RENORMALIZATION GROUP TREATMENT OF BOND PERCOLATION IN ANISOTROPIC AND "INHOMOGENEOUS" PLANAR LATTICES

by

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A B S T R A C T

The uncorrelated bond percolation problem is studied in three planar systems where there are two distinct occupancy probabilities. We apply two different real space renormalization group approaches (referred as the "canonical" (CRG) and the "parametric" (PRG) ones) to the anisotropic first-neighbour square lattice, and both of them exhibit the expected tendency towards the exactly known phase boundary (p+q=1). Then we introduce, within the context of PRG calculations for increasingly large cells, an extrapolation method which leads to analytic proposals for the other two lattices, namely p+q = 1/2 for the first-and second-neighbour square lattice (p and q are, respectively, the first and second neighbour occupancy probabilities), and $3(p-1/2) = 4\left[(1-q)^2 + (1-q)^3\right]$ (p and q are, respectively, the occupancy probabilities of the topologically different bonds which are in a 1:2 ratio) for the 4- 8 lattice.

1. INTRODUCTION

The percolation problem has been the subject of many recent works exploring its analogies and connections with termodynamic phase transitions (see, for instance, the reviews of Zallen 1978, Stauffer 1979 and Essam 1980). Many real-space renormalization group (RG) approaches have been developed for bond, site and site-bond percolation. Some authors (Harris et al 1975, Dasgupta 1976, Marland and Stinchcombe 1977, Kunz and Wu 1978, Burkhardt and Southern 1978) made use of the relationship between the percolation problem and the s-state Potts model in the limit $s \to 1$ (Kasteleyn and Fortuin 1969). Others (Young and Stinchcombe 1975, Kirkpatrick 1977, Reynolds et al 1977, 1978, Nakanishi and Reynolds 1979a, Shapiro 1979, Tsallis and Schwachheim 1979, Murase and Yuge 1979, Magalhães et al 1980) have constructed RG transformations which act directly on the space of occupancy probabilities. Concerning bond percolation problems where more than one probability appear, some exact or approximate results have been obtained by graph theory (Sykes and Essam 1963), series expansions (Redner and Stanley 1979), "ghost site" method (Turban 1980) and RG (Lage 1979, Ikeda 1979, Turban 1979).

In the present work we study, within a real space RG framework, uncorrelated bond percolation in three planar lattices, namely the anisotropic first-neighbour square lattice and the "inhomogeneous" first-and second-neighbour square and $4 - 8$ lattices, where the word "inhomogeneous" stands for the fact that two different occupancy probabilities p and q are introduced. The critical frontier in the $p - q$ space (as well as the connectivity critical exponent ν_p) associated to the anisotropic first-neighbour square lattice is discussed (§ 2) within two different RG approaches, to which we refer as the canonical (CRG) and the parametric (PRG) ones. Both of them exhibit the correct tendency towards the well known (Sykes and Essam 1963)

exact result $p + q = 1$. The first-and second-neighbour square lattice (§ 3) and the 4 - 8 one (§ 4) are discussed only within the PRG framework by using a convenient extrapolation procedure. Finally for both critical lines we propose analytical expressions which fit extremely well the numerical results, and might therefore be the exact ones.

2. ANISOTROPIC FIRST-NEIGHBOUR SQUARE LATTICE

Let us consider a first-neighbour square lattice where the independent occupation probabilities for "vertical" and "horizontal" bonds are, respectively, p and q. Its exact critical line (which separates the percolating and non-percolating regions) is already known (Sykes and Essam, 1963):

$$p + q = 1 \tag{1}$$

This equation has recently been confirmed by different approaches: series expansions (Redner and Stanley 1979), self-consistent decimation within the context of an effective medium theory (Lage 1979), anisotropic RG transformations using as a starting point the isotropic square lattice (Turban 1979). In this section, we present the already mentioned CRG and PRG approaches*: both of them support equation (1).

Let us first of all choose a family of clusters (whose size will be characterized by b) which completely cover the lattice (those associated to b = 1 and b = 2 are, respectively, indicated in Figs. (1a) and (1b)). Next we associate to each cluster a graph by collapsing the entries and exits into two terminals and by eliminating the irrelevant bonds (several examples of such graphs appear in Fig. 1). Finally, through the deletion-contraction rule (see Appendix A), we associate to each graph a polynomial (noted $R_b(p,q)$) which represents the probability of the two terminals being connected, in particular:

$$R_1(p,q) = p \tag{2}$$

$$R_2(p,q) = 2p^2 - p^4 + q\,(2p^2 - 4p^3 + 2p^4) \tag{3}$$

* The CRG procedure has independently been used by Nakanishi and Reynolds 1979 (b).

$$R_3(p,q) = 3p^3 - 3p^6 + p^9 + 8q(p^3-p^4-p^5+p^7+p^8-p^9)+q^2(10p^3-30p^4+$$
$$+12p^5+18p^6+24p^7-60p^8+26p^9) +4q^3(p^3-6p^4+8p^5+11p^6-36p^7+$$
$$+31p^8-9p^9) + q^4 (2p^3-19p^4+72p^5-140p^6+148p^7-81p^8+18p^9) \quad (4)$$

These expressions can be reobtained by explicit counting of all the percolating bond configurations*(Young and Stinchcombe 1975, Reynolds et al 1977). Notice that $R_2(p,p)$ and $R_3(p,p)$ reproduce , respectively, eq(12) of Reynolds et al (1977)and eq(5)of Magalhães et al(1980).

By considering "vertical" percolation and renormalizing a cell of side b into a smaller one of side b'(therefore the lattice expansion factor will be b/b') we get the following relation:

$$R_{b'}(p',q') = R_b(p,q) \quad (5)$$

In order to complete the RG recursive procedure we need a second relation (among p',q',p and q) which can be obtained through the CRG or the PRG procedures we present next.

a) Canonical RG Procedure (CRG)

Herein the second relation comes from consideration of percolation along another direction; in the present case we shall choose this direction to be the "horizontal" one,hence:

$$R_{b'}(q',p') = R_b(q,p) \quad (6)$$

The relations (5) and (6) define a two-parameter RG tranformation, whose flow lines in the p-q space yield the critical frontier we are looking for. The term "canonical" comes from the fact that this is the traditional procedure(Niemeyer and van Leeuwen 1974,1976 , Young and Stinchcombe 1975).

*For details of the method see, for instance,Magalhães et al (1980).

Let us, now, point out some general results related to the symmetry of this particular problem. First of all, we notice that the system of recursive relations (5) and (6) leads to a flow diagram which is symmetric with respect to the axis p=q (see Fig.2). Furthermore the clusters of the family we have chosen are self - dual(see the last part of Appendix A), hence:

$$R_b(p,q)+R_b(1-p,\ 1-q)=1 \qquad \forall b \qquad \forall (p,q) \qquad (7)$$

This relation immediately implies a second symmetry(in the flow diagram) namely a punctual one with respect to $(p,q)=(1/2,1/2)$, which together with the first one leads to a new symmetry axis (determined by p+q=1).

The RG transformation we are dealing with has nine fixed points for any finite values of b and b', namely: the point (1/2, 1/2) which corresponds to the exact result of the isotropic case, the trivial points (0,0) and (1,1), the linear chain points (0,1) and (1,0) and the four points dependent on b and b' $(p_o,0)$, $(1-p_o,1)$, $(0,p_o)$ and $(1,1-p_o)$, where p_o satisfies (cf the contraction-deletion rule)

$$(1-p_o^{\,b'})^{b'} - (1-p_o^{\,b})^{b} = 0 \qquad \forall\ (b,b') \qquad (8)$$

We verify that $p_o(b,b')$ tends towards unity if $b \to \infty$ (we recall that $b' < b$); the quickest convergence is obtained for $b'=b-1$. This fact is illustrated with the following examples :

$p_o(2,1)=(\sqrt{5}-1)/2\simeq 0.6180$; $p_o(3,1)=0.6823$; $p_o(20,1)\simeq 0.8939$; $p_o(200,1)\simeq 0.9805$; $p_o(1000,1)\simeq 0.9948$; $p_o(20,19)\simeq 0.9651$ and $p_o(200,199)\simeq 0.9965$.

