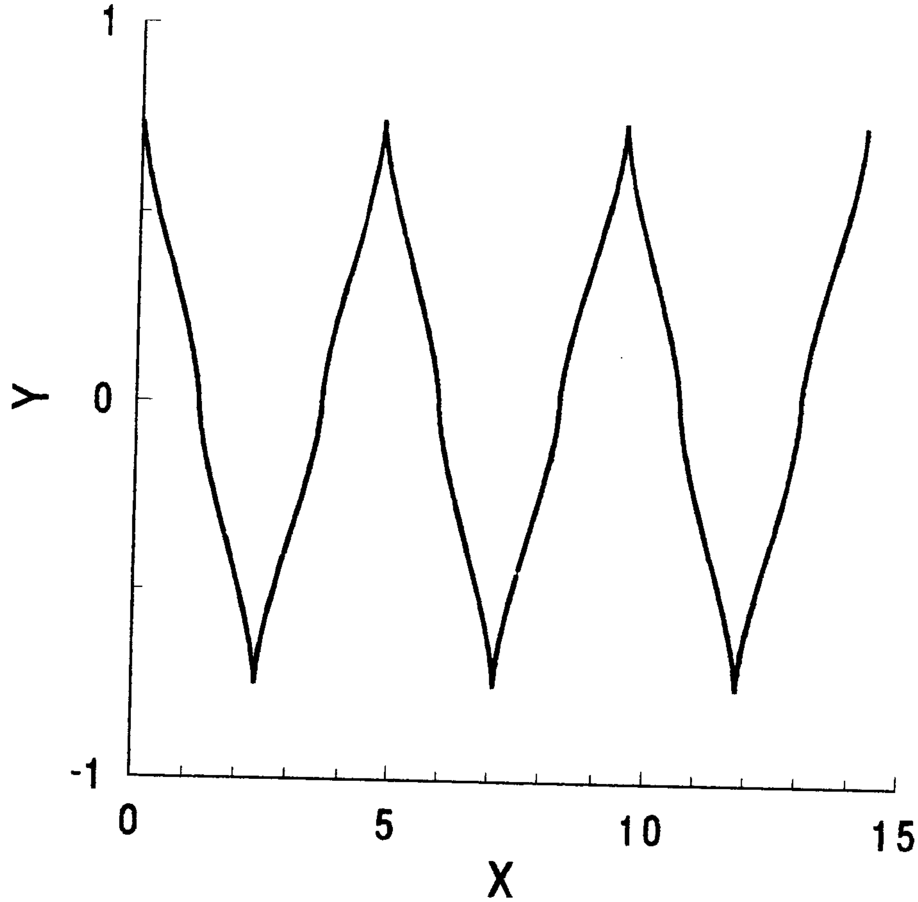

Fig. 4a

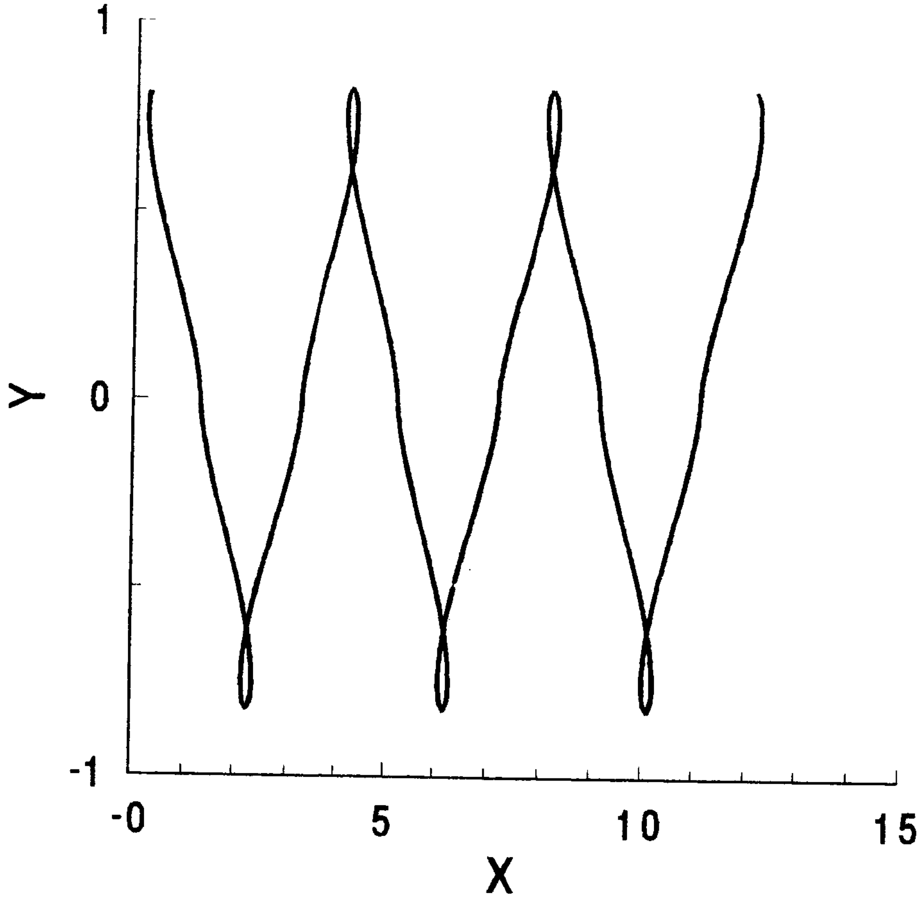

Fig. 4b

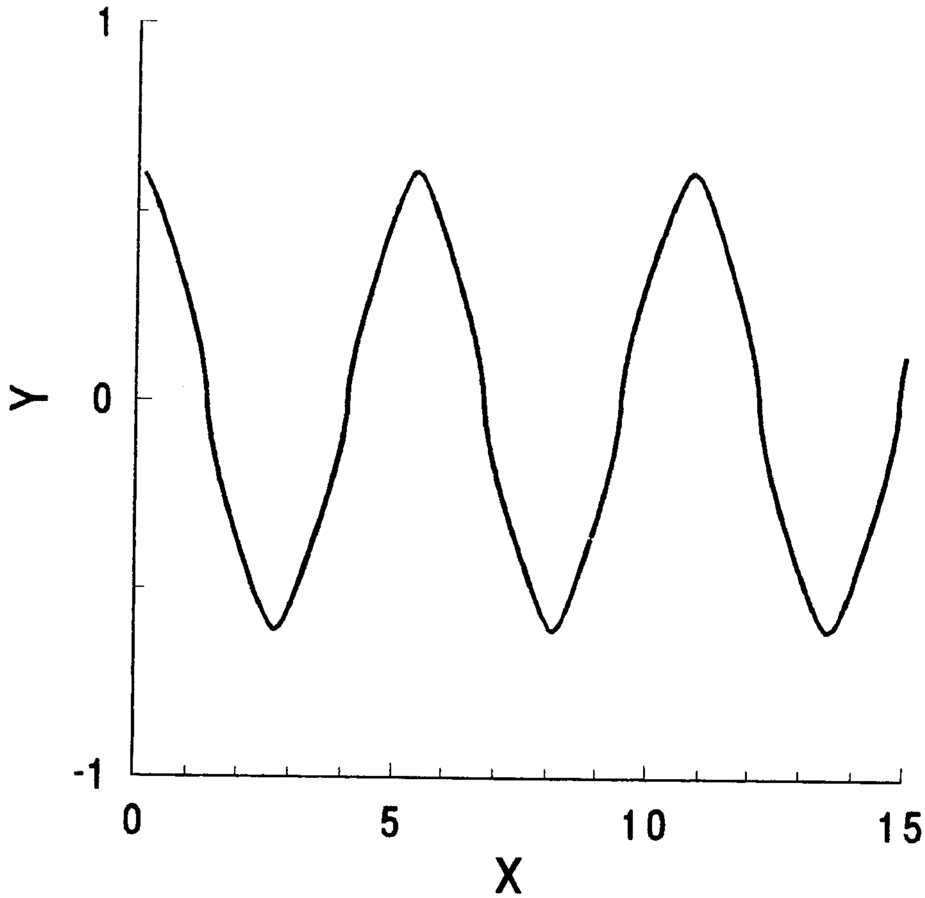

Fig. 4c

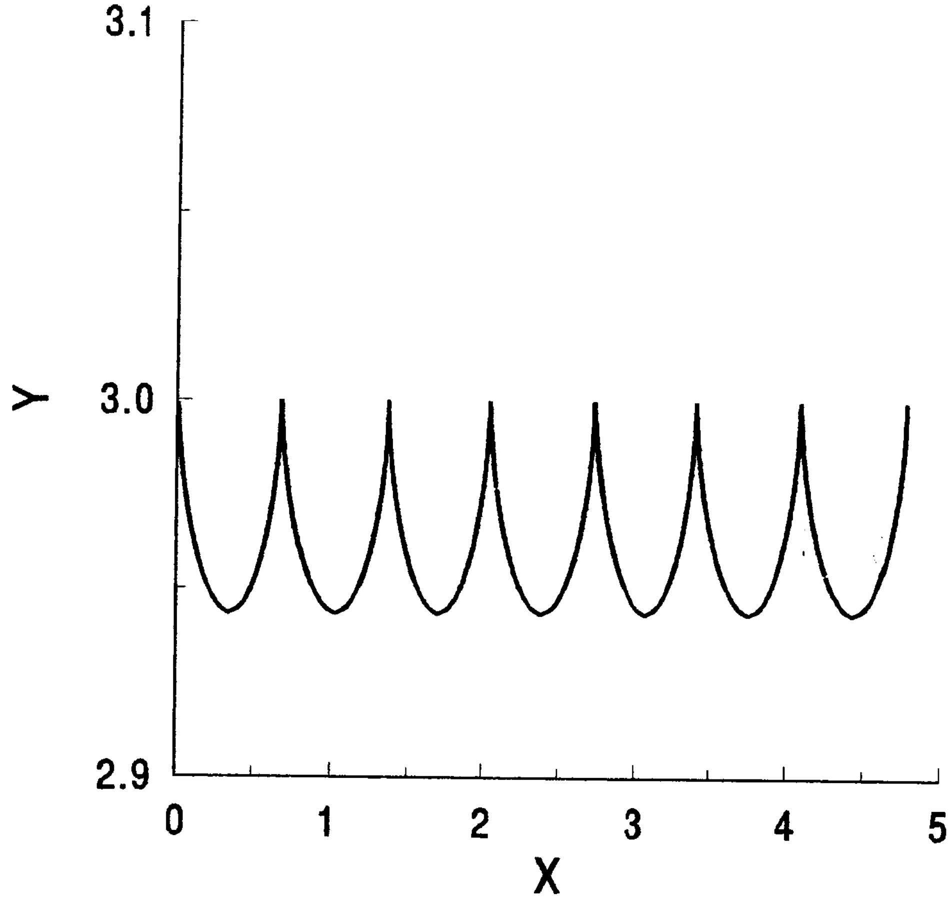

Fig. 5a

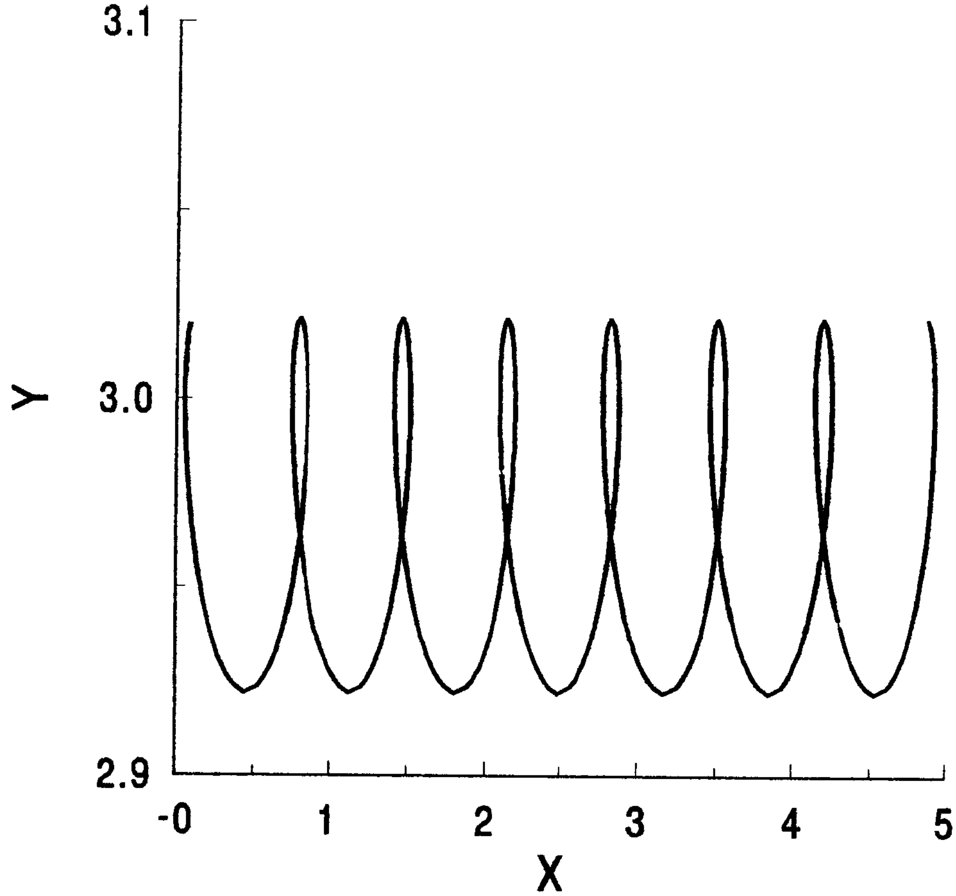

Fig. 5b

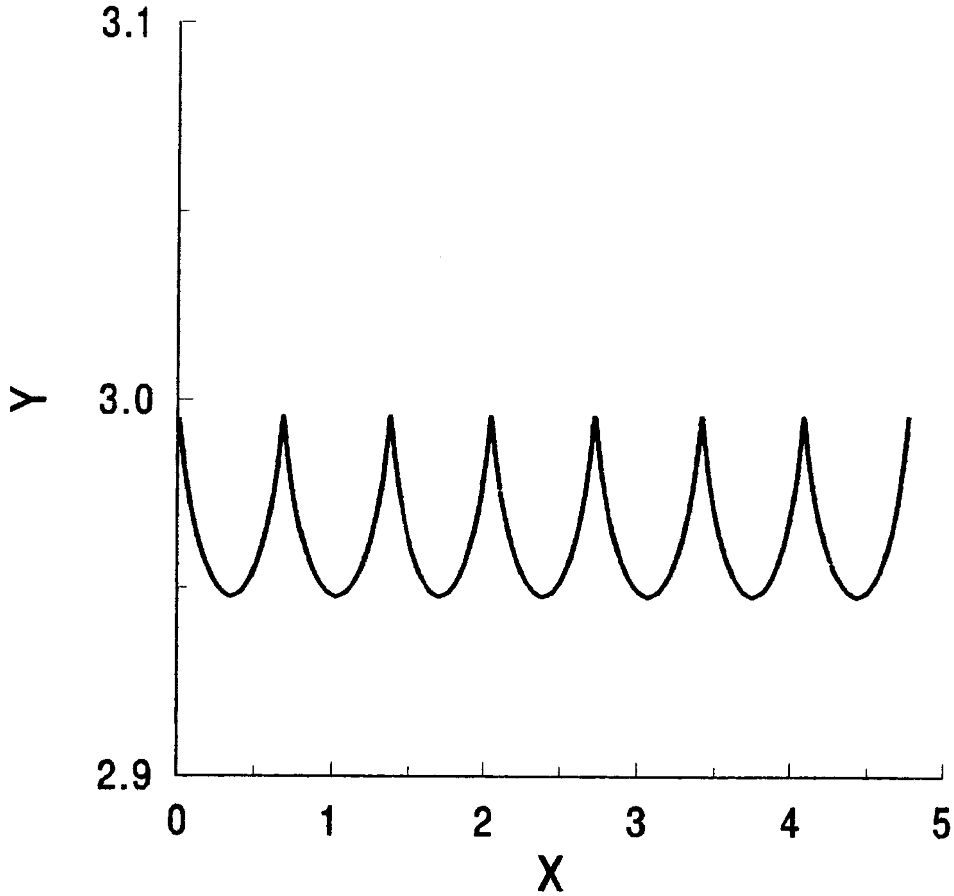

Fig. 5c

References

