CBPF-NF-013/93

Exact Solution for the Gravitational Field of a String With Non Null Cosmological Constant

by

M. Novello and M.C. Motta da Silva

Centro Brasileiro de Pesquisas Físicas — CBPF/CNPq
Rua Dr. Xavier Sigaud, 150
22290-180 – Rio de Janeiro, RJ – Brasil

Abstract

We generalise Hiscock's arguments [1] regarding the external gravitational field of a string, in order to allow for larger values of the string's linear density. In this way we are able to envision the string as being related to fundamental states of bosons of arbitrary mass.

Key-words: String; Cosmological constant; Grand unified theories.

Cosmic strings are thought to be stable configurations generated by some special behaviour of Higgs field $\phi_{(a)}$ that undergoes phase transition as the Universe cools down to a certain critical value [2]. A typical self-interacting potential for the scalar complex field $\phi(x)$ is provided by $V(\phi) = \frac{\sigma}{2}(\phi^*\phi - \xi^2)^2$ for a $U(1)$ symmetry. The state $< \phi >= \xi e^{i\lambda(x)}$ defines the string solution. It has been shown (see Vilenkin [3] for a complete review) that the structure of this solution can be represented by a unique parameter, μ, identified to the linear density of the string. The norm of the tension along the string, $|p_z|$, equals its density ρ. A remarkable property of this object is the behaviour of the geometry it induces in its exterior: the geometry is locally flat and possesses a conical topological defect.

Vilenkin [3] obtained the gravitational field of a static, cylindrically symmetric string as a linear approximation to General Relativity. Hiscock [1] showed that this result was correct to all orders in μ. It has been argued by many authors [1] [4] [5] that the string density μ must be lower than $1/2$. This argument is related to the fact that an integration of the string's density ρ to yield μ provides the relation $\alpha^2 = (1-4\mu)^2$. Thus, as the linear density μ tends to $1/2$, the external region becomes more and more inaccessible, since the string conical topology closes upon itself. This was not considered a real trouble, since a typical grand unified value for μ yields $\mu \sim 10^{-6}$. However, this imposes a real bound on the acceptable models of unification, disregarding the possibility of a unified theory which includes, for instance, gravity — since, in this case, $\mu \sim 1$. Thus, at least in principle, the aforementioned geometrical limitation on μ has been considered a puzzle. Why could not such conical (external) deformation of flat spacetime be produced by bosons of arbitrary mass?

There is a simple way to solve this difficulty: to modify the internal structure of the string. We show here a somewhat naïve model to deal with such problem, as presented by Novello and Oliveira [6]. We start by considering a doublet of fields, ϕ and ψ, such that when ϕ is in the "string " state (cf. above) the ψ field undergoes a phase transition and jumps to a constant (non vanishing) value ψ_0. The net effect of ψ_0 amounts to producing a distribution of energy with a constant value $T_{\mu\nu}(\psi_0) = \Lambda g_{\mu\nu}$. This will induce a perturbation on the string, e.g., a radial pressure $\hat{p}_r = \Lambda$, which tends to disrupt this configuration. The string counterbalances this effect (known as the $\phi - \psi$ coupling) by developing internal anisotropic pressures such that $p_r + \hat{p}_r = 0$.

The general form of this equilibrium configuration can thus be described by the energy-momentum tensor

$$T_{\mu\nu} = \rho V_\mu V_\nu - p h_{\mu\nu} + \pi_{\mu\nu} \tag{1}$$

in which p is the isotropic pressure and the traceless tensor $\pi_{\mu\nu}$ represents the anisotropic part of the pressure. Thus, in the local inertial frame, the non null components of $T_{\mu\nu}$ are:

$$\begin{aligned} T_{00} &= \rho \\ T_{11} = T_{22} &= p + \pi_0 \\ T_{33} &= p - 2\pi_0 \end{aligned} \tag{2}$$

in which the perturbed string maintains its cylindrical symmetry, that is,

$$\pi_{11} = \pi_{22} = -\frac{1}{2}\pi_{33} \equiv \pi_0 \tag{3}$$

At this point we proceed to match an interior solution to Einstein's equations (cosmic string with no rotation) to an exterior one (deformed Minkowskii with a conical topological defect), following the scheme used in reference [7]. We begin by defining the basic metric for both exterior and interior solutions:

$$ds^2 = dt^2 - dr^2 - \Delta^2(r)d\varphi^2 - dz^2 \tag{4}$$

where all the kinematic quantities (acceleration, shear, expansion and rotation) are zero in a comoving reference frame and Δ is a function of the coordinate r only.

