CBPF-NF-012/98

Feynman Path-Integral for the Damped Caldirola-Kanai Action

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Abstract

We propose a Local Feynman path integral representation for the damped Caldirola-Kanai action.

Key-words: Quantum dissipative systems; Feynman path integral.
PACS numbers: 71.20 Ad; 72.15 Rm, 72.70.+m

It is an interesting problem in Dissipative Quantum Mechanics [1], [2] to find a Local Feynman path integral for the classical system of a free electron in a medium with a frictional drag proportional to velocity.

In this note we propose a formal Path-Integral to the phenomenological Caldirola-Kanai action by following original heuristic Feynman procedure [2] to quantize classical systems by means of a suitable sum over paths.

Let us start our analysis by considering the Local Caldirola-Kanai Classical action [1] of a one-dimensional free electron of mass in moving on a medium with a frictional drag proportional to its velocity and with a positive viscosity (temperature dependent) coeficient ν.

$$L_\nu(x(\sigma);\dot{x}(\sigma)) = \int_{t'}^{t} d\sigma\; exp(\nu\sigma)\left(\frac{1}{2}m\dot{x}^2(\sigma)\right) \tag{1}$$

In order to write a Feynman path-integral representation for the Feynman Quantum Mechanical Propagator associated to the Lagrangean Eq. (1), we follow Feynman by postulating the asymptotic Green Function connecting the Wave functions for infinitesimal different times $t_{k+1} - t_k = -(t-t')/N \rightarrow {}^{N\rightarrow\infty} 0$

$$\psi(x_{k+1};t_{k+1}) = \int_{-\infty}^{+\infty} dx_k \tilde{G}[(x_{k+1},t_{k+1});(x_k,t_k)]\psi(x_k;t_k) \tag{2}$$

where the asymptotic Green Function used to define the short-time propagation is determined by the Classical action Eq. (1) with suitable pre-factors.

$$\begin{aligned}&\tilde{G}[(x_{k+1},t_{k+1});(x_k,t_k)]_{\varepsilon\rightarrow 0} \approx A(t_{k+1},t_k)\\ &exp\left\{\frac{i}{\hbar}\frac{1}{2}m\; exp[\nu(at_{k+1}+bt_k)]\left[\frac{(x_{k+1}-x_k)^2}{\varepsilon^2}\right]\varepsilon \rightarrow\right\}\end{aligned} \tag{3}$$

Note that in order to analize anomalous pre-factors in the Feynman Path Integral for dissipative system ([2]]), we have proposed to introduce a weighted rule ($a+b=1$) for the discretization of the damping Caldirola-Kanai term $exp(\nu t)$ into the Damping Classical action Eq. (1).

From Eq. (2) for $\varepsilon\rightarrow 0$, we can determine the pre-factor in Eq. (3) as originally done by Feynman in his heuristic description of the Feynman measure for time independent propagartors. Note that the origin of the above mentioned "Dissipative Anomaly" is a consequence of the appearence of the discretized Caldirola-Kanai term in the expression of this purely quantum object in the Feynman Path-Integral (propagator pre-factor)

$$A(t_{k+1},t_k) = exp\;\frac{\nu}{2}(at_{k+1}+bt_k)\left[\frac{m}{2\pi i\hbar(t_{k+1}-t_k)}\right]^{1/2} \tag{4}$$

As a consequence of Eq. (2)-Eq. (4), we can write the Green Function for arbitrary different times as a Feynman Path-Integral as done originally in the first reference of [2].

$$\begin{aligned}&\tilde{G}[(x,t);(x',t')] =\\ &\lim_{N\rightarrow\infty}\int\left(\prod_{k=1}^{N-1}dx_k\right) exp\;\frac{\nu}{2}\left[\sum_{k=0}^{N-1}\; a\left(t'+\frac{t-t'}{N}(k+1)\right)+b\left(t'+\frac{t-t'}{N}k\right)\right]\\ &\times\prod_{k=0}^{N-1}\left(\frac{m}{2\pi i\hbar(t_{k+1}-t_k)}\right)^{1/2}\; exp\left\{\frac{i}{\hbar}\prod_{k=0}^{N-1}\varepsilon\; exp\left[\frac{\nu}{2}(at_{k+1}+bt_x)\right]\frac{(x_{k+1}-x_k)^2}{\varepsilon^2}\right\}\end{aligned} \tag{5}$$

