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EINSTEIN'S THEORY OF GRAVITY IN FIERZ VARIABLES

by

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ABSTRACT

We present a complete theory of massive and massless spin-two field using Fierz variables $A_{\alpha\beta\mu}$; its relationship with the standard variable $\psi_{\mu\nu}$ and the bridge relation which allows the passage from one representation to the other. We develop the Hamiltonian formalism for both the massive and the massless cases.

The theory thus presented is nothing but Einstein's General Relativity in the Jordan-Lichnerowicz formulation. We then discuss an unification program in which a new short-range force is postulated. This force becomes the local counterpart of gravity in the same way we can think that weak interaction is the short-range counterpart of electrodynamics.

Key-words: Einstein's theory; Gravity; Fierz variables.

I - INTRODUCTION

The fundamental variable in Einstein's theory of General Relativity (GR) is the symmetric metric tensor $g_{\mu\nu}(x)$ of a four-dimensional Riemannian manifold. This can be thought of as the natural extension [endowed with a geometric interpretation] of the symmetric tensor $\psi_{\mu\nu}$ which is the most fashionable way to describe a spin two field.

In the realm of field theory, however, the description of a given field is not unique. The behavior of the distribution of the energy content of the continuum mediating a given interaction throughout the space time can be assigned in various alternative or equivalent manners. This depends on the choice of the fundamental variables one uses to describe the process. In the specific case we will analyse in this paper (which is restricted to spin two fields - massive and massless) this choice can be twofold. Either one uses the so called standard variables $\psi_{\mu\nu}$ or the Fierz variables $A_{\mu\nu\lambda}$.

Although such result is a very old one [1,2], in the last decades it has been almost forgotten - and the use of Fierz variables to describe spin two field seems to have lost its attractiveness. Our purpose here is to try to revert this situation and to change this unbalanced and arbitrary choice. However one should ask about the leitmotiv or the usefulness of a procedure which enlarge the number of fundamental variables. The same question, of course, could be adressed to Einstein's description. Indeed, why take 10 variables $g_{\mu\nu}(x)$ to describe gravity if we know that this interaction has only two (2) degrees of freedom (per each space-time point) ? The answer to both of these questions is the same: ... "Nature is not economical of structures: only of principles of fundamental applicability" - A Einstein quoted by A. Salam [3]. The new variable exhibits its formal qualities in an enlarged vision of the characterization of the symmetries of the theory, its conservation laws, and its relationship with others fields. Besides, we think that the main difficulties which physicists have faced in the last decades on the road of the unification of gravity with other forces are intimately related to the choice of fundamental variables. We will see that the passage to the new variables provides new insights into Einstein's general relativity, which can then led us to a more deep understanding of the theory and even to allow the production of an unification scheme with others interactions (see Chapter IV).

The plan of the paper is the following:

In the second chapter we review the eq. of motion of massive spin two field in Minkowskii space-time in the standard variable ($\psi_{\mu\nu}$) and in the Fierz variable ($A_{\mu\nu\lambda}$), set the scheme of equivalence and compare both representation.

Then we present the Lagrangian formalism for the free field, the interacting scheme and the massless limit which projects us into Einstein's road of geometrization. We exhibit the gauge freedom of the theory, its energy content and finally we present a complete description of the Hamiltonian formalism. We examine the constraints of the theory, show that they are all second class and led to the exhibition that the theory has only five degrees of freedom - as it should be for a massive spin two field. We then turn to the massless case in which some first class constraints appears which indicates the existence of gauge freedom.

In Section III we change to general curved space-time and present Jordan-Lichnerowicz representation of General Relativity. We exhibit the passage from Fierz variables $A_{\mu\nu\lambda}$ to Lanczos potential $L_{\mu\nu\lambda}$ - and show that this potential $L_{\mu\nu\lambda}$ (which contains all information of Weyl conformal tensor) is nothing but the form Fierz variables assume when passing to curved space time in the massless limit.

We then proceed to present in section IV some new examples of the aplications of Fierz variables beyond General Relativity. The main point here is the use of the electro-weak SU(2)xU(1) unification scheme of Salam-Glashow-Weinberg as a paradigm to supply a theory of a new short-range force in terms of Fierz variables. This force becomes the local counterpart of gravity in the same way we can think that weak interaction is the short-range counterpart of electrodynamics.

