

Integrable Multiparametric $SU(N)$ Chain

Angela Foerster^(1,a), Itzhak Roditi^(2,b) and Ligia M.C.S. Rodrigues^(2,c)

⁽¹⁾Instituto de Física da UFRGS
Av. Bento Gonçalves, 9500
91501-970 – Porto Alegre, RS – Brazil

⁽²⁾Centro Brasileiro de Pesquisas Físicas - CBPF
Rua Dr. Xavier Sigaud, 150
22290-180 – Rio de Janeiro, RJ – Brazil

ABSTRACT

We analyse integrable models associated to a multiparametric $SU(N)$ R -matrix. We show that the Hamiltonians describe $SU(N)$ chains with twisted boundary conditions and that the underlying algebraic structure is the multiparametric deformation of $SU(N)$ enlarged by the introduction of a central element.

Key-words: Integrable models; Quantum inverse scattering method; Quantum groups.

e-mail address:

^(a)ANGELA@IF1.UFRGS.BR

^(b)RODITI@CBPFSU1.CAT.CBPF.BR

^(c)LIGIA@CBPFSU1.CAT.CBPF.BR

The Quantum Inverse Scattering Method [1] provides a unified framework for the exact solution of classical and quantum models and led naturally to the new mathematical concept of quantum groups [2]. These were formulated as one-parameter deformation of a universal enveloping algebra. Multiparametric deformations have also been developed [3] and are currently attracting much attention: consistent multiparametric generalizations of the R -matrix and the corresponding quantum groups were discussed by several authors [4-11].

In this letter we obtain the Hamiltonian for an $SU(N)$ chain with twisted boundary conditions and find that the underlying algebraic structure is the multiparametric deformation of $SU(N)$ enlarged by the introduction of a central element.

We start with the following multiparametric generalization of the $SU(N)$ R -matrix, first introduced by Perk and Schultz [12],

$$R(x, q, \{p\}) = a(x, q) \sum_{\alpha}^N e^{\alpha\alpha} \otimes e^{\alpha\alpha} + b(x) \sum_{\alpha \neq \beta}^N p_{\alpha\beta} e^{\alpha\alpha} \otimes e^{\beta\beta} \quad (1)$$

$$+ c_{-}(x, q) \sum_{\alpha < \beta}^N e^{\alpha\beta} \otimes e^{\beta\alpha} + c_{+}(x, q) \sum_{\alpha > \beta}^N e^{\alpha\beta} \otimes e^{\beta\alpha} ,$$

where x is the spectral parameter and

$$a(x, q) = xq - \frac{1}{xq} , \quad b(x) = x - \frac{1}{x} , \quad c_{-}(x, q) = \frac{1}{x} (q - \frac{1}{q}) , \quad c_{+}(x, q) = x(q - \frac{1}{q}) . \quad (2)$$

q and $p_{\alpha\beta}$ are $1 + \frac{N(N-1)}{2}$ independent parameters with $p_{\alpha\beta}, \alpha, \beta = 1, \dots, N (\alpha < \beta)$, $p_{\beta\alpha} = (p_{\alpha\beta})^{-1}$. The $N \times N$ matrices $e^{\alpha\beta}$ have elements $(e^{\alpha\beta})^{\gamma\rho} = \delta^{\alpha\gamma} \delta^{\beta\rho}$.

It is easy to check that the R -matrix (1) satisfies the Yang-Baxter equation

$$R_{12}(x_{12}, q, \{p\}) R_{13}(x_{13}, q, \{p\}) R_{23}(x_{23}, q, \{p\}) = R_{23}(x_{23}, q, \{p\}) R_{13}(x_{13}, q, \{p\}) R_{12}(x_{12}, q, \{p\}) . \quad (3)$$

Notice that both sides of the equation above act on the tensor product of three N -dimensional auxiliary spaces $C_1^N \otimes C_2^N \otimes C_3^N$, whereas the matrix $R_{ik}(x_{ik}, q, \{p\})$ acts on $C_i^N \otimes C_k^N$.