Now we can define formally the limit in Eq. (5) as a well defined Feynman Measure over paths multiplied by a general damping anomaly factor $exp\ \frac{\nu}{4}\ (t-t')(a-b)$, namelly

$$e^{\frac{\nu}{4}(t-t')(a-b)} D^F[x(\sigma)] = \lim_{N\to\infty} \left[\left(\prod_{k=1}^{N-1} dx_k \right) \left(\frac{m}{2\pi i\hbar\bar{\varepsilon}(\nu,t,t')} \right)^{N/2} \right] e^{\frac{\nu}{4}(t-t')(a-b)} \tag{6}$$

Note that the infinitesimal step in the factorized Feynman measure, Eq. (8), is explicitly given by the expression below and is independent of our original weighted time-interval partition rule used for the dissipative term $exp\ (\nu t)$ in the Caldirola-Kanai action.

$$\bar{\varepsilon}(\nu,t,t') = \varepsilon\ exp\left[-\frac{\nu}{2}(t-t')\right] = \frac{(t-t')}{N}\ exp\left[-\frac{\nu}{2}(t+t')\right] \tag{7}$$

The above written results are simple consequence of the following evaluations

$$\begin{aligned}
& exp\ \frac{\nu}{2}\left[\sum_{k=0}^{N-1} a(t'+\varepsilon(k+1)) + b(t'+\varepsilon k))\right] \\
&= exp\ \frac{\nu}{2}\left[\sum_{k=0}^{N-1}(at') + \sum_{k=0}^{N-1}(a\varepsilon(k+1)) + \sum_{k=0}^{N-1}(bt') + \sum_{k=0}^{N-1}(b\varepsilon k)\right] \\
&= exp\ \frac{\nu}{2}\left[(at')(N-1) + (bt')(N-1) + b(t-t')\frac{(N-1)}{2} + a(t-t')\frac{(N+1)}{2}\right] \\
&= exp\ \frac{\nu}{2}\left[(N-1)t' + (a-b)\frac{(t-t')}{2} + \frac{a(t-t')N}{2} + \frac{b(t-t')N}{2}\right] \\
& exp\ \left[\frac{\nu}{2}(a-b)(t-t')\right] \times\ exp\ \left[\frac{\nu}{2}(t+t')N\right]
\end{aligned}$$

By substituting Eq. (8) into eq. (5) we get our above displayed Eq. (6).

The propagator Eq. (5) has, thus, the Dissipative anomaly found in the second reference of [2] factored out by an overall anomaly factor which exact value depends on the rule used to discretize, in eq. (1), the term $exp\ (\nu t)$ and of the initial and final time propagation. For the weighted rule it yields the result

$$\tilde{G}(x,t);(x',t')) = e^{\frac{\nu}{4}(t-t')(a-b)} \int_{x(t')=x';x(t)=x} \mathcal{D}^{F[x(\sigma)]e^{\frac{i}{\hbar}\int_{t'}^{t} d\sigma e^{\nu\sigma}(\frac{1}{2}m\dot{x}^2(\sigma))}} \tag{8}$$

Note that our main result Eq. (6)-Eq. (9) differs somewhat from that similar obtained in the above cited second reference Eq. (2.16) of [2]. Another point to stress is the similarity between the existence of a dissipative anomaly in the formal path-integral Eq. (9) and the famous De-Witt anomaly in curved space-time propagator. Let us remark the usefulness of our proposed Path-Integral Eq. (9) with the viscosity anomally effects factored out by calling attention that the combined Green function

$$G^{(0)}((x,t);(x',t')) = e^{-\frac{\nu}{4}(t-t')(a-b)}\tilde{G}((x,t);(x',t')) \tag{9}$$

satisfies now the usual time dependent Schroedinger equation initial value – problem for t and t' finite times (see Eq. (7)) here

$$i\hbar\frac{\partial}{\partial t}G^{(0)}((x,t);(x',t')) = -\frac{\hbar^2}{2me^{\nu t}dx^2}\ \frac{d^2}{dx^2}G^{(0)}((x,t);(x',t')) \tag{10}$$

here

$$\lim_{t\to t'} G^{(0)}((x,t);(x',t')) = \delta(x-x') \tag{11}$$

The above cited claim is a general consequence of Eq. (6) defining a well defined Feynman product measure (see appendix A for details). At this point, we remark that by choosing the Feynman middle point rule $a=b=1/2$, in the latticized prescription for the Caldirola-Kanai Path Integral propagator, we may suppress the anomaly into Eq. (9) (see Ref.[5] for similar phenomena in Feynman Path-Integral for curved space-time).

