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SPIN TWO FIELD DESCRIPTION IN FIERZ VARIABLES. I - THE CASE OF MINKOWSKII SPACE TIME

by

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ABSTRACT: We present the Hamiltonian formulation of massive spin 2 field using Fierz variables.

Key-words: Spin two field; Hamiltonian formalism.

1. SPIN TWO FIELD

The description of the distribution of the energy content of a given field is not unique. It depends on the choice of the fundamental variables one is using to describe the process. The best and simplest way to see this is to examine the Fierz alternative description[1]. In the case we are interested here we will show this for the spin two field. We limit all our consideration to flat space-time. We decided to present this theory in such simple form once it seems that there are some doubts on the coherence of the uses of Fierz variable to describe spin two field. The generalization to arbitrary curved space time is straightforward and will be presented by us in a forthcoming paper.

Let us define a tensor $C_{\alpha\beta\mu\nu}$ in terms of the potential $A_{\alpha\beta\mu}$ by the relation:

$$\begin{aligned} C_{\alpha\beta\mu\nu} = {} & \partial_\nu A_{\alpha\beta\mu} - \partial_\mu A_{\alpha\beta\nu} + \partial_\beta A_{\mu\nu\alpha} - \partial_\alpha A_{\mu\nu\beta} + \\ & + \frac{1}{2} A_{(\nu\alpha)}\eta_{\beta\mu} + \frac{1}{2} A_{(\beta\mu)}\eta_{\nu\alpha} - \frac{1}{2} A_{(\alpha\mu)}\eta_{\beta\nu} - \\ & - \frac{1}{2} A_{(\beta\nu)}\eta_{\alpha\mu} + \frac{2}{3}\partial_\lambda A^{\sigma\lambda}{}_\sigma(\eta_{\alpha\mu}\eta_{\beta\nu} - \eta_{\alpha\nu}\eta_{\beta\mu}) \end{aligned} \tag{1}$$

in which

$$A_{(\alpha\mu)} = A_{\alpha\mu} + A_{\mu\alpha}$$

$$A_{\alpha\mu} = \partial_\lambda A_\alpha{}^\lambda{}_\mu - \partial_\mu A_\alpha{}^\lambda{}_\lambda \quad .$$

The potential $A_{\alpha\beta\mu}$ has only 9 degrees of freedom, as it satisfies the properties

(2a) $$A_{\alpha\beta\mu} = -A_{\beta\alpha\mu}$$

(2b) $$A_{\alpha\beta\mu} + A_{\beta\mu\alpha} + A_{\mu\alpha\beta} = 0$$

(2c) $$\overset{*}{A}{}^{\alpha\beta\mu}{}_{,\beta} = 0 \quad ,$$

or equivalently

(2c)' $$A_{\alpha\beta}{}^{\mu}{}_{,\nu} + A_{\beta\nu}{}^{\mu}{}_{,\alpha} + A_{\nu\alpha}{}^{\mu}{}_{,\beta} = 0 \quad .$$

Then, it follows that the tensor $C_{\alpha\beta\mu\nu}$ has only 10 degrees of freedom, once it satisfies identically (from definition (1) and (2a)):

(3) $$C_{\alpha\beta\mu\nu} = -C_{\beta\alpha\mu\nu} = -C_{\alpha\beta\nu\mu} = C_{\mu\nu\alpha\beta}$$

has no trace

(4) $$C_{\alpha\beta\mu\nu}\eta^{\alpha\mu} = 0$$

and, as it follows from (2b) has no pseudo-trace:

(5) $$C^{\alpha}{}_{\beta\mu\nu} + C^{\alpha}{}_{\mu\nu\beta} + C^{\alpha}{}_{\nu\beta\mu} = 0 \quad .$$

2. DYNAMICS

The Lagrangian is taken to be given

(6) $$L = -\frac{1}{8} C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu} + \frac{m^2}{2} A_{\alpha\beta\mu}A^{\alpha\beta\mu}$$

which yields the equation

$$\partial_\nu C^{\alpha\beta\mu\nu} + m^2 A^{\alpha\beta\mu} = 0 \tag{7}$$

Compatibility of this dynamics with properties (3,4,5) is achieved by imposing that $A^{\alpha\beta\mu}$ is trace-free

$$A^{\alpha\beta\mu}\eta_{\beta\mu} = 0 \tag{8}$$

which reduces from 9 to 5 the degrees of freedom. There is another compatibility condition from (7), that is

$$\partial_\mu A^{\alpha\beta\mu} = 0 \tag{9}$$

which is identically satisfied from (2c) (make $\beta = \mu$ in (2c)' and use (8)).