The Lax operator associated to (1) is

$$L(x, q, \{p\}) = x \sum_{\alpha}^N q^{W_{\alpha}} \prod_{\beta \neq \alpha} (p_{\alpha\beta})^{W_{\beta}} e^{\alpha\alpha} - \frac{1}{x} \sum_{\alpha}^N q^{-W_{\alpha}} \prod_{\beta \neq \alpha} (p_{\alpha\beta})^{W_{\beta}} e^{\alpha\alpha} +$$

$$+ c_{+}(x, q) \sum_{\alpha > \beta} X_{\beta\alpha}^{+} e^{\alpha\beta} + c_{-}(x, q) \sum_{\alpha < \beta} X_{\beta\alpha}^{-} e^{\alpha\beta} , \quad (4)$$

where in the fundamental representation $W_{\alpha} = e^{\alpha\alpha}$ and $X_{\alpha\beta} = e^{\alpha\beta} (\alpha \neq \beta)$; $X_{\alpha\beta}^{+} (X_{\alpha\beta}^{-})$ has non-zero elements above (below) the diagonal. These W_{α} can be written as combinations

of the matrices H , representing the Cartan sub-algebra, and the identity matrix I ,

$$W_\alpha = \frac{1}{N} \sum_{\rho=1}^N w_{\alpha\rho} H_{\rho-1} \quad , \quad H_0 \equiv I \quad , \quad (5)$$

$$w_{\alpha\rho} = \begin{cases} 1 & \text{if } \rho = 1 \\ 1 - \rho & \text{if } \rho \leq \alpha \\ N + 1 - \rho & \text{if } \rho > \alpha \end{cases} \quad (6)$$

such that $H_\alpha = W_\alpha - W_{\alpha+1}$ and $I = \sum_{\alpha=1}^N W_\alpha$. The R matrix (1) and the Lax operator (4) obey the Fundamental Commutation Relation (FCR) [1],

$$R_{12}(x_{12}, q, \{p\}) L_n^1(x_1, q, \{p\}) L_n^2(x_2, q, \{p\}) = L_n^2(x_2, q, \{p\}) L_n^1(x_1, q, \{p\}) R_{12}(x_{12}, q, \{p\}) \quad . \quad (7)$$

Here the L -matrix is defined on the tensor product of the N -dimensional auxiliary space and the local quantum space $C_1^N \otimes C_2^d$, with d the dimension of the representation of the associated algebra satisfied by the operators in the entries of the matrix L .

Following the general procedure of Faddeev et al [1], we introduce the monodromy matrix

$$T(x, q, \{p\}) = L_{n_0}(x, q, \{p\}) \cdots L_2(x, q, \{p\}) L_1(x, q, \{p\}) \quad , \quad (8)$$

acting on the tensor product of an auxiliary (horizontal) space C^N and a quantum (vertical) space $\Omega = C_1^N \otimes \cdots \otimes C_{n_0}^N$.

The transfer matrix is defined as the trace of the monodromy matrix (8) in the auxiliary space:

$$\tau(x, q, \{p\}) = \text{tr} T(x, q, \{p\}) = \sum_{\alpha=1}^N T_\alpha^\alpha(x, q, \{p\}) \quad . \quad (9)$$

The Yang-Baxter equations (3) imply the commutativity of the transfer matrix for different spectral parameters,

$$[\tau(x, q, \{p\}), \tau(y, q, \{p\})] = 0 \quad , \quad (10)$$

which reflects the integrability of the model. In fact, the eigenvalue problem for a N -state integrable model is exactly solved by the Algebraic Nested Bethe Ansatz method [13]. This procedure is carried out in $(N - 1)$ steps and the Bethe Ansatz equations for the level “ l ” ($l = 1, \dots, N - 1$) are given by

$$\prod_{r \neq l+1}^N p_{r,l+1}^{N_r} \prod_{s \neq l}^N p_{l,s}^{N_s} \prod_{i=1}^{n_{l-1}} \frac{a(x_k^{(l)}/x_i^{(l-1)})}{b(x_k^{(l)}/x_i^{(l-1)})} \prod_{i'=1}^{n_l} \left(\frac{a(x_{i'}^{(l)}/x_k^{(l)})}{b(x_{i'}^{(l)}/x_k^{(l)})} \frac{b(x_k^{(l)}/x_{i'}^{(l)})}{a(x_k^{(l)}/x_{i'}^{(l)})} \right) \prod_{i''=1}^{n_{l+1}} \frac{b(x_{i''}^{(l+1)}/x_k^{(l)})}{a(x_{i''}^{(l+1)}/x_k^{(l)})} = -1 \quad , \quad (11)$$

$$k = 1, \dots, n_l \quad .$$

Here $n_N = 0, x^{(0)} = 0, N_i = n_{i-1} - n_i$ and $\prod_{i=1}^0$ is assumed to be one. Therefore, the eigenvalue problem of the transfer matrix (9) is reduced to a system of coupled algebraic

equations for the Bethe Ansatz parameters $x_k^{(l)}$, ($l = 1, \dots, N-1$; $k = 1, \dots, n_l$). We observe that each parameter $x^{(l)}$ couples just with its neighbour-level parameter $x^{(l\pm 1)}$ (except $x^{(1)}(x^{(N-1)})$, which couples only with $x^{(2)}(x^{(N-2)})$).