We conclude in Section V with a resumé of our results and some perspectives for the future of the theory.

II - THE EQUATION OF MOTION OF SPIN-TWO FIELD IN MINKOWSKI SPACE-TIME

The Standard Variable

In the standard theory the field of spin two is described by a symmetric tensor function $\psi_{\mu\nu}$ which obeys the equation of motion:

$$\text{(1)} \qquad \Box\, \psi_{\mu\nu} - \psi_{(\mu}{}^{\varepsilon}{}_{,\nu),\varepsilon} + \psi_{,\mu,\nu} + \eta_{\mu\nu}\,(\psi^{\alpha\beta}{}_{,\alpha,\beta} - \Box\,\psi) + m^2\,(\psi_{\mu\nu} - \psi\eta_{\mu\nu}) = 0$$

in which $\psi \equiv \psi^{\mu}{}_{\mu}$. In this work we mean the parenthesis ( ) to denote symmetrization, that is:

$$\psi_{(\mu\nu)} = \psi_{\mu\nu} + \psi_{\nu\mu}.$$

Taking the trace and the divergence of (1) we obtain two compatibility conditions:

$$\text{(2a)} \qquad \psi = 0$$

$$\text{(2b)} \qquad \psi^{\mu\nu}{}_{,\nu} = 0$$

which guarantees then that $\psi_{\mu\nu}$ has only 10-1-4 = 5 degrees of freedom, as it should be in order to describe a spin-two field.

Since the early days of field theory physicists have employed this scheme which in turn has acquired a status of uniqueness. However, this is not the case, as we shall see soon. The theory we will present is based in part in the work of Fierz in the thirties, when it was already known that it is possible to describe a spin-two field (and indeed any field of higher order spin) by equivalent representations. Let us turn to a specific example for this alternative scheme.

The Fierz Variable

We define a third order tensor $A_{\mu\nu\lambda}$ which has twenty (20) independent components once it obey the properties of being anti-symmetric in the first two indices and has no pseudo-trace:

(3a) $$A_{\mu\nu\lambda} = - A_{\nu\mu\lambda}$$

(3b) $$\overset{*}{A}{}^{\alpha\beta}{}_{\beta} = 0$$

or, equivalently,

$$A_{\alpha\beta\mu} + A_{\beta\mu\alpha} + A_{\mu\alpha\beta} = 0 \quad .$$

The star $*$ operator represents the dual, constructed by means of the Levi-Civita completely anti-symmetric symbol $\varepsilon_{\alpha\beta\mu\nu}$ through the expression

(4) $$\overset{*}{A}_{\alpha\beta\mu} = \frac{1}{2}\, \varepsilon_{\alpha\beta}{}^{\rho\sigma} A_{\rho\sigma\mu}$$

Such tensor $A_{\mu\nu\lambda}$ can be used to describe a spin-two field, as we will now show.

From the potential $A_{\alpha\beta\mu}$ we construct the field $C_{\alpha\beta\mu\nu}$ through first order derivatives by the formula

$$C_{\alpha\beta\mu\nu} = A_{\alpha\beta[\mu,\nu]} + A_{\mu\nu[\alpha,\beta]} + \tag{5}$$
$$+ \frac{1}{2} A_{(\alpha\nu)} \eta_{\beta\mu} + \frac{1}{2} A_{(\beta\mu)} \eta_{\alpha\nu} -$$
$$- \frac{1}{2} A_{(\beta\nu)} \eta_{\alpha\mu} - \frac{1}{2} A_{(\alpha\mu)} \eta_{\beta\nu} +$$
$$+ \frac{2}{3} A^{\sigma\lambda}{}_{\sigma,\lambda} (\eta_{\alpha\mu}\eta_{\beta\nu} - \eta_{\alpha\nu}\eta_{\beta\mu})$$

in which $\eta_{\mu\nu}$ denotes the Minkowskii metric, $\eta_{\mu\nu} = \text{diag}(+1,-1,-1,-1)$. The symbol [ ] means anti-symmetrisation

$$f_{[\mu\nu]} \equiv f_{\mu\nu} - f_{\nu\mu} \quad .$$

The tensor $A_{\mu\nu}$ is given by

$$A_{\mu\nu} = A_{\mu}{}^{\varepsilon}{}_{\nu,\varepsilon} - A_{\mu}{}^{\varepsilon}{}_{\varepsilon,\nu} \quad .$$