The linearization of equations(5) and(6) in the neighbourhood of the point (1/2,1/2) leads to a Jacobian matrix whose two eigenvectors are in the directions p=q (associated to the largest eigenvalue, which coincides with the isotropic one, namely

$$\lambda \equiv \left\{ \frac{dR_b(p,p)}{dp} \qquad \left[\frac{dR_{b'}(p,p)}{dp} \right]^{-1} \right\}_{p=\frac{1}{2}}$$

and p+q=1, as can be seen in Fig. 2. The connectivity critical exponent is given, within the present approximation, by $\nu_p(b,b') = \ell n(b/b')/\ell n\ \lambda$ (cf. Magalhães et al 1980, $\nu_p(2,1) \simeq 1.428$, $\nu_p(3,1) \simeq 1.380$, $\nu_p(4,1) \simeq 1.363$ and the extrapolated value is $\nu_p = 1.351 \pm 0.012 \pm 0.020$). We verify that the flow diagram associated to b=3,4 are very similar to that of b=2 (Fig. 2). In all of them this fixed point is unstable in all directions in particular along the line p+q=1, contrarily to what we (and Nakanishi and Reynolds 1979 (b)) were expecting on account of universality. We were not able to find any other explanation for this fact than the possibility of reversal of this tendency for sufficiently high b.

In Fig. 2 we can see that "division" lines (broken lines) separate the p-q space into four regions (namely I,II,III and IV), each of them associated to an attractive fixed point ((1,0), (1,1), (0,1) and (0,0), respectively). As b increases, for let us say fixed b', the regions (I) and (III) shrink; this fact, together with the tendency of p_o towards unity when $b \to \infty$, strongly suggest that the CRG frontiers coalesce onto the exact one p+q=1.

b) Parametric RG Procedure (PRG)

Another choice, instead of equation (6), which completes (with equation (5)) the RG recursive set of transformations we are looking for, might be

$$Q(p',q') = Q(p,q) \equiv \varepsilon \qquad (9)$$

where ε is a parameter and the function Q (p,q) is an arbitrary one. Simple examples are* :

$$Q(p,q) = q/p \qquad (10)$$

$$Q(p,q) = q \qquad (11)$$

$$Q(p,q) = p \qquad (12)$$

* The choice (10) has been used for a square lattice by Ikeda (1979) in the quasi unidimensional and nearly isotropic cases.

It is important to remark that within the present approximative procedure every point of the critical line is a fixed one. Furthermore the equation of this critical line (referred hereafter as Γ (b,b')), namely

$$R_{b'}(p^*, q^*) = R_b(p^*, q^*) \quad , \tag{13}$$

<u>independs</u> on the choice of Q (p,q). On the other hand, the critical exponents (in particular ν_p) <u>depend</u> on the choice of Q(p,q) as this function determines the recursion in the neighborhood of the critical frontier. Moreover the approximate critical exponents exhibit, along this frontier, an unphysical dependence on the parameter ε, which, in order to support universality, should disappear with increasing cluster sizes.

We have obtained, by using equation (13), several frontiers Γ (b,b') ($2 \leqslant b \leqslant 5$ and $1 \leqslant b' \leqslant 4$): some of them are indicated in Fig.3. All the frontiers Γ (b,b') were derived from closed forms of eq.(13), with the exception of $\Gamma(5,b')$ ($1 \leqslant b' \leqslant 4$) which were treated by a two-variable version of the Monte Carlo method presented elsewhere (Magalhães et al 1980). Throughout this paper, we have worked, for every (b,b'), around $N_b\, n_b(p)\, n_b(q) \gtrsim 10^7$ (N_b being the total number of Monte Carlo runs and $n_b(p)$, $n_b(q)$ standing, respectively, for the number of relevant p-and q-bonds of the cluster characterized by b).

Let us notice that all the frontiers Γ(b,b') include the exact isotropic point (1/2,1/2), and preserve, like the CRG frontiers, the punctual symmetry with respect to this point (see equations (7) and (13)). On the other hand, their derivatives evaluated at this point, contrarily to what happens in the CRG, are not the exact one (namely -1), but they tend to it for increasing cluster sizes in all the cases we have examined. Let us also observe that the curves Γ(b,b') are not symmetric with respect to the line p=q since equation (13) is

not invariant under the transformation $p^* \leftrightarrow q^*$ (see Fig. 3). It is worth remarking that the point $(p_o(b,b'),0)$ (see eq. (8)) belongs to $\Gamma(b,b')$: this is due to the fact that both sides of equation (6) identically vanish for $q=0$, therefore both the CRG and PRG treatment rely on one and the same equation (5). Let us finally remark that the CRG frontiers are for the same values of (b,b'), better (though harder to work out) than the PRG ones (see Fig. 2); nevertheless all of them tend towards the exact line $p + q = 1$.

In what concerns the calculation of the approximate critical exponent $\nu_p(b,b';\varepsilon)$ we observe that, for the small examined clusters ($2 \leqslant b \leqslant 4$, $1 \leqslant b' \leqslant 3$), the artificial dependence* of ν_p on ε (for the three choices of ε, namely, q/p, q and p) increases, instead of disappearing, as we use bigger cells; we ignore if this tendency will be reversed for larger clusters.

* Ikeda's treatment (1979) presents a similar variation of ν_p in the neighbourhood of the isotropic case $p=q=1/2$. He obtains $\nu_p(q^*=\varepsilon p \simeq 0.543) \simeq 1.25$ and $\nu_p(q^*=\varepsilon p=o.5) \simeq 1.10$; our results on the same points (for $b=2$, $b'=1$) are, respectively, 1.40 and 1.43. All these numbers are to be compared with the nearly exact value 1.35.

3. "INHOMOGENEOUS" FIRST- AND SECOND-NEIGHBOUR SQUARE LATTICE

In this section, we study the bond percolation on isotropic first- and second-neighbour square lattice where p and q are, respectively, the independent occupancy probabilities for the first and second neighbour bonds. We remark that in both particular cases q=0 and p=0 this lattice reduces to, respectively, one and two disconnected first-neighbour square lattices, whose critical probability is 1/2.

We will use the same family of cells we considered (Magalhães et al 1980) for the p=q case (see Figs. (1c) and (1d)). Application of the contraction-deletion rule to their respective graphs yields the following vertical renormalized probabilities $R_b(p,q)$:

$$R_1(p,q) = p+2q(1-p)-q^2(1-p) \tag{14}$$

and

$$\begin{aligned} R_2(p,q) = {} & 2p^2+2p^3-5p^4+2p^5+q(6p+2p^2-30p^3+26p^4-4p^6)+q^2(6-4p-56p^2+ \\ & 92p^3-2p^4-64p^5+28p^6)+q^3(-4-38p+128p^2-24p^3-248p^4+270p^5- \\ & -84p^6)+q^4(-8+64p-36p^2-338p^3+708p^4-530p^5+140p^6)+q^5 \\ & (8-6p-198p^2+690p^3-942p^4+588p^5-140p^6)+q^6(2-52p+272p^2- \\ & -608p^3+682p^4-380p^5+84p^6)+q^7(-4+38p-140p^2+260p^3-260p^4+ \\ & +134p^5-28p^6)+q^8(1-8p+26p^2-44p^3+41p^4-20p^5+4p^6) \end{aligned} \tag{15}$$

These equations, respectively, reproduce, if p=q, eqs. (7) and (8) of Magalhães et al (1980), as it should be. As illustrated in the previous section, both CRG and PRG procedures tend, for increasingly large cells, towards the exact frontiers; however, because of its operational simplicity we shall, from now on, use the latter.

We notice that for q=0 expressions (14),(15) (and those corresponding to higher values of b) reduce to those associated to the first-neighbour square lattice (Reynolds et al 1977, Magalhães et al 1980). Therefore, all the frontiers $\Gamma(b,b')$ we are looking for will contain the point $(p,q)=(1/2,0)$. On the other hand, the case p=0 deserves a few comments, as here every graph $R_b(0,q_o)$ becomes a parallel array of a certain graph (noted $G_b(q)$) with its dual (noted $G_b^D(q)$); see Fig. 4 and Appendix A. Therefore the equation (13) becomes

$$G_{b'}(q_o)+G_{b'}^D(q_o) - G_{b'}(q_o)G_{b'}^D(q_o) = G_b(q_o)+G_b^D(q_o)-G_b(q_o)G_b^D(q_o) \qquad (16)$$

hence

$$G_{b'}(q_o) = G_b(q_o) \qquad (16')$$

Furthermore we can see that $G_b^D(q) = G_b(q)$ if and only if b is an odd number, and consequently if both b and b' are odd numbers the equation (16') admits the root $q_o(b,b')=1/2$ which is the exact answer. On the other hand, if b and/or b' is even, $q_o(b,b')$ is different from 1/2 but it tends to it for increasingly large cells, as can be seen in the Table.