From the transfer matrix (9), we get a multiparametric version of a quantum Hamiltonian for a n_0 length $SU(N)$ chain through

$$\mathcal{H} \propto \frac{\partial}{\partial x} \ln(\tau) |_{x=1} . \quad (12)$$

This yields:

$$\mathcal{H} = \sum_{i=1}^{n_0} h_{i,i+1} \quad (n_0 + 1 \equiv 1) , \quad (13)$$

where

$$h_{i,i+1} = \frac{q + q^{-1}}{2} - \frac{q + q^{-1}}{2} \sum_{\alpha=1}^N e_i^{\alpha\alpha} e_{i+1}^{\alpha\alpha} - \frac{q - q^{-1}}{2} \sum_{\alpha \neq \beta}^N \text{sign}(\alpha - \beta) e_i^{\alpha\alpha} e_{i+1}^{\beta\beta} - \sum_{\alpha \neq \beta}^N p_{\alpha\beta} e_i^{\alpha\beta} e_{i+1}^{\beta\alpha} . \quad (14)$$

Let us now show that the general Hamiltonian (13) describes a $SU(N)$ chain with twisted periodic boundary conditions. For that sake we perform the similarity transformations generated by

$$U = e^{-\frac{1}{2} \sum_{i=1}^{n_0} (i-1) \sum_{\alpha=1}^{N-1} \xi^\alpha H_\alpha^i} . \quad (15)$$

The coefficients ξ^α are fixed when we impose the conditions

$$p_{\alpha\beta} U e_i^{\alpha\beta} e_{i+1}^{\beta\alpha} U^{-1} = e_i^{\alpha\beta} e_{i+1}^{\beta\alpha} ; \quad \alpha = 1, \dots, N-1 , \quad (16)$$

under which (13) particularizes to the Hamiltonian for the $SU(N)$ spin chain with twisted periodic boundary conditions

$$\begin{aligned} \mathcal{H} &= \sum_{i=1}^{n_0-1} \left\{ \frac{q + q^{-1}}{2} - \frac{q + q^{-1}}{2} \sum_{\alpha=1}^N e_i^{\alpha\alpha} e_{i+1}^{\alpha\alpha} - \frac{q - q^{-1}}{2} \sum_{\alpha \neq \beta}^N \text{sign}(\alpha - \beta) e_i^{\alpha\alpha} e_{i+1}^{\beta\beta} \right\} + \\ &- \sum_{i=1}^{n_0-1} \sum_{\alpha \neq \beta}^N e_i^{\alpha\beta} e_{i+1}^{\beta\alpha} - \sum_{\beta \neq \alpha}^N (p_{\alpha\beta}^{n_0} e_{n_0}^{\alpha\beta} e_1^{\beta\alpha} + p_{\alpha\beta}^{-n_0} e_1^{\alpha\beta} e_{n_0}^{\beta\alpha}) \end{aligned} \quad (17)$$

The case $N = 2$ was previously discussed in the literature [14,15].

The multiparametric Hamiltonian (13) is not quantum group invariant. Nevertheless, the quantum group structure also appears in this context. The underlying algebraic structure is obtained directly from the FCR (7) noting that now W_i and $X_{\alpha\beta}$ are considered as abstracts elements of the algebra. Besides, the W_α are now combinations of a central element Z and operators \hat{H}_α ,

$$W_\alpha = \frac{1}{N} \sum_{\alpha=1}^N w_{\alpha\rho} \hat{H}_{\rho-1} , \quad \hat{H}_0 \equiv Z , \quad (18)$$

that respectively coincide with I and H_α in the fundamental representation and the coefficients $w_{\alpha\rho}$ are the same as given in relation (6). The commutation relations are then