This tensor $C_{\alpha\beta\mu\nu}$ is a new object. We have introduced it here because it is a good tool to generate a scheme which describes a spin two field in a very similar way as

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} \quad .$$

does for spin one field. Such tensor $C_{\alpha\beta\mu\nu}$ has only 10 independent components. Indeed, from definition (5) it has the symmetries

$$C_{\alpha\beta\mu\nu} = -C_{\beta\alpha\mu\nu}$$
$$C_{\alpha\beta\mu\nu} = -C_{\alpha\beta\nu\mu} \tag{6}$$
$$C_{\alpha\beta\mu\nu} = C_{\mu\nu\alpha\beta}$$

and besides, it has no trace:

$$C_{\alpha\beta\mu\nu}\eta^{\alpha\mu} = 0 \quad .$$

The dynamics of the field is given by the equation

(7) $$C^{\alpha\beta\mu\nu}{}_{,\nu} + m^2 A^{\alpha\beta\mu} = 0 \quad .$$

From eq. (7) it follows two compatibility conditions: (i) $A_{\alpha\beta\mu}$ is trace-free; and (ii) it has no divergence (in the last index):

(8a) $$A^{\alpha\beta\mu}\eta_{\beta\mu} = 0$$

(8b) $$A^{\alpha\beta\mu}{}_{,\mu} = 0$$

Thus from the previous 20 components we are left only with 20-4-6 = 10 degrees of freedom. In order to eliminate the excedent 5 degrees we follow Fierz by imposing

(9) $$\overset{*}{A}{}^{\alpha\beta\mu}{}_{,\beta} = 0$$

or, equivalently,

(9)' $$A_{\alpha\beta}{}^{\mu}{}_{,\lambda} + A_{\beta\lambda}{}^{\mu}{}_{,\alpha} + A_{\lambda\alpha}{}^{\mu}{}_{,\beta} = 0 \quad .$$

Note that (8b) and (9) are not independent. Indeed, we can set $\mu = \lambda$ in expression (9)' to obtain expression (8b). A direct counting then shows that there remains only 5 degrees of freedom (see later on the Hamiltonian analysis which provides the proof of this result). Indeed, this corresponds to the separation of condition (9) into its irreducible parts:

$$A^{\overset{*}{\alpha}\beta\mu}{}_{;\beta} = -\frac{1}{2} A^{\overset{*}{\beta}[\alpha\mu]} - \frac{1}{2} A^{\overset{*}{\beta}(\alpha\mu)}$$

The antisymmetric part corresponds to the 6 conditions (8b). The symmetric part to 5 conditions. In order to prove this last statement, define the symmetric tensor $M^{\alpha\mu} \equiv A^{\overset{*}{\beta}(\alpha\mu)}{}_{;\beta}$. This tensor has 10 independent components due to the symmetry. It is traceless which means 10-1 = 9 quantities left. Finally $M^{\alpha\mu}{}_{,\mu} = 0$ which impose 4 more conditions leaving 9-4 = 5 as we claimed. Using equations (5), (7), (8) and (9) one arrives at the equivalent wave equation

$$(10) \qquad (\Box + m^2)\, A_{\alpha\beta\mu} = 0 \quad .$$

The Equivalence

The Fierz (1939) result claims that the above two representations $\psi_{\mu\nu}$ and $A_{\mu\nu\lambda}$ have the same rights to describe a spin-two field. They imply the same theory and can be shown to be completely equivalent. There is no best way to prove this than to exhibit the formula which allows the passage from one representation to the other and vice-versa. We claim that the bridge formuli are given by

$$(11a) \qquad A_{\mu\varepsilon\nu} = \psi_{\nu[\mu,\varepsilon]} + B\psi_{,[\mu}\eta_{\varepsilon]\nu} -$$

$$- B\psi_{[\mu}{}^{\alpha}{}_{,\alpha}\eta_{\varepsilon]\nu}$$

and conversely,

(11b) $$\psi_{\mu\nu} = - \frac{1}{2m^2} A_{(\mu}{}^{\varepsilon}{}_{\nu),\varepsilon} + \frac{Q}{2m^2} A_{(\mu}{}^{\varepsilon}{}_{\varepsilon,\nu)} - \frac{1}{3m^2} A^{\alpha\beta}{}_{\beta,\alpha}\, \eta_{\mu\nu}$$

in which

$$Q \equiv \frac{1-B}{1-3B}$$

and B is an arbitrary constant.