The lines $\Gamma(b,1)$ for $2\leqslant b\leqslant 5$ (obtained through eq. (13)) are indicated in Fig. 5. We shall next specify an extrapolation procedure we have developed in order to obtain our best proposal Γ_∞ for the critical line. We first cut the curves $\Gamma(b,1)$ with let us say straight lines parametrized by ε (e.g. $\varepsilon=p$ or $\varepsilon=q$ or $\varepsilon=q/p$); we shall call $Y_b^*(\varepsilon)$ the abscissa (or equivalently the ordinate) values which are thus obtained. Next we plot $Y_b^*(\varepsilon)$ vs. b^{-1/ν_p} (where we choose for ν_p the best available value), and through linear extrapolation to the origin we obtain the limit $Y_c(\varepsilon)$ for that particular value of ε. The set of $\{Y_c(\varepsilon)\}$ will be our proposal of Γ_∞ for that particular family of straight lines (e.g. $\varepsilon=p$). We repeat this procedure with other families (e.g. $\varepsilon=q$ or $\varepsilon=q/p$). We have verified that $\Gamma_\infty(\varepsilon=p)$, $\Gamma_\infty(\varepsilon=q)$ and $\Gamma_\infty(\varepsilon=q/p)$ are extremely close to each other (see Fig.5). Another test of self-consistency of the present procedure can be performed by verifying, on a plot $\log [Y_b^*(\varepsilon)-Y_c(\varepsilon)]$ vs. $\log b$, that we obtain parallel straight lines for different choices of families

of straight lines (e.g. $\varepsilon=p$, etc) and of values of the parameter ε (see Fig. 6) — this fact supports the basic assumption that ν_p independs on such choices. Let us point out that the whole above procedure is nothing but a generalization, for a two-dimensional probability space, of the finite-size scaling law (Fisher 1971, Sur et al 1976, Suzuki 1977, Reynolds et al 1978) which is frequently used for the one-dimensional case, i.e. we are admiting the validity of an equation of the type

$$Y_b^*(\varepsilon) - Y_c(\varepsilon) \sim b^{-1/\nu_p} \qquad (17)$$

Such a behaviour is expected to hold for very large b, nevertheless it turns out that surprisingly small values of b already satisfy it. Let us recall that, for the critical probability of bond percolation on simple cubic lattice, we have obtained (Magalhães et al 1980), by using $b \leqslant 7$, $p_c=0.2526$, which is extremely close to the value $p_c=0.2495$ obtained by Kirkpatrick (1978), by using clusters with $b \leqslant 80$.

Let us now turn back to the critical line of the first- and second neighbour square lattice. We have verified that the three families we have chosen (namely, $\varepsilon=p$, $\varepsilon=q$ and $\varepsilon=q/p$) lead to points (see Fig. 5) which lie on straight lines within a linear correlation coefficient of 0.99995. Furthermore we have previously estimated (Magalhães et al 1980) the critical probability for the particular case $p=q$, and have found $p_c=0.250 \pm 0.003$. And finally we remark that the 1:1 ratio of the numbers of p-and q-bonds for the infinite lattice is not exactly respected for small clusters, hence the curves $\Gamma(b, 1)$ exhibit a little assymmetry* (with respect to the line $p=q$) which disappears for increasingly large cells. All these reasons strongly suggest that the exact critical frontier is given by

$$p + q = \frac{1}{2} \qquad (18)$$

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* This assymetry is evident on the Fig. 6 : the lines corresponding to let us say $\varepsilon=q/p=2/3$ and 3/2 do not coincide (they would do so if the curves $\Gamma(b, 1)$ were symmetric with respect to the line $p=q$).

though we have not succeeded to find a rigorous argument to prove it.

We have also calculated $\nu_p(b,b';\varepsilon)$ for different parameters ($\varepsilon = p,q$ and q/p) and for $2 \leqslant b \leqslant 3$, $1 \leqslant b' \leqslant 2$. We observed the same tendency as before (§ 2), e.g., the increase of the percentual variation of ν_p with ε for increasingly large cells. This fact shows that the PRG method is not suitable for calculating critical exponents.

4. "INHOMOGENEOUS" 4-8 LATTICE

Our last calculation refers to the critical frontier of the 4-8 lattice where the two topologically different bonds (see Fig. (1f)) have occupancy probabilities p and q. Notice that for either q=1 or p=1 the system reduces to the simple square lattice. The probability polynomials $R_b(p,q)$ corresponding to the cells shown in Figs. (1e) (b=1) and (1f) (b=2) are, respectively:

$$R_1(p,q) = 2pq^2-pq^4 \tag{19}$$

and

$$\begin{aligned} R_2(p,q) = {} & q^4(8p^2+8p^3+2p^4)+q^5(-4p^3-6p^4-2p^5)+q^6(-8p^2+4p^3-4p^4-6p^5) \\ & +q^7(-10p^3-12p^4-14p^5+4p^6)+q^8(2p^2-26p^4+14p^5+20p^6)+ \\ & +q^9(6p^3+24p^4+130p^5-4p^6)+q^{10}(-2p^3+42p^4-58p^5-108p^6)+ \\ & +q^{11}(18p^4-162p^5+16p^6)+q^{12}(-40p^4-4p^5+232p^6)+ \\ & +q^{13}(-24p^4+144p^5-144p^6)+q^{14}(12p^4+32p^5-112p^6)+ \\ & +q^{15}(16p^4-104p^5+128p^6)+q^{16}(-7p^4+32p^5-32p^6) \end{aligned} \tag{20}$$

Similarly to the previous section, we have applied only the PRG method to the cells of the type mentioned just above, obtaining the curves $\Gamma(b,1)$ (b=2,3,4) indicated in Fig. (7). Since the chosen cells reduce, for q=1, to the self-dual H-shaped clusters, all curves $\Gamma(b,b')$ contain the exact point (p,q) = (1/2,1). Furthermore, we can show (see Appendix B) that at this particular point, due to the fact that the 4-8 lattice (as well as the clusters we have chosen) is constituted by q-bond-polygons linked to each other by p-bonds*, we have:

$$\left.\frac{dp}{dq}\right|_{\substack{p=1/2 \\ q=1}} = 0 \qquad \forall\,(b,b') \tag{21}$$

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* The "inhomogeneous" 3-12 lattice also has this topological property, and hence eq. (21) holds also for this lattice.

It is also worth remarking that in the particular case of p=1, our clusters reduce for b=2 and 3, respectively, to the last ones drawn in Fig. 4 of our previous work (Magalhães et al 1980) (where they were introduced to illustrate a "biased" percolation on square lattice); we can see in Fig. 7 that q* approaches the exact value 1/2 as b increases, as it should be.

Following the same extrapolation procedure of §3 (see eq. (17)), we have obtained, through three different families of straight lines, the points shown in Fig. 7; the self-consistency of the extrapolation method for this particular lattice can be checked on Fig. 8. Extrapolation in the neighbourhood of p=1 yields the following estimate for the derivative

$$\left.\frac{dq}{dp}\right|_{\substack{p=1\\q=1/2}} = -0.45 \pm 0.10 \qquad (22)$$

In the particular case p=q, we were able to calculate $p^*(b,1)$ for bigger clusters ($b \leqslant 7$). From the plot of $p^*(b,1)$ versus b^{-1/ν_p} (see Fig. 9) we estimate that:

$$p_c = q_c = 0.681 \pm 0.005 \qquad (23)$$

which differs very little from $Y_c(q/p=1) \simeq 0.684$ (where $b \leqslant 4$).

Let us conclude by saying that the simple analytic relation

$$v = \frac{1}{3}(2u^2 + u^3) \qquad (24)$$

where

$$v \equiv 2(p-1/2)$$

and

$$u \equiv 2(1-q)$$

determines a critical frontier which contains the exact points (1,1/2) and (1/2,1), satisfies the property (21), leads to

$$\left.\frac{dq}{dp}\right|_{\substack{p=1\\p=1/2}} = -\frac{3}{7} \simeq -0.43 \text{ (in agreement with estimate (22))}$$

and to $p_c = q_c \simeq 0.680$ (in accordance with eq. (23)), and fits very satisfactorily the rest of the extrapolated points (see Fig. 7); no doubt that eq. (24) has a good chance of being the exact one.

5. CONCLUSION

We have verified in the anisotropic first-neighbour square lattice that the canonical renormalization group (CRG) and the parametric renormalization group (PRG) procedures we have used yield satisfactory results in what concerns the percolation critical line. While the convergence properties are better in the CRG method, the PRG one is operationally much easier; however both procedures lead to critical lines which tend, for increasingly large clusters, towards the exact critical frontier $p+q=1$.

The critical frontiers corresponding to the "inhomogeneous" first- and second-neighbour square and 4-8 lattices are estimated within the PRG framework, by using an extrapolation procedure which applies for two- (or higher) dimensional probability space and which exhibits quite satisfactory self-consistent properties. All of our extrapolated numerical results are well fitted by the possibly exact equations $p+q=1/2$ for the former and

$$3(p-1/2)=4\left[(1-q)^2+(1-q)^3\right] \quad \text{for the latter.}$$

The PRG method does not give satisfactory results for the critical exponents.