$$\begin{aligned}
[Z, \hat{H}_\alpha] &= [\hat{H}_\alpha, \hat{H}_\beta] = 0 \\
[Z, X_{\alpha\beta}^\pm] &= 0, \\
[\hat{H}_\gamma, X_{\alpha\beta}^\pm] &= a_{\alpha\beta}(\hat{H}_\gamma) X_{\alpha\beta}^\pm, \\
p_{\beta\alpha} X_{\beta\alpha}^+ X_{\alpha\beta}^- - p_{\alpha\beta} X_{\alpha\beta}^- X_{\beta\alpha}^+ &= \\
\Omega^{-1} [q^{W_\beta - W_\alpha} - q^{-W_\beta + W_\alpha}] \prod_{\rho \neq \beta} (p_{\beta\alpha})^{W_\rho} \prod_{\delta \neq \alpha} (p_{\alpha\delta})^{W_\delta}, \\
p_{\alpha\beta} X_{\alpha\beta}^\pm X_{\beta\rho}^\pm - p_{\beta\rho} X_{\beta\rho}^\pm X_{\alpha\beta}^\pm &= q^{\pm W_\beta} \prod_{\delta \neq \beta} (p_{\beta\delta})^{W_\delta} X_{\alpha\rho}^+, \\
p_{\alpha\beta} X_{\alpha\beta}^+ X_{\beta\rho}^- - p_{\beta\rho} X_{\beta\rho}^- X_{\alpha\beta}^+ &= q^{W_\beta} \prod_{\delta \neq \beta} (p_{\beta\delta})^{W_\delta} X_{\alpha\rho}^-, \alpha < \rho \\
p_{\alpha\beta} X_{\alpha\beta}^+ X_{\beta\rho}^- - p_{\beta\rho} X_{\beta\rho}^- X_{\alpha\beta}^+ &= q^{-W_\alpha} \prod_{\delta \neq \beta} (p_{\beta\delta})^{W_\delta} X_{\alpha\rho}^+, \alpha > \rho \\
X_{\beta\alpha}^\epsilon X_{\beta\rho}^\epsilon &= q^{\text{sign}(\alpha-\rho)} p_{\alpha\rho} X_{\beta\rho}^\epsilon X_{\beta\alpha}^\epsilon, \epsilon = \pm, \\
X_{\alpha\beta}^- X_{\alpha\rho}^+ &= q^{-1} p_{\beta\rho} X_{\alpha\rho}^+ X_{\alpha\beta}^-, \beta > \rho, \\
X_{\alpha\beta}^+ X_{\alpha\rho}^- &= q p_{\beta\rho} X_{\alpha\rho}^- X_{\alpha\beta}^+, \beta < \rho, \\
X_{\alpha\beta}^\epsilon X_{\rho\beta}^\epsilon &= q^{\text{sign}(\alpha-\rho)} p_{\alpha\rho} X_{\alpha\rho}^\epsilon X_{\alpha\beta}^\epsilon, \epsilon = \pm, \\
X_{\alpha\beta}^- X_{\alpha\beta}^+ &= q p_{\alpha\rho} X_{\alpha\beta}^+ X_{\rho\beta}^-, \alpha < \rho, \\
X_{\alpha\beta}^+ X_{\alpha\beta}^- &= q^{-1} p_{\alpha\rho} X_{\alpha\beta}^- X_{\rho\beta}^+, \beta > \rho;
\end{aligned} \tag{19}$$

Finally,

$$p_{\rho\alpha} X_{\alpha\beta}^\pm X_{\rho\lambda}^\pm - p_{\lambda\beta} X_{\rho\lambda}^\pm X_{\alpha\beta}^\pm = -\Omega X_{\rho\beta}^\pm X_{\alpha\lambda}^\pm, \tag{20}$$

where $\Omega = q - q^{-1}$, and $\beta > \lambda > \alpha > \rho$ and all their cyclic permutations and

$$p_{\rho\alpha} X_{\alpha\beta}^\pm X_{\rho\lambda}^\pm - p_{\lambda\beta} X_{\rho\lambda}^\pm X_{\alpha\beta}^\pm = \Omega X_{\rho\beta}^\pm X_{\alpha\lambda}^\pm, \tag{21}$$

with $\beta < \lambda < \alpha < \rho$ and all their cyclic permutations; otherwise, (for example, $\beta > \lambda < \alpha < \rho$), we have

$$p_{\rho\alpha} X_{\alpha\beta}^\pm X_{\rho\lambda}^\pm - p_{\lambda\beta} X_{\rho\lambda}^\pm X_{\alpha\beta}^\pm = 0. \tag{22}$$

The coefficients $a_{\alpha\beta}(\hat{H}_\gamma)$ are obtained by solving

$$W_\rho X_{\alpha\beta}^\epsilon = X_{\alpha\beta}^\epsilon (W_\rho + \delta), \tag{23}$$

where

$$\delta = \begin{cases} 1 & , \quad \rho = \alpha \\ -1 & , \quad \rho = \beta \\ 0 & , \quad \text{otherwise,} \end{cases} \tag{24}$$

remembering that $\hat{H}_\alpha = W_\alpha - W_{\alpha+1}$.