Inserting expression (11a) into (11b) yields precisely equation (1) for the compatibility; and using (11b) into (11a) gives the equation of evolution (10), exhibiting the coherence of the system.

We can even go one step further and propose to consider (11a,b) as the basic expressions of the theory which contains, as we saw above, the evolutionary equations of $\psi_{\mu\nu}$ (eq. (1)) and $A_{\mu\nu\lambda}$ (eq. (10)) as compatibility requirements.

Remark that constraints (2a,b) and (8a,b) of each representation are not independent but are consequences of each other through the bridge relations (11a,b). Let us point out one more remark. Inserting the value of $A_{\mu\nu\lambda}$ in terms of $\psi_{\mu\nu}$ as given by (11a) into the definition relation (5), implies the presence of higher order derivative terms in the equation of motion (7). This, of course, is a direct consequence of the bridge formuli. There is no effect in the causal properties of the theory. The Cauchy problem remains well posed, as we will see later on. If we remain in one representation - which one, does not matter - we have to deal only with well behaved second order derivative equations. However, if we use one set of equations (say, eq. (7))

and apply the bridge formula to pass to the complementar set of variables, then, of course, we jump into a theory which contains necessarily higher order derivatives. this we will call the Jordan-Lichnerowicz (JL) approach (see section III).

Thus if one does not want to be involved with a higher-order derivatives theory one should not use the bridge formuli and deal only within one representation. Which one does not matter. They are completely equivalent when described in itself.

TABLE 1 - Comparison of the dynamical scheme of massive spin-one and spin-two fields.

Element	Spin One	Spin Two	
		Fierz Representation	Standard Representation
Field	$F_{\mu\nu}$	$C_{\mu\nu\rho\sigma}$	$\psi_{\mu\nu}$
Potential	A_{μ}	$A_{\mu\nu\lambda}$	—
Lagrangian	$L=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{m^2}{2}A_{\mu}A^{\mu}$	$L=-\frac{1}{8}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}+$ $+\frac{m^2}{2}A_{\mu\nu\rho}A^{\mu\nu\rho}$	$L=-\frac{1}{2}\psi^{\mu\nu}\Big[\Box\psi_{\mu\nu}-$ $-\psi_{(\mu}{}^{\beta}{}_{,\nu),\beta}+\psi_{,\mu,\nu}+\eta_{\mu\nu}\big(\psi^{\alpha\beta}{}_{,\alpha,\beta}-$ $-\Box\psi\big)-\frac{m^2}{2}(\psi_{\mu\nu}\psi^{\mu\nu}-\psi^{\alpha}{}_{\alpha})\Big]$
Dynamics	$F^{\mu\nu}{}_{,\nu}+m^2A^{\mu}=0$	$C^{\alpha\beta\mu\nu}{}_{,\nu}+m^2A^{\alpha\beta\mu}=0$	$\Box\psi_{\mu\nu}-\psi_{(\mu}{}^{\beta}{}_{,\nu),\beta}+$ $+\psi_{,\mu,\nu}+\eta_{\mu\nu}(\psi^{\alpha\beta}{}_{,\alpha\beta}-\Box\psi)+$ $+m^2(\psi_{\mu\nu}-\psi\eta_{\mu\nu})=0$
Algebraic Compatibility Condition	none	$A^{\alpha\beta}{}_{\beta}=0$	$\psi^{\alpha}{}_{\alpha}=0$
Differential Compatibility Condition	$A^{\mu}{}_{,\mu}=0$	$A^{\alpha\beta\mu}{}_{,\mu}=0$	$\psi^{\alpha\mu}{}_{,\mu}=0$
Additional Constraint	none	$\overset{*}{A}{}^{\alpha\beta\mu}{}_{,\beta}=0$	none
Number of Degrees of Freedom	3	5	5

The Lagrangian

The free Lagrangian which gives the equation of motion of $C_{\alpha\beta\mu\nu}$ (eq. (7)) is given by

$$(12) \qquad L = -\frac{1}{8} C^{\alpha\beta\mu\nu} C_{\alpha\beta\mu\nu} + \frac{m^2}{2} A_{\alpha\beta\mu} A^{\alpha\beta\mu}$$

The resemblance with the Lagrangian for spin-1 field is striking. [Note that we have assumed dim $A_{\alpha\beta\mu}$ = (length $)^{-1}$ and that we are using natural units in which $\hbar = c = 1$.]