A simple solution of Eq. (11)-Eq. (12) is easily obtained for non-zero initial time t':

$$G^{(0)}((x,t);(x',t')) = \left(\frac{4\pi\nu m i e^{\nu t'}}{\hbar(1-e^{-\nu(t-t')})}\right)^{1/2} e^{i\frac{(x-x')^2 m\nu e^{\nu t'}}{\hbar(1-e^{-\nu(t-t')})}} \tag{12}$$

The complete scheme-dependent Functional propagator will, thus, be given by the result

$$\tilde{G}^{(0)}((x,t);(x',t')) = e^{\frac{\nu}{4}(t-t')(a-b)} \times \tilde{G}^{(0)}((x,t);(x',t')) \tag{13}$$

It is worth point out that only for $a=b=1/2$, we have that the quantum probability $\tilde{G}^{(0)}((x,t);(x',t'))|^2$ decay to zero at the equilibrium limit $t\to\infty$.

It is important to remark that the presence of time-dependent potentials does not modify the Path-Integral representation above displayed

$$\tilde{G}^{(0)}((x,t);(x',t')) = e^{\frac{\nu}{4}(t-t')(a-b)} \int_{x(t')=x';x(t)=x} \mathcal{D}^F[x(\sigma)] e^{\frac{i}{\hbar}\int_{t'}^{t} d\sigma e^{\nu\sigma}(\frac{1}{2}m\dot{x}^2(\sigma)-V(x(\sigma),\sigma)]} \tag{14}$$

Let us exemplify Eq. (14) by applying it to the case of existence of a constant magnetic field perpendicular to the plane containing the particle trajectory ([4])

$$\vec{A}(x,y) = (-\frac{1}{2}Hy)\vec{i} + \left(\frac{1}{2}Hx\right)\vec{j} \tag{15}$$

Here, the particle vector positron is

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} \tag{16}$$

In this two-dimensional case we have the following structure for the scheme dependent propagator

$$\tilde{G}((\vec{r},t);(\vec{r},t')) = e^{\frac{\nu}{2}(t-t')(a-b)}\tilde{G}((\vec{r},t);(\vec{r},t')) \tag{17}$$

with $G((\vec{r},t);(\vec{r},t'))$ satisfying now the Schroedinger time-dependent problem in view of our previous reslts Eq. (6)–Eq. (9).

$$\begin{aligned} & i\hbar\frac{\partial}{\partial t}G((\vec{r},t);(\vec{r},t')) = \\ & -\frac{\hbar^2}{2me^{\nu t}}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)G((\vec{r},t);(\vec{r},t')) + \\ & \frac{\hbar e H}{2icm}e^{\nu t}\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)G((\vec{r},t);(\vec{r},t')) \end{aligned} \tag{18}$$

with

$$\lim_{t \to t'} G((\vec{r}, t); (\vec{r}, t')) = \delta^{(2)}(\vec{r} - \vec{r}) \tag{19}$$

In order to solve exactly Eq. (10)-Eq. (20), we perform the following transformation to map the above written Green Functions in a free function ([3]). Namelly

$$\begin{aligned} x &= (\rho(\sigma)cos\theta(\sigma))u + (\rho(\sigma)sin\theta(\sigma))\nu \\ y &= -(\rho(\sigma)sin\theta(\sigma))u + (\rho(\sigma)cos\theta(\sigma))\nu \\ sigma &= f(t) - f(0) \end{aligned} \tag{20}$$