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APPENDIX A

Some useful formulae*

- Series Configuration:

An array of n bonds in series with probabilities $p_1, p_2, \ldots, p_n$ is equivalent to a single bond with probability p_s given by:

$$p_s = \prod_{i=1}^{n} p_i \qquad \text{(A-1)}$$

- Parallel Configuration:

The equivalent probability p_p for n parallel bonds with probabilities $p_1, p_2, \ldots, p_n$ satisfies:

$$1 - p_p = \prod_{i=1}^{n} (1-p_i) \qquad \text{(A-2)}$$

- Deletion-Contraction Rule:

Consider a two-terminal graph (noted $G(p_1, \ldots, p_j, \ldots)$) whose bonds are present with probabilities $p_1, p_2, \ldots, p_j, \ldots$ If we choose an arbitrary bond, let us say the j-th bond, the following relation (called the "deletion-contraction rule") holds:

$$G(p_1, \ldots, p_j, \ldots) = (1-p_j) G_j^b(p_1, \ldots, p_{j-1}, p_{j+1}, \ldots) +$$

$$+ p_j G_j^c(p_1, \ldots, p_{j-1}, p_{j+1}, \ldots) \qquad \text{(A-3)}$$

where G_j^b (G_j^c) is the equivalent probability of the graph obtained from G by breaking (collapsing) its j-th bond (or in equivalent words, by making $p_j=0$ ($p_j=1$)).

Let us illustrate this method on an example:

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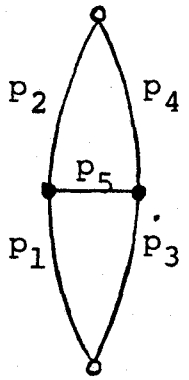
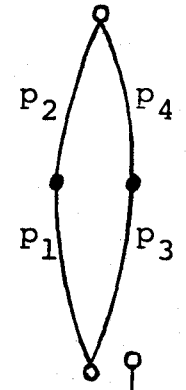
* The following properties are discussed in Tsallis 1979.

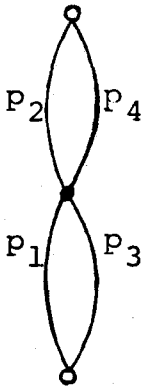
$G(p_1,p_2,p_3,p_4,p_5)$ =



$= (1-p_5)$

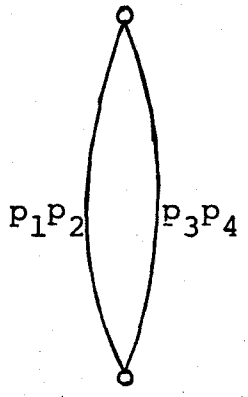


$+ p_5$



=

$= (1-p_5)$



$+ p_5$

$p_2+p_4-p_2p_4$

=

$p_1+p_3-p_1p_3$

$$= (1-p_5)\ (p_1p_2+p_3p_4-p_1p_2p_3p_4)+p_5\,(p_2+p_4-p_2p_4)\,(p_1+p_3-p_1p_3)$$

(A-4)

- Duality

The two- terminal planar graphs G and G^d are said to be dual if they can be superimposed in such a way that each bond of a graph crosses one and only one bond of the other, and also that each internal (non-terminal) node of one graph is surrounded by an elementary mesh of the other. In particular, if G is self-dual (see, for example, Fig. 1b and Fig. 4) its equivalent probability satisfies

$$G(p_1,p_2,\ \ldots) + G(1-p_1,1-p_2,\ \ldots) = 1 \qquad \text{(A-5)}$$

which implies, if $p_i=p\ \forall\ i$, that the graphical representation of G against p admits the point (1/2,1/2) as a symmetry center.

APPENDIX B

- 4-8 Lattice

In the present Appendix we prove, for all the frontiers $\Gamma(b,b')$ associated to the 4-8 lattice, that $\left.\frac{dp}{dq}\right|_{\substack{p=1/2\\q=1}} = 0$

Every probability polynomial $R_b(p,q)$ associated to a b- cluster of any lattice can be written as:

$$R_b(p,q)=F_o^{(b)}(p)\ q^m+F_1^{(b)}(p)q^{m-1}(1-q)+A^{(b)}(p,q)(1-q)^2 \quad \forall b \qquad (B\text{-}1)$$

where m is the number of relevant q- bonds. $F_o^{(b)}(p)$ and $F_1^{(b)}(p)$ stand for the probabilities that a b-cluster with, respectively, m and (m-1) unblocked q- bonds (and any number of present p- bonds) percolates.

From eq. (B-1) it follows that

$$\left.\frac{\partial R_b}{\partial q}\right|_{q=1} = m\ F_o^{(b)}(p) - F_1^{(b)}(p) \qquad \forall b,\ \forall p \qquad (B\text{-}2)$$

Now it is easy to see that, due to the fact that the "inhomogeneous" 4-8 lattice (as well as the clusters we have chosen for it) is constituted by q- bond- polygons linked among them by p- bonds (the same topological property appears in the 3-12 lattice, for instance), if a given configuration percolates (does not percolate) when all the q- bonds are present, it continues percolating (not percolating) if we block one of them. As we have m such possibilities we conclude that

$$F_1^{(b)}(p) = m\ F_o^{(b)}(p) \qquad \forall b,\ \forall p \qquad (B\text{-}3)$$

which substituted in eq. (B-2) leads to:

$$\left.\frac{\partial R_b}{\partial q}\right|_{q=1} = 0 \qquad \forall b,\ \forall p \tag{B-4}$$

Furthermore, we know from eq. (13) that, along the curve $\Gamma(b,b')$, it holds

$$\left.\frac{dp}{dq}\right|_{(p^*,q^*)} = -\left\{\left(\frac{\partial R_b}{\partial q} - \frac{\partial R_{b'}}{\partial q}\right)\left(\frac{\partial R_b}{\partial p} - \frac{\partial R_{b'}}{\partial p}\right)^{-1}\right\}_{(p^*,q^*)} \tag{B-5}$$

Although we did not succeed to rigorously prove that, in general, $\left.\frac{\partial R_b}{\partial p}\right|_{q=1} \neq \left.\frac{\partial R_{b'}}{\partial p}\right|_{q=1}$, it was so in all the cases we examined.

Therefore we conclude that the expression (B-5) evaluated at $q^*=1$ is always zero for every (b,b').

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CAPTION FOR FIGURES AND TABLE

Fig. 1 - Examples of clusters (b=1,2) and their associated graphs, used for the treatment of the anisotropic first-neighbour square lattice ((a) and (b)) and the "inhomogeneous" first- and second-neighbour square ((c) and (d)) and 4-8((e) and (f)) lattices. The arrows indicate the entries and exits of the clusters. Open (full) circles denote the terminal (internal) nodes of the graphs.

Fig. 2 - An example (b=2,b'=1) of flow diagram in the p-q space for the anisotropic first-neighbour square lattice. Each full line represents a CRG trajectory followed by some arbitrary initial point in the neighbourhood of the fixed point (1/2, 1/2). The successive points exhibit the velocity of convergence within flow lines. The broken and chain lines respectively indicate the CRG and PRG critical frontiers. The fixed points can be fully stable (■), fully unstable (◊) and partially stable (●).

Fig. 3 - Examples of PRG critical frontiers for the first-neighbour square lattice ("E" refers to closed form frontiers; the other is a Monte Carlo one). The q=0 points, namely $p(2,1) < p(3,1) < p(4,1) < p(5,1) \simeq p(3,2) < p(4,2) < p(5,2) < p(4,3) < p(5,3) < p(5,4)$ and their respective symmetric q=1 points are also indicated. The dashed line represents the exact critical line.

Fig. 4 - Example of a cluster with odd b (b=3), associated to the case p=0 of the first- and second-neighbour square lattice; it can be decomposed into a self-dual graph in parallel with itself.

Fig. 5 - PRG critical frontiers and extrapolated points for the first- and second-neighbour square lattice ("E" refers to closed form frontiers; the others are Monte Carlo ones).

The extrapolated points were obtained through three different families of straight lines, where, because of universality, the nearly exact value 1.35 has been adopted for ν_p. The proposal p+q=1/2 is indicated by the dashed line.

Fig. 6 - Log-log plot of $\left[Y_b^*(\varepsilon)-Y_c(\varepsilon)\right]$ as a function of b for the first- and second-neighbour square lattice (for simplicity only a few examples are shown). A straight chain line with slope equal to $-(1.35)^{-1}$ is indicated.

Fig. 7 - PRG critical frontiers and extrapolated points (with ν_p=1.35) for the 4-8 lattice ("E" refers to a closed form frontier; the other two are Monte Carlo ones). The dashed line represents our analytic proposal (eq.(22)).

Fig. 8 - Log-log plot of $\left[Y_b^*(\varepsilon)-Y_c(\varepsilon)\right]$ as a function of b for the 4-8 lattice (for simplicity only a few examples are shown). A straight chain line with slope equal to $-(1.35)^{-1}$ is indicated.