[1] K. von Klitzing, G. Dorda and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980)

[2] R. B. Laughlin, *"Elementary theory: The Incompresible Quantum Fluid"* in "The Quantum Hall Effect", eds. R. E. Prange and S. Girvin, 233 (Springer Verlag, Heidelberg 1988)

[3] F. Claro, Phys. Rev. **B 35**, 7980 (1987)

[4] H. Fukuyama, P. M. Platzman, and P. W. Anderson,Phys. Rev. **B 19**, 5211 (1979)

[5] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983)

[6] T. Toyoda, V. Gudmundsson and Y. Takahashi, Phys. Lett. **113 A**, 482 (1986)

[7] F. D. M. Haldane and E. H. Rezayi, Phys. Rev. Lett. **54**, 237 (1985)

[8] M. Taut, J. Phys. A: Math. Gen. **27**, 1045 (1994)

[9] D. Weiss, K. von Klitzing, K. Ploog, and G. Weimann, Europhys. Lett. **8**, 179 (1989)

[10] Keith R. Symon, *"Mechanics"*, Second Edition, Addison-Wesley Publishing Company, Inc., 145 (1965)

[11] Milton Abramowitz and Irene A. Stegun, *"Handbook of Mathematical Functions"*, Ninth Printing, Dover Publications, Inc., New York, 17 (1970)