with

$$\begin{aligned} \frac{d\sigma}{dt} &= f'(t) = me^{\nu t}\rho^2(t) \\ \theta(t) &= \frac{1}{2}\left(\frac{eH}{mc}\right)t \\ \rho(t) &= e^{-\frac{\nu t}{2}}cos[\sqrt{\frac{1}{4}((\frac{eH}{mc})^2 - \nu^2)t}] \end{aligned} \tag{21}$$

and under the classical damped condition

$$\left(\frac{eH}{mc}\right)^2 > \nu^2 \tag{22}$$

The Green function Eq. (19)-Eq. (20) is, thus, given explicitly by

$$\begin{aligned} &G((\vec{r}, t); (\vec{r}, t')) = e^{iF(u(t,x,y),v(t,x,y);\sigma(t))}\frac{m}{2\pi i\hbar(\sigma(t) - \sigma(t'))} \times \\ &e^{\frac{im}{2\hbar(\sigma(t)-\sigma(t'))}[(u(t,x,y)-u(t',x',y'))^2+(v(t,x,y)-\nu(t',x',y'))^2]}e^{-iF(u(t',x',y'),v(t',x',y');\sigma(t'))} \end{aligned} \tag{23}$$

Note that in Eq. (24), we have used the fact that the Jacobian of the spatial coordinates Eq. (21) is the unity and the functions $u(t, x, y)$ and $v(t, x, y)$ are explicitly given by inverting Eq. (21) and Eq. (22). The complex phase function $F(u, v, \sigma)$ is explicitly given by

$$\begin{aligned} &F(u, v, \sigma) = \frac{1}{2}me^{vt} \times \\ &\{[\rho(t)sin(\theta(t)) + \frac{d}{dt}(\rho(t)sin(\theta(t))) + \rho(t)cos(\theta(t)) + \frac{d}{dt}\rho(t)cos(\theta(t))] \\ &times(u^2(t, x, y) + v^2(t, x, y) + iln\rho(t))\} \end{aligned} \tag{24}$$

For a general procedure to solve time-dependent harmonic oscilator in any dimension see appendix C.

Work on the introduction of non-linearity λx^4 in these phenomenological dissipative quantum systems to understand quantum chaos are in progress.

Acknowledgements

Luiz C.L. Botelho is thankfull to Dr. A. Nassar (UFPa) and Professor J. Bassallo (UFPa) for calling his attention to their work on similar quantization problems of quantum dissipative systems, Ref.[6]. One of us, (E.P.S.), is thankfull to the Physics Department of Pontificia Universidade Católica do Rio de Janeiro for the warm hospitality.

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Appendix A

The Feynman Path-Integral with a Pure Product Measure Leads to the Schroedinger Equation

In this appendix we give a simple argument that if one defines the Feynman path measure by the (formal) infinite product

$$\mathcal{D}^F[x(\sigma)] = \lim_{N\to\infty} \prod_{k=1}^{N} dx_k, \tag{A.1}$$

where

$$x - k = x(t' + \frac{t-t')}{N}k) \tag{A.2}$$

the Feynman propagator

$$G((\vec{r},t);(\vec{r},t')) = \int_{x(t')=x';x(t)=x} \mathcal{D}[x(\sigma)]e^{\frac{i}{\hbar}S[x(\sigma)]} \tag{A.3}$$

satisfies the Schroedinger equation

$$i\hbar\frac{\partial G((x,t);(x',t'))}{\partial t} = \left(-\frac{\hbar^2}{2mdx^2} + V(x)\right)G((x,t);(x',t')) \tag{A.4}$$

In order to show this in an elementary way, we have make a general trajectory end point variation of Eq. (A-2) $(x(t) =)$

$$\Delta_x G((x,t);(x',t')) = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial x}\Delta\dot{x} + \frac{\partial G}{\partial x}\Delta t = \frac{\partial G}{\partial x}\Delta t, \tag{A.5}$$

since the Feynman propagator does not depends on the end point velocity and Δx is zero as the x-points is fixed.