The coproduct is obtained by considering the product of two L 's acting on two internal spaces; we find

$$\Delta Z = Z \otimes \mathbf{1} + \mathbf{1} \otimes Z \quad (25)$$

$$\Delta \hat{H}_\alpha = \hat{H}_\alpha \otimes \mathbf{1} + \mathbf{1} \otimes \hat{H}_\alpha \quad (26)$$

$$\begin{aligned} (\Delta X_{\nu\mu}^+) &= q^{W_\mu} \prod_{\beta \neq \mu} p_{\mu\beta}^{W_\beta} \otimes X_{\nu\mu}^+ + X_{\nu\mu}^+ \otimes q^{W_\nu} \prod_{\beta \neq \nu} p_{\nu\beta}^{W_\beta} \\ &\quad + \Omega \sum_{\beta < \mu} X_{\beta\mu}^+ \otimes X_{\nu\beta}^+ \end{aligned} \quad (27)$$

$$\begin{aligned} (\Delta X_{\nu\mu}^-) &= q^{-W_\mu} \prod_{\beta \neq \mu} p_{\mu\beta}^{-W_\beta} \otimes X_{\nu\mu}^- + X_{\nu\mu}^- \otimes q^{-W_\nu} \prod_{\beta \neq \nu} p_{\nu\beta}^{-W_\beta} \\ &\quad + \Omega \sum_{\beta > \mu} X_{\beta\mu}^- \otimes X_{\nu\beta}^- \end{aligned} \quad (28)$$

Thus, the introduction of a central element has allowed us to construct a coherent coproduct which makes appear the underlying algebraic structure of the $SU(N)$ chain with twisted boundary conditions.

Acknowledgements

Angela Foerster thanks CBPF for its kind hospitality and CNPq for financial support. Ligia M.C.S. Rodrigues thanks S. Sciuto and M.R-Monteiro for useful discussions and also thanks M.R-Monteiro for constant encouragement.

References

- [1] N.Yu. Reshetikhin, L.A. Takhtajan and L.D. Faddeev, Algebra and Analysis I, 178 (1989); L.D. Faddeev, “New Problems, Methods and Techniques in Quantum Field Theory and Statistical Mechanics” ed M. Rasetti (World Scientific Publ.), (1990); E. Sklyanin, L. Takhtajan and L. Faddeev, Leningrad Math. J. **1**, 193 (1979); Teor. Matem. Fiz. **40**, 194 (1979).
- [2] M. Jimbo, Lett. Math. Phys. **10**, 63 (1985); V.G. Drinfeld, Sov. Math. Dokl. **32**, 254 (1986).
- [3] N. Reshetikhin, Lett. Math. Phys. **20**, 331 (1990).
- [4] A. Sudbery, J. Phys. A: Math. Gen. **23**, L697 (1990).
- [5] M. Takeuchi, Proc. Jap. Acad. **66**, 112 (1990).
- [6] C. Burdick and L. Hlavatý, J. Phys. A: Math. Gen. **24**, L165 (1991).
- [7] J. Schirrmacher, J. Wess and B. Zumino, Z. Phys. C – Particles and Fields **49**, 317 (1991); J. Schirrmacher, Z. Phys. C – Particles and Fields **50**, 321 (1991).
- [8] B. Basu Mallick and A. Kundu, J. Phys. A: Math. Gen. **25**, 4147 (1992).
- [9] A. Aghamohammadi, V. Karimipour and S. Rouhani, J. Phys. A: Math. Gen. **26**, L75 (1993).
- [10] M.R-Monteiro, I. Roditi, L.M.C.S. Rodrigues and S. Sciuto, Mod. Phys. Lett. A **10**, 419 (1995); Phys. Lett. **B354**, 389 (1995).
- [11] J. Abad and M. Rios, J. Phys. A: Math. Gen. **28**, 3319 (1995).
- [12] J.H.H. Perk and C.L. Schultz, Phys. Lett. **A84**, 407 (1981); C.L. Schultz, Physica **A112**, 71 (1983).
- [13] O. Babelon, H.J. De Vega and C.M. Viallet, Nucl. Phys. **B200**, 266 (1982).
- [14] F.C. Alcaraz, M.N. Barber and M.T. Batchelor, Annals of Phys. **182** 280, (1988).
- [15] V. Pasquier and H. Saleur, Nucl. Phys. B **316**, 523 (1990).