In terms of this potential, equation (7) can be equivalently written as

$$\Box\, A^{\alpha\beta\mu} + m^2 A^{\alpha\beta\mu} = 0 \tag{10}$$

in which $\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$.

We claim that the above theory represents a spin-two massive field. The best way to see this is to show by an hamiltonian treatment that the above counting of degrees of freedom (5) is indeed the correct one. Let us consider $A_{\alpha\beta\mu}$ as given by (2a) and (2b) and introduce the constraint (2c) in the Lagrangian. To simplify our notation we redefine the irreducible parts of $A_{\alpha\beta\mu}$ through the dictionary:

$$
(11)\quad \begin{aligned}
\phi &= A_{\ell o}{}^{\ell} \\
\xi_i &= A_{oi}{}^{o} \\
\gamma_i &= A_{\ell i}{}^{\ell} \\
\beta_{ij} &= A_{ij}{}^{o} \\
\alpha_{ij} &= A_{ioj} + A_{joi} - \frac{2}{3}\,\phi\,\eta_{ij} \\
\Delta_{ijk} &= A_{ijk} - \frac{1}{2}\,\eta_{ki}\,A^{\ell}{}_{j\ell} + \frac{1}{2}\,\eta_{kj}A^{\ell}{}_{i\ell}
\end{aligned}
$$

We then write the full Lagrangian:

$$
(12)\quad \begin{aligned}
L_T = {} & -C^{ioko}C_{ioko} - \frac{1}{2}\,C^{ijko}C_{ijko} + \\
& + m^2[\xi_i\xi^i + \frac{1}{4}\,\alpha_{ij}\alpha^{ij} + \frac{3}{4}\,\beta_{ij}\beta^{ij} + \frac{1}{2}\,\Delta_{ijk}\Delta^{ijk} + \\
& + \frac{1}{2}\,\gamma^i\gamma_i + \frac{1}{3}\,\phi^2] + \Omega^{ijk}[\dot{\Delta}_{ijk} + \frac{1}{2}\,\alpha_{kj,i} \\
& - \frac{1}{2}\,\alpha_{ki,j} - \frac{1}{3}\,\beta_{ij,k} + \\
& + \frac{1}{6}\,\beta_{ki,j} + \frac{1}{6}\,\beta_{jk,i} - \frac{1}{4}\,\eta_{ki}\,(\alpha^{\ell}{}_j - \beta^{\ell}{}_j)_{,\ell} + \\
& + \frac{1}{4}\,\eta_{kj}(\alpha^{\ell}{}_i - \beta^{\ell}{}_i)_{,\ell}] + Q\,\Omega^{ijk}\Omega_{ijk}
\end{aligned}
$$

in which a dot means time derivative and Ω^{ijk} is a Lagrange multiplier. We introduce the extra term in Q to eliminate dynamical residue of Ω_{ijk}. The definitions of the momenta canonically conjugate give:

$$
(13a)\quad \Pi_{ij} = \frac{\delta L}{\delta\dot{\alpha}_{ij}} = -\,C_{iojo}
$$

$$
(13b)\quad \Pi^{ijk} = \frac{\delta L}{\delta\dot{\Delta}_{ijk}} = -C^{ijko} + \Omega^{ijk} \approx -C^{ijko}
$$