Fig. 9 - The approximate critical probability $p^*(b,1)=q^*(b,1)$ for the 4-8 lattice as a function of b^{-1/ν_p} with ν_p=1.35 ("E" denotes a closed form point; the others are Monte Carlo ones).

TABLE - RG closed form values of $q_o(b,b')$ associated to the p=0 bond percolation on the first- and second-neighbour square lattice.

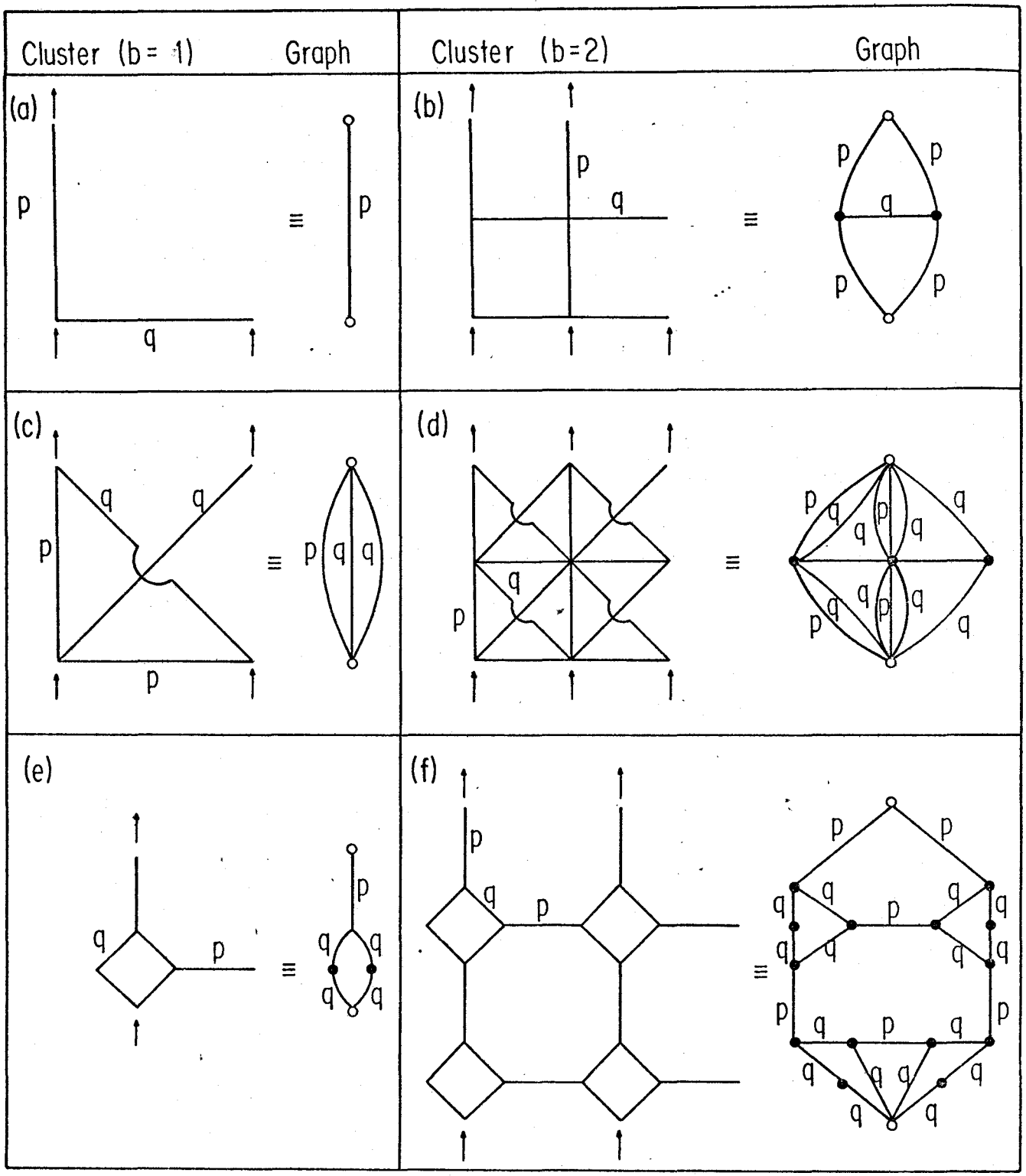

FIG. 1

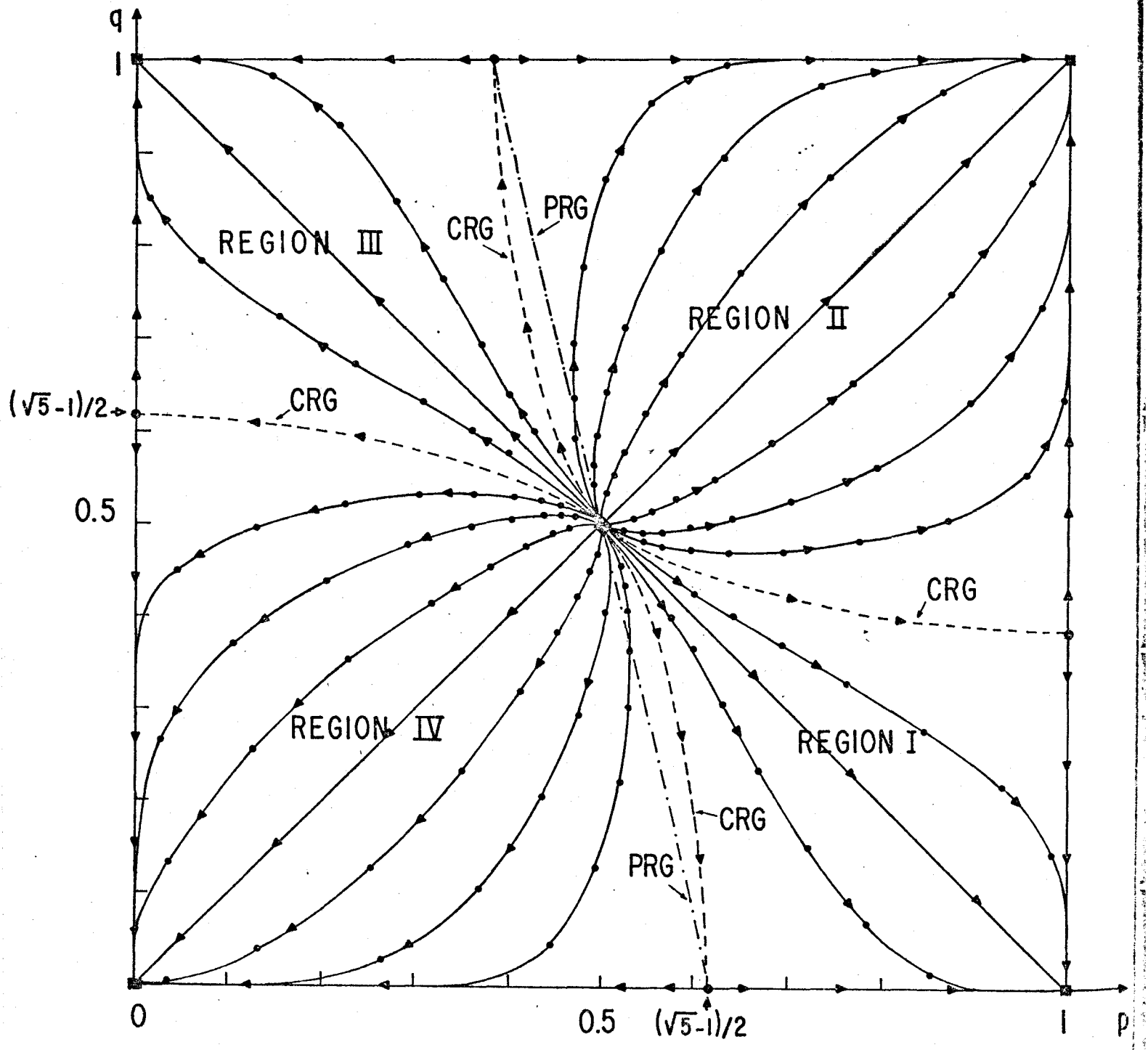
q
1
REGION III
CRG
PRG
REGION II
(√5-1)/2
CRG
0.5
CRG
REGION IV
REGION I
CRG
PRG
0
0.5
(√5-1)/2
1
p