At this point we note that

$$\begin{aligned}
&\Delta_x\left(\int_{x(t')=y;x(t)=x} \mathcal{D}^F[x(\sigma)]e^{\frac{i}{\hbar}S[x(\sigma)]}\right) \\
&= \int_{x(t')=y;x(t)=x} \mathcal{D}^F[x(\sigma)](\frac{i}{\hbar}\Delta_x)e^{\frac{i}{\hbar}S[x(\sigma)]} \\
&= \frac{i}{\hbar}\int_{x(t')=y;x(t)=x} \mathcal{D}^F[x(\sigma)](H\Delta t)e^{\frac{i}{\hbar}S[x(\sigma)]}
\end{aligned} \tag{A.6}$$

As a consequence of the commutation relation

$$[P_x, x] = i\hbar \tag{A.7}$$

which holds true only for Feynman product measures of the form Eq. (A-1), we get the following result

$$i\hbar\frac{\partial G}{\partial t}((x,t);(x',t')) = H(i\hbar\frac{\partial}{\partial x}, x)G((x,t);x',t')) \tag{A.8}$$

Appendix B

In Ref. Phys. Rev. E56, 1230 (1997), A.B. Nassar et al has proposed the following *time non-local* Caldirola-Kanai action as a physical classical action which should be associated to their proposed quantum hydrodynamical propagator *in one-dimension*

$$\tilde{S}[x(\sigma)] = \int_0^t d\sigma e^{v(\sigma-t)} \left(\frac{1}{2} m \left(\frac{dx}{d\sigma} \right)^2 - V(x(\sigma)) \right). \tag{B.1}$$

One can, in principle, apply the same procedure of this paper to analyse the associated dissipative anomally factor. Unfortunately, Eq. (B.1) can be mapped in the following negative viscosity non-local Caldirola-Kanai action

$$\tilde{S}[x(\sigma)] = \int_0^t d\sigma e^{\nu\sigma} \left(\frac{1}{2} m \left(\frac{dx}{d\sigma} \right)^2 - V(x(t-\sigma)) \right). \tag{B.2}$$

which, by its turn, has the problem of a potential involving all derivatives of the path $x(\sigma)$, namelly,

$$V(x(t-\sigma)) = V\left(\sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{d^k x}{d\sigma} \right) |_{-\sigma} t^k \right) = V\left(x((-\sigma) + x'(-\sigma)t + \frac{x''(-\sigma)}{2} t^2 + \ldots \right). \tag{B.3}$$

This result leads to the fact the approximations involved in the Nassar et al work is meaningless as a quantization process.

Appendix C

The Exact Solubility of the Path-Integral of the General Harmonic Oscillator

In this paper we have faced the problem of determining the Feynman propagator of a general harmonic oscillator with time dependent variable mass and a time dependent frequency

$$i\hbar \frac{\partial G(x,x',t)}{\partial t} = \left[-\frac{1}{2m(t)dx^2} \frac{d^2}{dx^2} + \omega^2(t) x^2 \right] G(x,x',t) \tag{C.1}$$

with the initial condition

$$G(x,x',t) = \delta(x-x') \tag{C.2}$$

where we have considered the one-dimensional case solubility for simplicity.

In order to solve Eq. (C1) by pure Feynman path-integral techniques straightfowardly generalizable to the three-dimensional case, we consider first the following pure time transformation on Eq. (C1).

$$\begin{aligned} x &= x \\ \zeta &= f(t) = \int_0^t \frac{d\sigma}{m(\sigma)} \end{aligned} \tag{C.3}$$

The propagator equation takes, thus, the simple form below, where the time dependence of the mass is not present anymore.

$$i\hbar\frac{\partial G(x,x',t)}{\partial\zeta}=\left[-\frac{1}{2}\frac{d^2}{2dx^2}+\bar{\omega}^2(\zeta)x^2\right]G(x,x',\zeta) \tag{C.4}$$

The above propagator has a representation in terms of the *usual* Feynman path-integral (see appendix A of this paper), after considering the decomposition of the quantum trajectories in terms of classical and purely quantum paths entering into functional measure.