Choosing a tetrad frame $e_A^{(\alpha)}$ as

$$\begin{aligned} e_0^{(0)} &= e_1^{(1)} = e_3^{(3)} = 1 \\ e_2^{(2)} &= \tfrac{1}{\Delta} \end{aligned} \tag{5}$$

we obtain the non null components of Einstein tensor $G_{AB} = R_{AB} - \frac{1}{2}R\eta_{AB}$:

$$\begin{aligned} G_{00} &= \tfrac{\Delta''}{\Delta} \\ G_{33} &= -\tfrac{\Delta''}{\Delta} \end{aligned} \tag{6}$$

where a prime ($'$) denotes, as usual, the derivative with respect to r.

Einstein's field equations with cosmological constant are given by

$$G_{AB} = -T_{AB} - \Lambda\eta_{AB}$$

in which we adopt the system of units in which $G = c = 1$. In this way, the resulting equations obtained with (2) and (6) are:

$$\Delta'' + (\rho + \Lambda)\Delta = 0 \tag{7}$$

$$\Lambda = p + \pi_0 \tag{8}$$

$$\Delta'' - (p - 2\pi_0 - \Lambda)\Delta = 0 \tag{9}$$

Using equations (3) and (8), (9) yields:

$$\Delta'' + 3\pi_0\Delta = 0 \tag{10}$$

which, along with equation (7), yields immediately:

$$\rho + \Lambda = 3\pi_0 \tag{11}$$

It is now possible to build a model to describe this space-time. The external region is a static, cylindrically symmetric, vacuum solution, with

$$\rho = \Lambda = 0$$

The internal solution, though, may be considered in two ways: the first (analysed by Hiscock [1] and Gott [4] [5]) possesses:

$$\rho + \Lambda > 0$$

and the condition that the metric be flat in the z-axis (i.e., with no cone singularity) must be valid. The other possible internal solution presents

$$\rho + \Lambda < 0$$

We proceed at this point to study these two models, as well as their properties.

Both models are constructed by joining two cylindrically symmetric solutions to Einstein's equations through the hypersurface $r = r_b \equiv constant$. The exterior solution (referred to, from now on, as region II) is the same for the two models: since ρ and Λ are zero, equations (8) and (11) yield

$$\pi_0 = p = 0$$

— which is consistent with a vacuum solution — and also

$$\Delta''_{II} = 0$$

which is easily integrated as

$$\Delta_{II}(r) = \alpha r + b$$

(α and b are integration contants). Now, in order to make our calculations easier, we drop the constant b, choosing it to be zero. In this case, the metric for region II is written as:

$$ds^2_{II} = dt^2 - dr^2 - \alpha^2 r^2 d\varphi^2 - dz^2 \tag{12}$$

Let us consider the angular coordinate φ; it ranges from 0 to 2π as usual. But, from (12), we can define a new angle $\varphi' = \alpha\varphi$, since α is a constant. We verify that the new angular range becomes:

$$0 < \varphi' < 2\pi\alpha \tag{13}$$

We will shortly see that this new choice for the angular coordinate is relevant to the validity considerations for the two possible interior solutions.

The first interior solution (to which we will refer as region I) possesses, as already stated, $(\rho + \Lambda) > 0$. Equation (7) gives thus:

$$\Delta_I(r) = \gamma \sin(\sqrt{\rho + \Lambda} r) + \lambda \cos(\sqrt{\rho + \Lambda} r) \tag{14}$$

with γ and λ as the integration constants. On the other hand, by imposing that

$$\Delta_I(r = 0) = 0$$

we obtain immediately that $\lambda = 0$ and so,

$$\Delta_I(r) = \gamma \sin(\sqrt{\rho + \Lambda} r)$$

The constant γ may also be determined in terms of ρ and Λ if we impose that ds_I^2 tends to Minkowskii metric for small r. This condition yields thus:

$$\Delta_I(r) \simeq \gamma\sqrt{\rho+\Lambda}r$$

$$ds_I^2 \simeq dt^2 - dr^2 - \gamma^2(\rho+\Lambda)^2 d\varphi^2 - dz^2$$

and therefore

$$\gamma^2 = \frac{1}{(\rho+\Lambda)} \tag{15}$$

This results in the following interior metric:

$$ds_I^2 = dt^2 - dr^2 - \gamma^2 \sin^2\left(\frac{r}{\gamma}\right) d\varphi^2 - dz^2 \tag{16}$$