FIG. 2

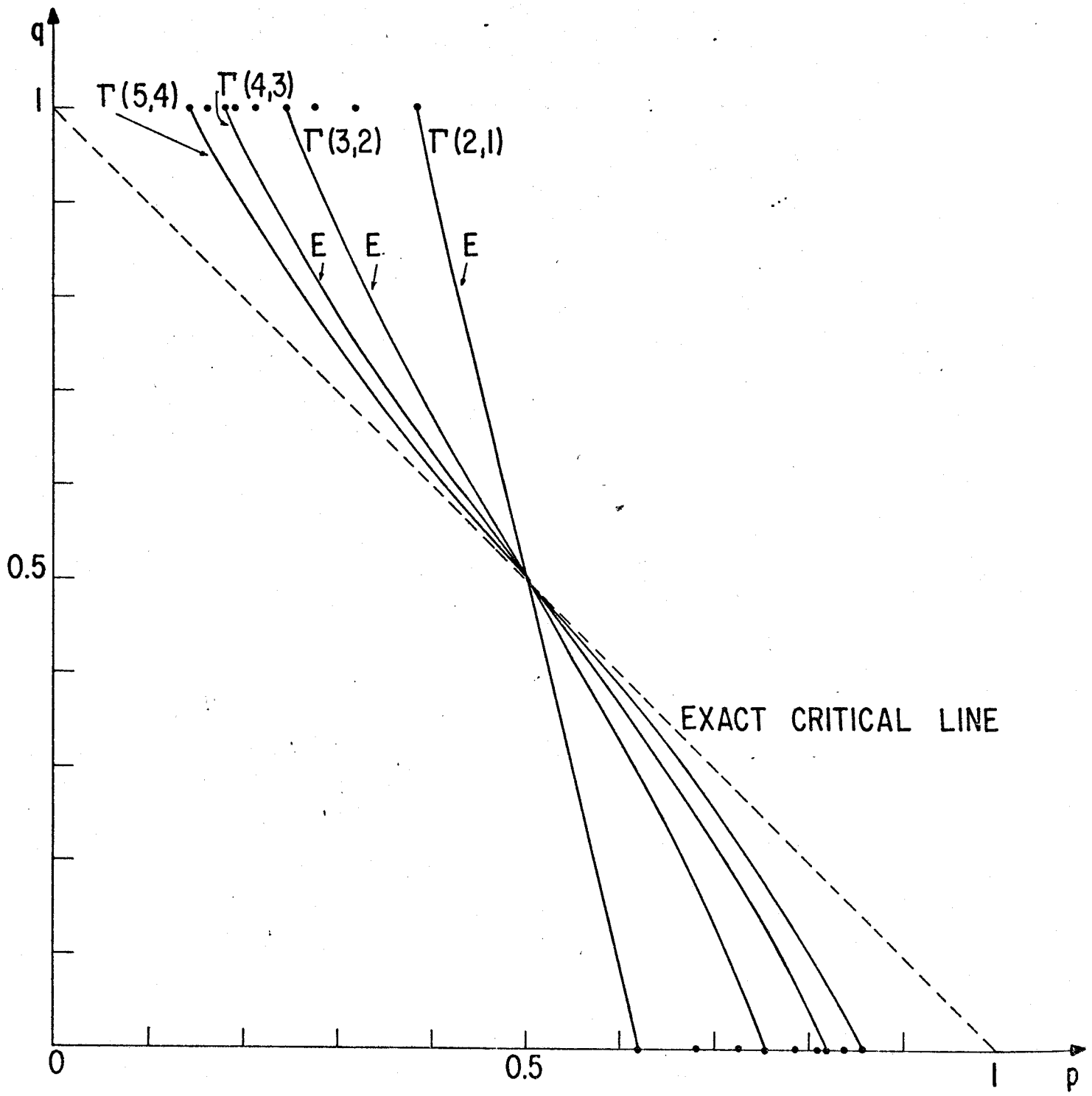

FIG. 3

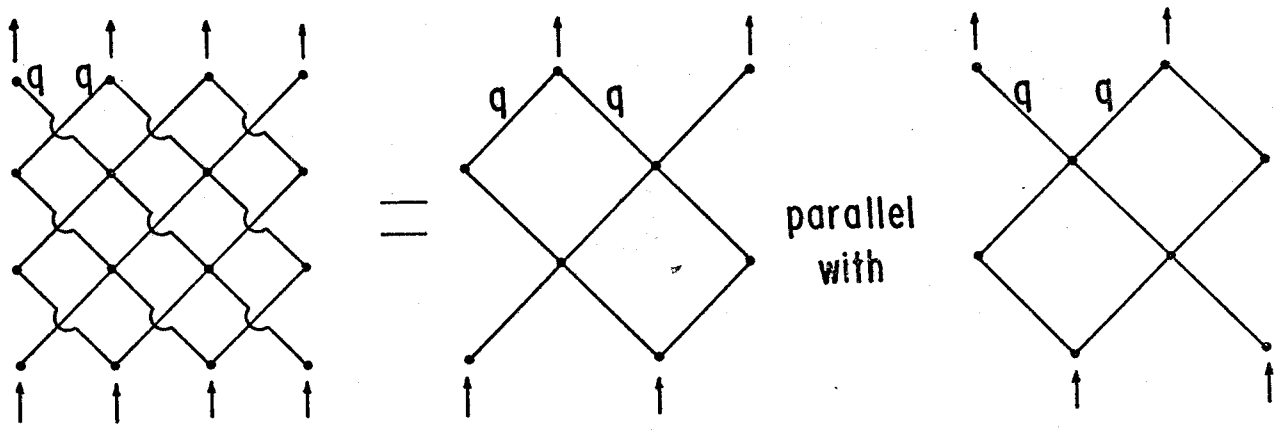

FIG. 4

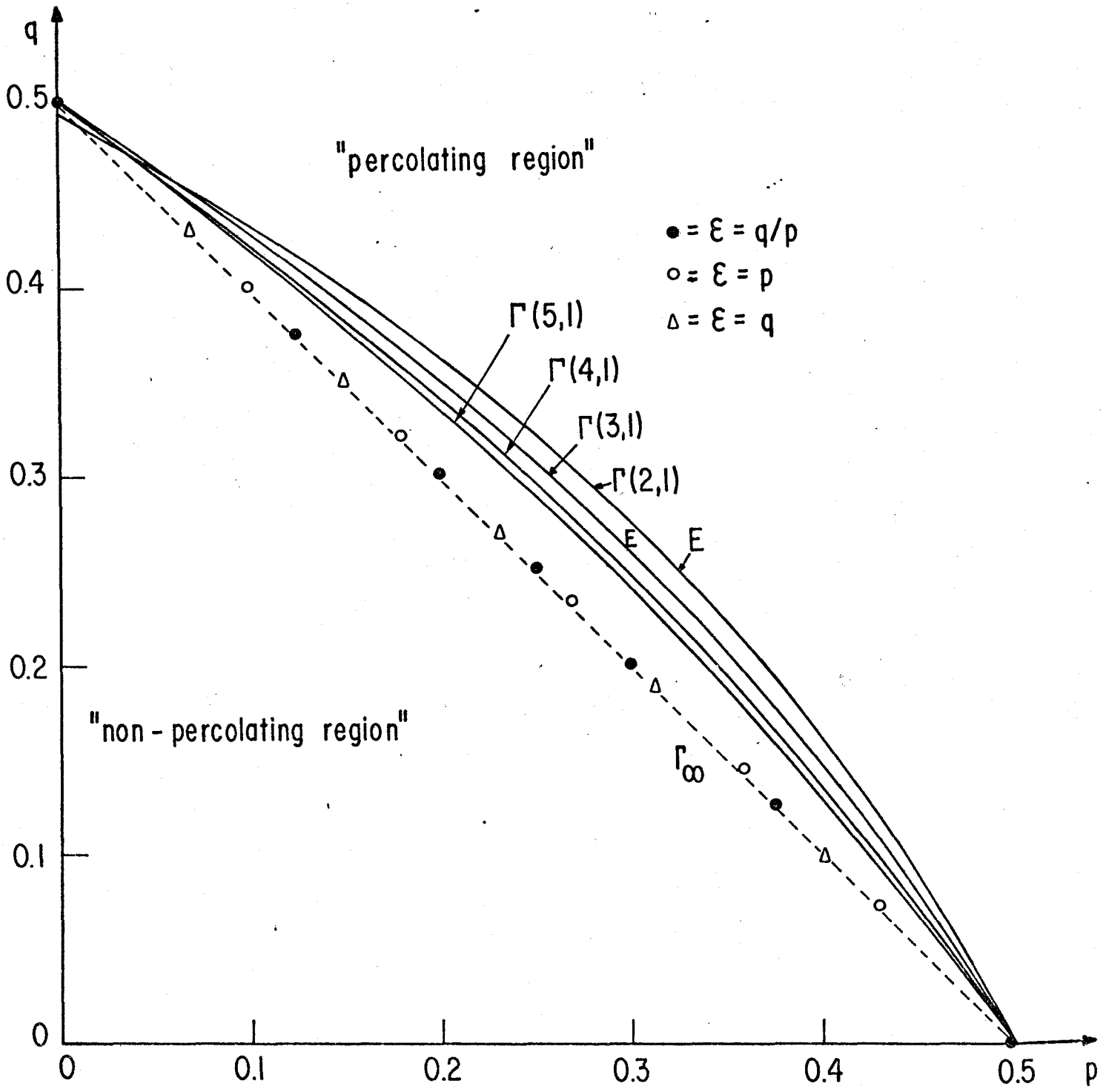
q
0.5
0.4
0.3
0.2
0.1
0
"percolating region"
"non - percolating region"
● = ε = q/p
○ = ε = p
△ = ε = q
Γ(5,1)
Γ(4,1)
Γ(3,1)
Γ(2,1)
E
E
Γ∞
0
0.1
0.2
0.3
0.4
0.5
p