As a consequence, we should evaluate only the pure quantum path-integral

$$G(0,0,\zeta)=\int_{x^q(0)=0;x^q(\zeta)=0}\mathcal{D}^F[x^q(\beta)]e^{\frac{i}{\hbar}\int_0^\zeta d\beta[\frac{1}{2}(\dot{x}^q)^2(\beta)-\bar{\omega}^2(\beta)(x^q)(\beta)]}. \tag{C.5}$$

Let us, thus, consider the following change of variable in Eq. (C5)

$$\dot{x}^q(\beta)-\frac{\dot{k}(\beta)}{k(\beta)}x^q(\beta)=\dot{y}^q(\beta) \tag{C.6}$$

where $k(\beta)$ is a classical trajectory of the system with fixed end point zero velocities

$$\begin{aligned}&\frac{d^2}{d\beta^2}k(\beta)+\bar{\omega}(\beta)k(\beta)=0\\&\dot{k}(0)=\dot{k}(\zeta)=0.\end{aligned} \tag{C.7}$$

One can easily see that

$$\mathcal{D}^F[x^q(\beta)]=\mathcal{D}^F[y^q(\beta)]det\left[1-\left(\frac{d}{dt}\right)^{-1}\left(\frac{\dot{k}}{k}\right)\right]=\mathcal{D}^F[y^q(\beta)] \tag{C.8}$$

Since the above written functional determinant is expressed in terms of loops with propagators of the form $\left(\frac{d}{dt}\right)^{-1}(t,t')=\theta(t-t')$ and we have considered the prescription $\theta(0)=0$ to evaluate the above equation.

We have additionally the following path-integral new end points to the transformed path-integral

$$0=y^q(\zeta)\quad;\quad 0=y^q(0) \tag{C.9}$$

together with the compatibility condition (constraint) to be imposed on the Feynman path measure $\mathcal{D}^F[y^q(\beta)]$

$$\int_0^\zeta d\beta y^q(\beta)\left[\frac{\dot{k}(\beta)}{k(\beta)}\right]+\int_0^\zeta d\beta\frac{\dot{k}(\beta)}{k(\beta)}\int_0^\beta d\beta'\left(\frac{\dot{k}}{k^2}\right)(\beta')y^q(\beta')=0 \tag{C.10}$$

By grouping together Eq. (C6) and Eq. (C10), we are able to write the quantum prefactor of the harmonic oscillator into a straightforward Feynman path integral of a free particle in the presence of a external source

$$\tilde{G}(0,0,\zeta) = \int_{-\infty}^{+\infty} d\lambda$$

$$\left[\int_{y^q(0)=0=y^q(\zeta)} \mathcal{D}^F[y^q(\beta)] e^{i\int_0^\zeta d\beta \frac{1}{2}(\dot{y}^q(\beta))^2} e^{i\lambda[\int_0^\zeta d\beta y^q(\beta)\left(\frac{\dot{k}(\beta)}{k(\beta)}\right)}\right.$$

$$\left. + \int_0^\zeta \frac{\dot{k}(\beta)}{k(\beta)} \int_0^\beta d\beta' \left(\frac{\dot{k}}{k^2}\right)(\beta') y^q(\beta')]\right] \tag{C.11}$$

which, by its turn, may easily evaluated.

In the general 3D-case below,

$$\tilde{G}(0,0,\zeta) = \prod_{i=1}^{3} \int_{x^q(0)=0;x^q(\zeta)=0} \mathcal{D}^F[x_i^q(\beta)] e^{\frac{i}{\hbar}(\int_0^\zeta d\beta[\frac{1}{2}(\dot{x}_i^q)^2(\beta) - \bar{\omega}_{ij}^2(\beta)(\dot{x}_j^q)^2(\beta)])}, \tag{C.12}$$

the transformation, Eq. (C5), takes the 3-D form

$$(\dot{x}_i^q(\dot{k}k^{-1})_{i\ell} x_\ell^q(\beta) = \dot{y}_i^q(\beta). \tag{C.13}$$

with

$$\frac{d^2}{d\beta^2} k_{ij}(\beta) = \bar{\omega}_{i\ell}(\beta) k_{\ell i}(\beta). \tag{C.14}$$

with

$$\dot{k}_{ij}(0) = \dot{k}_{ij}(0) \tag{C.15}$$

and, thus, leading to analogous exact solubility of the resulting 3-D path-integral.

Note that the above expused procedure is an alternative to the formal procedure of ref.[3] which, by its turn is heavily based on the existence of global solutions of ordinary differential equations with variable coefficients.