Applying now the Darmois-Lichnerowicz (DL) junction conditions (as, for example, in reference [8]) through the hypersurface $r = r_b$ above mentioned, we find that they are reduced to:

$$[\Delta]_{r=r_b} \equiv \Delta_{II} - \Delta_I = 0$$

$$\left[\Delta'\right]_{r=r_b} \equiv \Delta'_{II} - \Delta'_I = 0$$

which, for (12) and (16), yield:

$$\gamma \sin\left(\frac{r_b}{\gamma}\right) = \alpha r_b \tag{17}$$

$$\cos\left(\frac{r_b}{\gamma}\right) = \alpha \tag{18}$$

Conditions (17) and (18) define the junction radius r_b as:

$$r_b = \gamma\sqrt{\frac{1}{\alpha^2} - 1} \tag{19}$$

and, from (19), we obtain that

$$\alpha^2 < 1 \tag{20}$$

Equation (20), applied in (13), leads to the conclusion that the range of φ' is **smaller** than 2π. This is what is characterized as a **topological defect**, as defined by [3] [4] [5].

We proceed now to investigate the other possible solution and take $(\rho+\Lambda) < 0$. In this case, (7) applied to the new region I yields:

$$\Delta_I(r) = \gamma \sinh(\sqrt{-(\rho+\Lambda)}r) + \lambda \cosh(\sqrt{-(\rho+\Lambda)}r)$$

where $\Delta_I(r=0) = 0$ again yields $\lambda = 0$ and also — imposing that the metric tends to Minkowskii for small r:

$$\sinh(\sqrt{-(\rho+\Lambda)}r) \simeq -(\rho+\Lambda)r$$

$$\gamma^2 = -\frac{1}{(\rho+\Lambda)} \tag{21}$$

The internal solution for the second model is thus given by:

$$ds_I^2 = dt^2 - dr^2 - \gamma^2 \sinh^2\left(\frac{r}{\gamma}\right) d\varphi^2 - dz^2 \tag{22}$$

and DL junction conditions must be again applied through the hypersurface $r = r_b$, yielding as results:

$$\alpha r_b = -\gamma \sinh\left(\frac{r_b}{\gamma}\right) \tag{23}$$

$$\alpha = -\cosh\left(\frac{r_b}{\gamma}\right) \tag{24}$$

Equations (23) and (24) give then

$$r_b = \gamma\sqrt{1 - \frac{1}{\alpha^2}} \tag{25}$$

where the inequality

$$\alpha^2 > 1 \tag{26}$$

must be valid.

In applying condition (26) to equation (13), we discover that the superior limit of the new angular coordinate φ' in region II is **greater** than 2π — the maximum possible value. Therefore, the second model is **not** acceptable and we must only consider the first model, with interior and exterior metrics given by equations (12) and (16) respectively, for $(\rho + \Lambda) > 0$.

Keeping this in mind, the relation between the constant α and the string's density μ (or its mass per unit length) is easily obtained. Repeating the exact calculation made for instance as in [1], we integrate the energy density over the source's volume of the string:

$$\begin{aligned}\mu &= \frac{1}{8\pi}\int_0^{2\pi} d\varphi \int_0^{r_b} \rho\gamma \sin\left(\frac{r}{\gamma}\right) dr \\ &= \frac{\rho}{4(\rho+\Lambda)}\left\{1 - \cos\left(\frac{r_b}{\gamma}\right)\right\}\end{aligned} \tag{27}$$

where the factor $\left(\frac{1}{8\pi}\right)$ appears due to the relation which gives the energy density in terms of γ (see [3]) and also because the angular deficit $\delta\varphi$ is given by $\delta\varphi = 8\pi\mu$ [1] [3]. Equation (27) yields finally:

$$4\mu\left(\frac{\rho+\Lambda}{\rho}\right) = 1 - \cos\left(\frac{r_b}{\gamma}\right) \tag{28}$$