and the primary constraints

(13c) $$P_{ij} = \frac{\delta L}{\delta \dot{\beta}_{ij}} \approx 0$$

(13d) $$\Pi_i = \frac{\delta L}{\delta \dot{\xi}^i} \approx 0$$

(13e) $$P_i = \frac{\delta L}{\delta \dot{\gamma}^i} \approx 0$$

(13f) $$\Pi = \frac{\delta L}{\delta \dot{\phi}} \approx 0$$

(13g) $$P = \frac{\delta L}{\delta \dot{Q}} \approx 0$$

(13h) $$P^{ijk} = \frac{\delta L}{\delta \dot{\Omega}_{ijk}} \approx 0$$

We then obtain as secondary constraints the set

(14a) $$\phi \approx 0$$

(14b) $$\Pi^{ik}{}_{,k} - m^2 \xi^i \approx 0$$

(14c) $$\Pi^{ik}{}_{,k} + m^2 \gamma^i \approx 0$$

(14d) $$\Pi^{ijk}{}_{,k} + m^2 \beta^{ij} \approx 0$$

(14e) $$\Pi^{ijk} + \alpha_{ki,j} - \alpha_{kj,i} - \frac{2}{3} \beta_{ij,k} - \frac{1}{3} \beta_{ik,j} - \frac{1}{3} \beta_{kj,i} + \frac{1}{2} (\alpha_j{}^{\ell} + \beta^{\ell}{}_j)_{,\ell}\, \eta_{ik} - \frac{1}{2} (\alpha_i{}^{\ell} + \beta^{\ell}{}_i)_{,\ell}\, \eta_{jk} \approx 0$$

(14f) $$\Omega_{ijk} \approx 0$$

(14g) $$\Pi_{ki,j} - \Pi_{kj,i} - \gamma_{i,jk} + \gamma_{j,i,k} + $$
$$+ \Delta_k{}^m{}_{j,i,m} - \Delta_k{}^m{}_{i,j,m} + \Delta_j{}^m{}_{k,i,m} -$$
$$- \Delta_i{}^m{}_{k,j,m} - \frac{2}{3}\Delta_{ij}{}^{\ell}{}_{,\ell,k} - \frac{1}{3}\Delta_{ik}{}^{\ell}{}_{,\ell} -$$
$$- \frac{1}{3}\Delta_{kj}{}^{\ell}{}_{,\ell,i} + \frac{1}{2}\eta_{ki}\,(- \gamma_{j,\ell,m}\,\eta^{\ell m} +$$
$$+ \gamma^{\ell}{}_{,\ell,j} + \Pi_j{}^{\ell}{}_{,\ell} + 2\Delta^{\ell}{}_j{}^m{}_{,\ell,m}) -$$
$$- \frac{1}{2}\eta_{kj}\,(-\gamma_{i,\ell,m}\,\eta^{\ell m} + \gamma^{\ell}{}_{,\ell,i} + \Pi_i{}^{\ell}{}_{,\ell} +$$
$$+ 2\Delta^{\ell}{}_i{}^m{}_{,\ell,m}) - m^2\Delta_{ijk} \approx 0 \quad .$$

It is a rather long but straightforward calculation to show that all constraints are of second class. We can easily show that P and Q can be discarded from the hamiltonian analysis once they provoke no dynamics. We have thus the following set of quantities to deal with: 20 $A_{\alpha\beta\mu}$ and 20 corresponding momenta; 5 independent Ω_{ijk} and its 5 corresponding momenta. Thus the total number of variables are 50[(2)]. All constraints being of second class the number of freedom to be removed are given by

3 for (13c)

3 for (13d)

3 for (13e)

1 for (13f)

5 for (14f)

5 for (13h)

5 for (14e)

5 for (14g)

3 for (14d)

1 for (14a)

3 for (14b)

3 for (14c)

Adding all this number gives 40 second class constraints yielding $\frac{50-40}{2} = 5$ degrees of freedom, as it should be to describe a massive spin two field.

We have thus accomplished our task to prove that the present theory gives a complete Lagrangean-Hamiltonian formulation of spin 2 field in Fierz coordinates. The case of massless field goes along the same lines although, as one should expect, in this case there appears also first class constraints meaning gauge freedom. We present this case elsewhere.

References

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M. Fierz - W. Pauli - Proc. Roy. Soc. A173(1939),211.

(2) A. Salam quote A. Einstein: "Nature is not economical of structures: only of principles of fundamental applicability" in Nobel Lectures, Rev. Mod. Phys. 52(1980)525.