FIG. 5

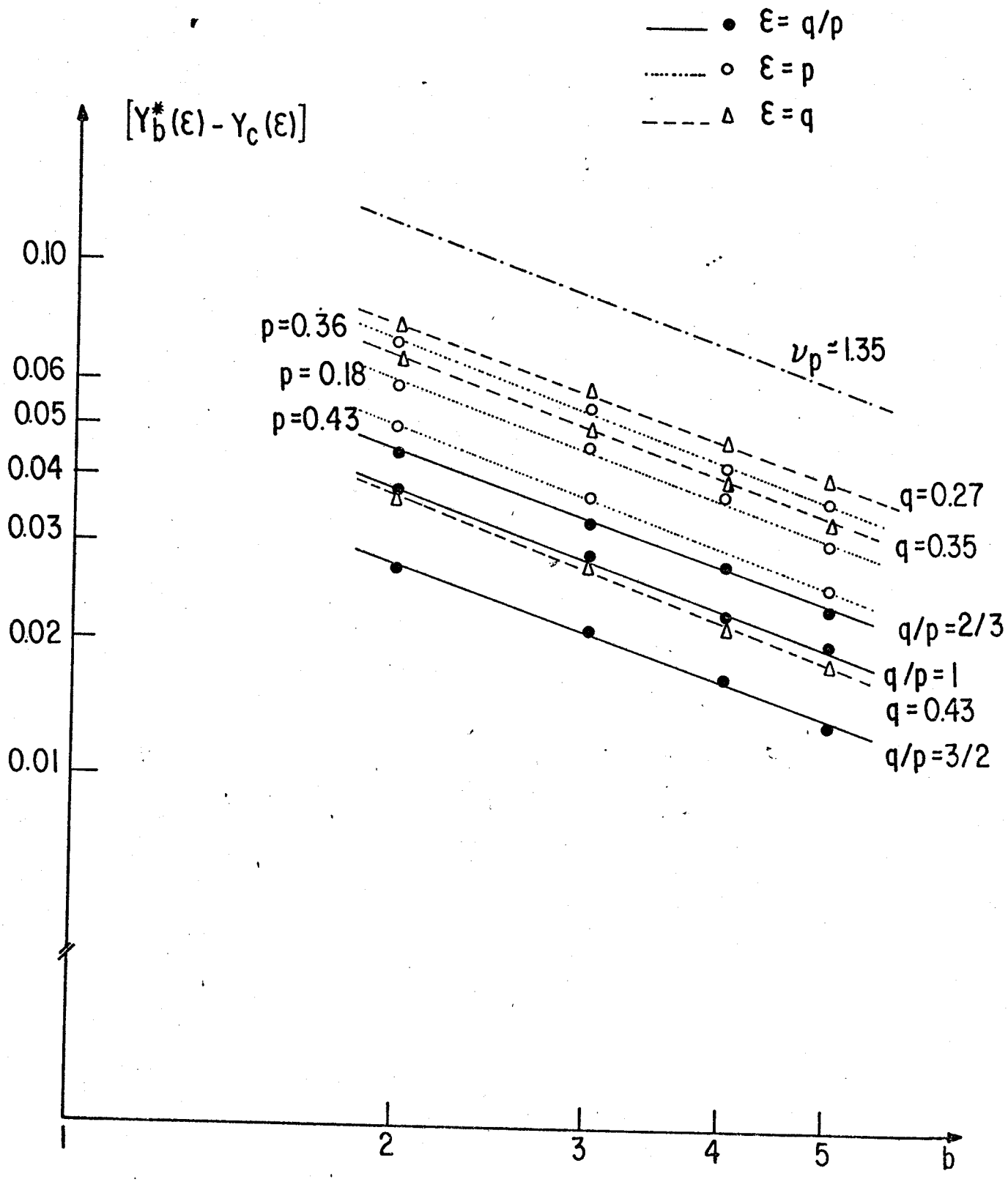
ε = q/p
ε = p
ε = q
$[Y_b^*(\varepsilon) - Y_c(\varepsilon)]$
0.10
0.06
0.05
0.04
0.03
0.02
0.01
p=0.36
p = 0.18
p=0.43
$\nu_p = 1.35$
q=0.27
q=0.35
q/p =2/3
q/p = 1
q = 0.43
q/p =3/2
1
2
3
4
5
b

FIG. 6

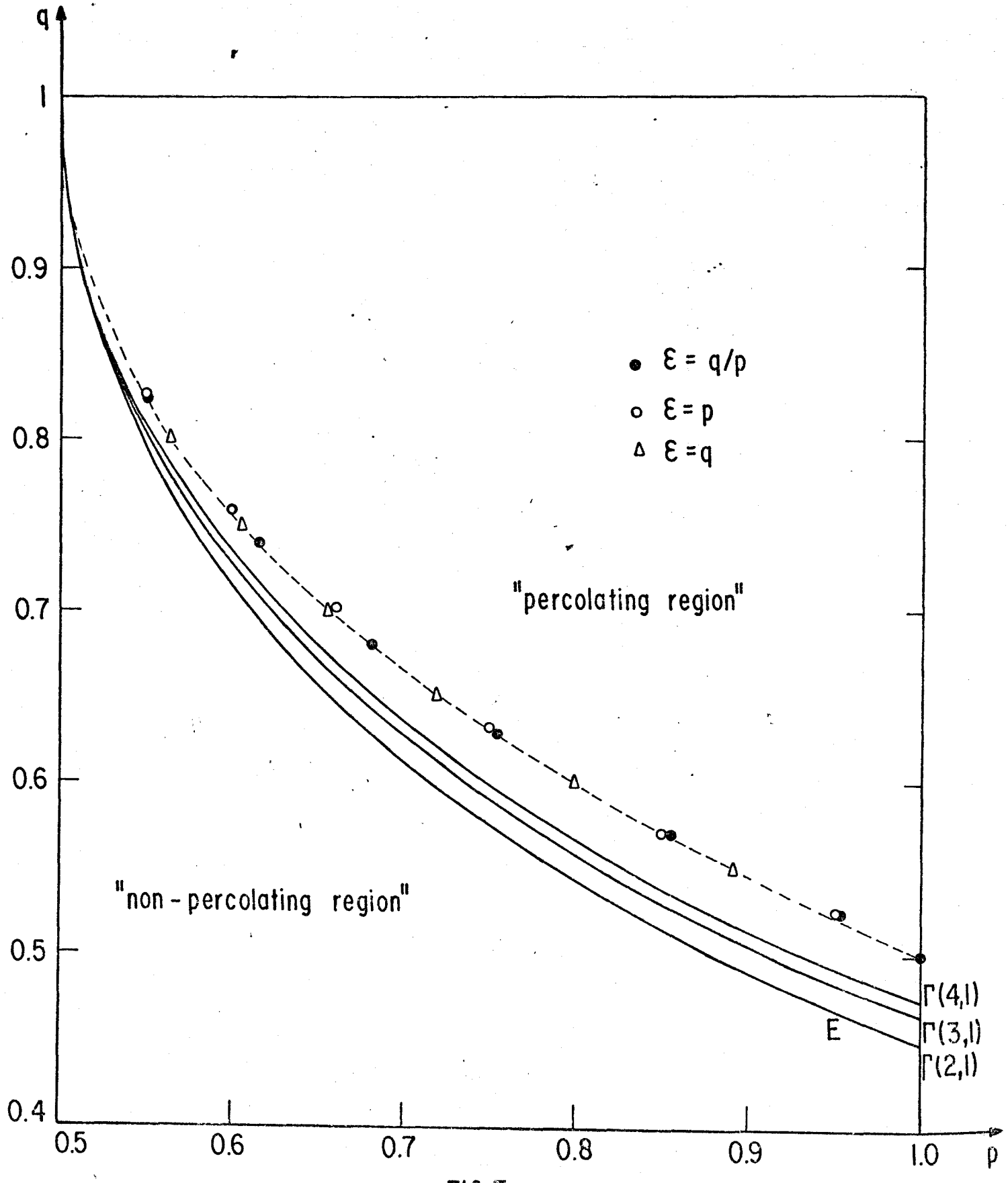
q
1
0.9
0.8
0.7
0.6
0.5
0.4
0.5
0.6
0.7
0.8
0.9
1.0
p
ε = q/p
ε = p
ε = q
"percolating region"
"non - percolating region"
Γ(4,1)
Γ(3,1)
Γ(2,1)
E