From (28) it follows that:

$$\mu \leq \frac{\rho}{2(\rho+\Lambda)} \tag{29}$$

and, from (18), we discover that

$$\alpha = 1 - 4\mu\left(\frac{\rho+\Lambda}{\rho}\right) \tag{30}$$

Consequently the external metric is written as:

$$ds_{II}^2 = dt^2 - dr^2 - \left\{1 - 4\mu\left(\frac{\rho+\Lambda}{\rho}\right)\right\}^2 r^2 d\varphi^2 - dz^2 \tag{31}$$

It is easy to see that, by making $\Lambda = 0$ in the above equation, we reobtain Hiscock's result [1]:

$$ds^2 = dt^2 - dr^2 - (1-4\mu)^2 r^2 d\varphi^2 - dz^2$$

What are the possible values for μ? It is possible to determine them if we remember that inequality (29) must be simultaneously valid with the condition

$$\rho + \Lambda > 0 \Longrightarrow \rho > -\Lambda \tag{32}$$

From (29) we have thus

$$\rho \leq \frac{2\mu}{(1-2\mu)}\Lambda \tag{33}$$

where $\mu = \frac{1}{2}$ is not accceptable, since the quantity $\left(\frac{2\mu}{2\mu-1}\right)$ then diverges. It is then easily proved that, in order that $\rho > 0$ and both (29) and (32) are valid, we must have the following possibilities:

$$\Lambda > 0 \Longrightarrow \mu < \frac{1}{2} \tag{34}$$

$$\Lambda < 0 \Longrightarrow \mu > \frac{1}{2} \tag{35}$$

where, from equation (11), we also have:

$$\pi_0 = \frac{(\rho+\Lambda)}{3} > 0 \tag{36}$$

and the anisotropic pressure is always positive. Nevertheless we obtain from equation (8) that, for $\mu > \frac{1}{2}$, there occurs:

$$p < 0 \tag{37}$$

for positive π_0 and negative Λ. This does not exclude our solution, though, for it can be proved that — even if p is negative — the dominant energy condition is still satisfied, as it can be seen.

Indeed, the weak energy condition (as stated by [9]) requires that the energy density, measured by any observer, must be non negative, that is:

$$T_{AB}W^A W^B \geq 0$$

where W^A is a timelike vector. On the other hand, the dominant energy condition requires that besides the weak energy condition be valid, the pressure cannot exceed the energy density. This means that:

$$\rho + p \geq 0 \tag{38}$$

Making use of equations (8) and (36) respectively, we obtain:

$$\begin{aligned} \rho + p &= \rho + \Lambda - \pi_0 \\ &= \rho + \Lambda - \frac{(\rho + \Lambda)}{3} \\ &= \frac{2}{3}(\rho + \Lambda) \end{aligned} \tag{39}$$

which is always positive, in view of the original condition (30). Therefore our solution with $\mu > \frac{1}{2}$ is still valid, despite p is negative, because the dominant energy condition is indeed satisfied, allowing that larger values for the string's density be considered as valid, in the framework of Grand Unified Theories.

Acknowledgement

One of us (M.C.Motta da Silva) would like to thank CNPq for a grant.

References

[1] Hiscock,W.A. Phys. Rev. D **31**, 12(1985), 3288-3290.

[2] Linde,A.D. Rep. Prog. Phys. **42**, 25(1979).

[3] Vilenkin,A. Phys. Rev. D **23**, 852(1981).

[4] Gott III,R.J. Astrophys. J. **288**, 422(1985).

[5] Gott III, R.J. Phys. Rev. Letters **66**, 9(1991), 1126-1129.

[6] Novello,M. and Oliveira,L.A.R. Revista Brasileira de Física **17**, 3(1987).

[7] Novello,M. and Motta da Silva,M.C. *"Cosmic Spinning String and Causal Protecting Capsules "*. Preprint. Notas de Física, CPPF, 1993.

[8] Lake,K. in V^{th} *Brazilian School of Cosmology and Gravitation.* Editor M.Novello, Rio de Janeiro, 1987. World Scientific.

[9] Hawking,S. and Ellis,G.F.R. *"The Large Scale Structure of Space-Time "*. Cambridge University Press (1977); pages 88-96.