FIG. 7

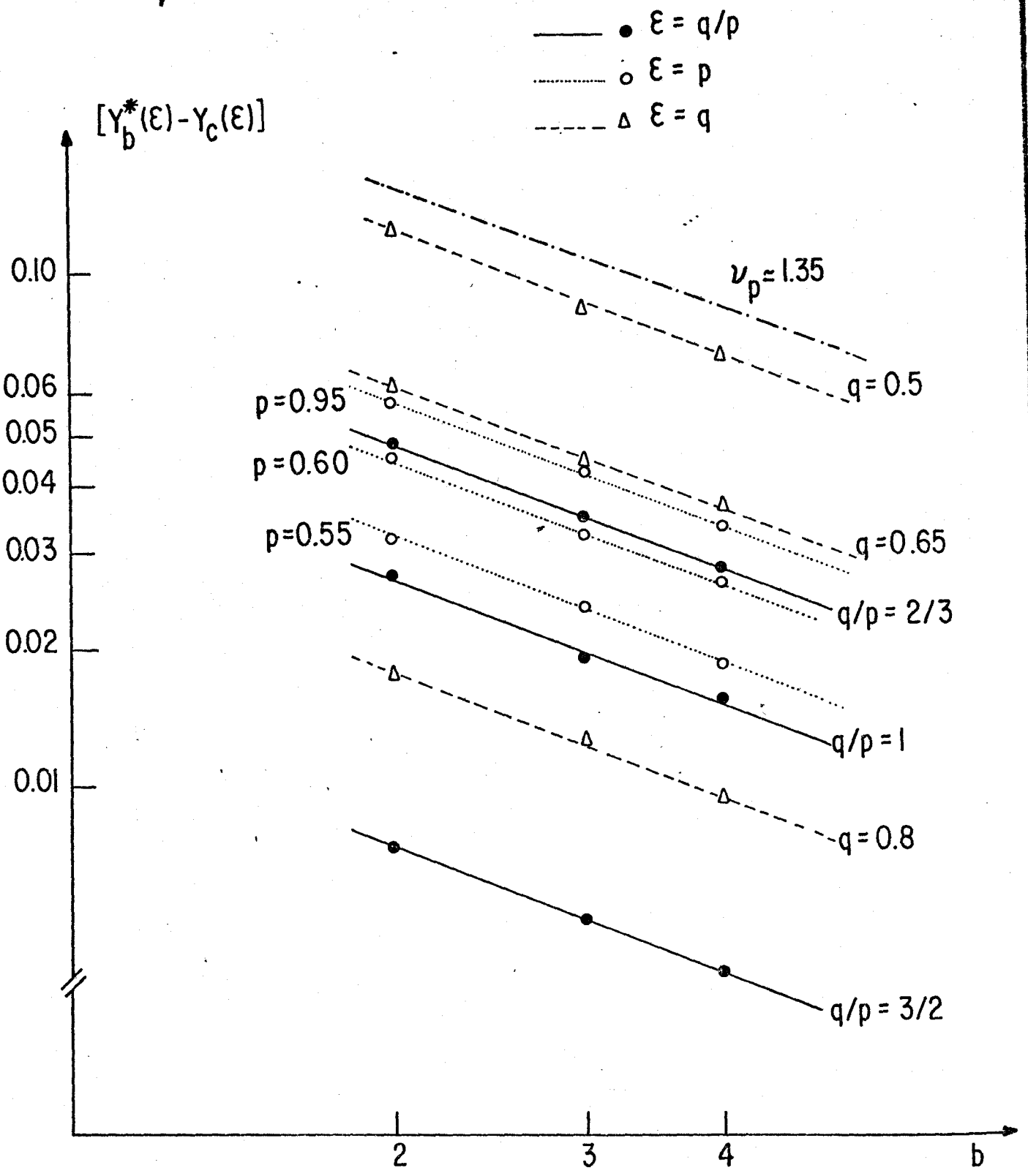
ε = q/p
ε = p
ε = q
$[Y_b^*(\varepsilon) - Y_c(\varepsilon)]$
0.10
0.06
0.05
0.04
0.03
0.02
0.01
$\nu_p = 1.35$
q = 0.5
p = 0.95
p = 0.60
p = 0.55
q = 0.65
q/p = 2/3
q/p = 1
q = 0.8
q/p = 3/2
2
3
4
b

FIG. 8

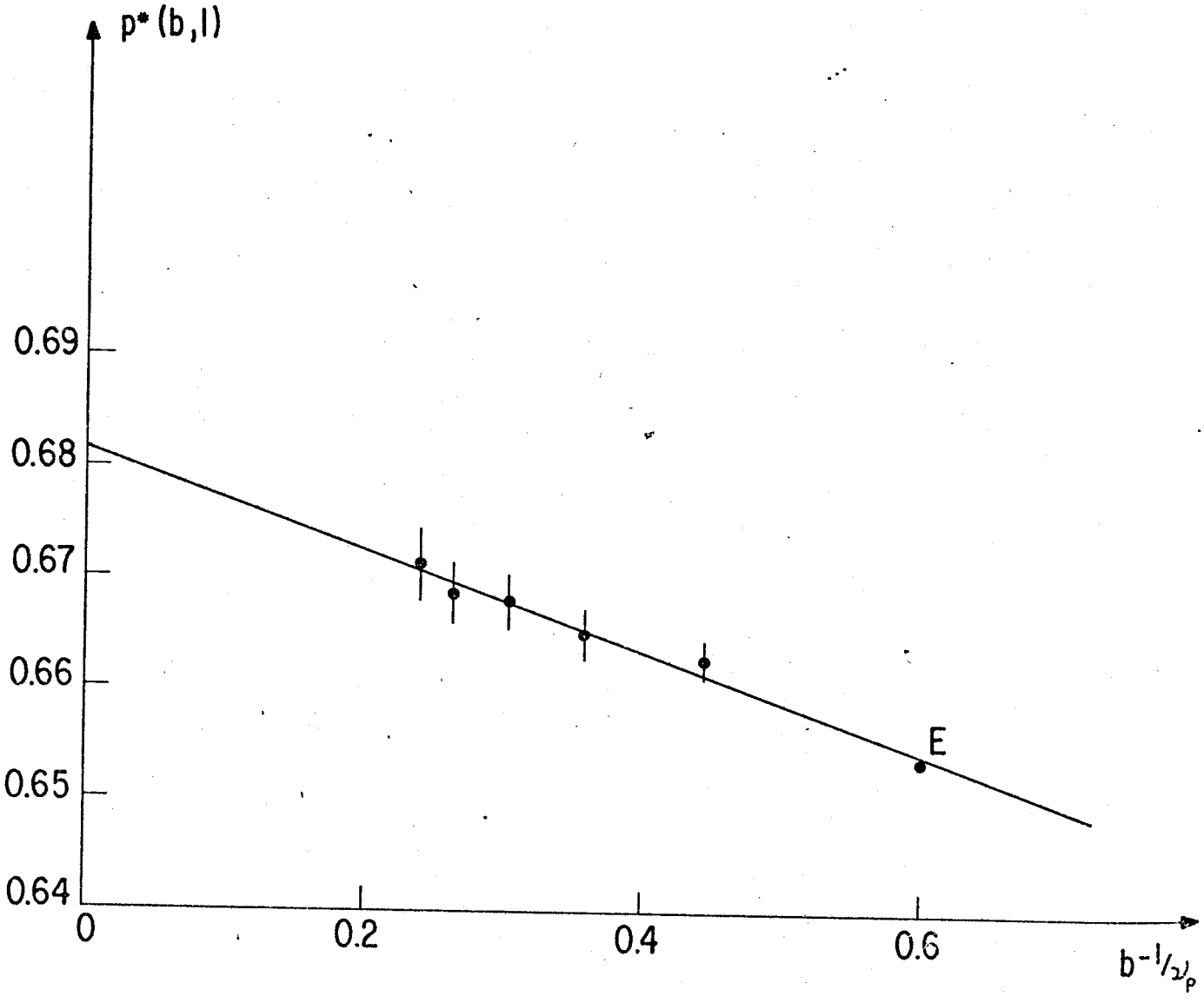

FIG. 9

T A B L E

b' \ b	2	3	4	5
1	0.4923	1/2	0.4997	1/2
2	-	0.5090	0.5042	0.5032
3	-	-	0.4989	1/2
4	-	-	-	0.5011

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