Passing to a general scheme, interaction with other fields can be made through a current in the same way as $j^{\mu}A_{\mu}$ in electrodynamics, for instance.

Let us examine the case of a scalar field interacting with $A_{\mu\nu\lambda}$.

The free part of the dynamics of the ϕ field is given by

$$(13) \qquad L_{\phi} = \overline{\phi}_{,\mu} \phi_{,\nu} \eta^{\mu\nu} - V(\phi)$$

in which $\overline{\phi}$ means the complex conjugate of ϕ and $V(\phi)$ is a potential which is not necessary to specify.

We can re-write L_{ϕ} in a form more suitable for our purposes.

Define a current $S_{\mu\nu\lambda}$ by

$$(14) \qquad S_{\mu\nu\lambda} = \phi_{,\mu} \eta_{\nu\lambda} - \phi_{,\nu} \eta_{\mu\lambda}$$

and

$$\overline{S}_{\mu\nu\lambda} = \overline{\phi}_{,\mu} \eta_{\nu\lambda} - \overline{\phi}_{,\nu} \eta_{\mu\lambda}$$

This current has two interesting properties (compare with (3a,b)):

(15a) $S_{\mu\nu\lambda} = -S_{\nu\mu\lambda}$

(15b) $S_{\mu\nu\lambda} + S_{\nu\lambda\mu} + S_{\lambda\mu\nu} = 0$

We can then prove immediately that

$$\frac{1}{6}\,\overline{S}_{\mu\nu\lambda}S^{\mu\nu\lambda} = \overline{\phi}_{,\mu}\phi_{,\nu}\eta^{\mu\nu}$$

with allows us to re-write the Lagrangian L_ϕ as

$$L_\phi = \frac{1}{6}\,\overline{S}_{\mu\nu\lambda}S^{\mu\nu\lambda} - V(\phi)$$

The fact that the current $S_{\mu\nu\lambda}$ has precisely the same symmetries than $A_{\mu\nu\lambda}$ induces us to set the interaction of ϕ with $A_{\mu\nu\lambda}$ in terms of the minimal coupling principle, that is

(16) $$L_{tot} = \frac{1}{6}\,(\overline{S}_{\mu\nu\lambda} + ig\overline{\phi}A_{\mu\nu\lambda})(S^{\mu\nu\lambda} - ig\phi A^{\mu\nu\lambda}) - V(\phi)$$

After some algebraic manipulation this L_{tot} becomes

(16)' $$L_{tot} = (\partial_\mu + \frac{i}{3}\,gA_\mu)\phi(\partial_\nu - i\,\frac{g}{3}\,A_\nu)\phi\eta^{\mu\nu} + \\ + \frac{1}{6}\,g^2\,\overline{\phi}\phi\;a_{\mu\nu\lambda}a^{\mu\nu\lambda} - V(\phi)$$

in which gA^μ is the trace $A^{\mu\alpha}{}_{\alpha}$ and $a_{\mu\nu\lambda}$ is the traceless part of $A_{\mu\nu\lambda}$, that is $a_{\mu\nu\lambda} = A_{\mu\nu\lambda} - \frac{g}{3}\,A_\mu\eta_{\nu\lambda} + \frac{g}{3}\,A_\nu\eta_{\mu\lambda}$.

We can then state that the net effects of the minimal coupling of $A_{\mu\nu\lambda}$ and ϕ are two: (i) the appearance of the trace $A_{\mu\lambda}{}^{\lambda}$ as a vector field minimally coupled to ϕ, and (ii) a mass-term proportional to $g^2\,|\phi|^2$ for the $A_{\mu\nu\lambda}$ field.

The extrapolation of the above result for other fields with spin greater than zero is straightforward in Minkowskii space-time as in an arbitrary Riemannian curved one.

The Case of Spinor Field

The kinematical part of the free Lagrangian for the spinor field ψ is given by ,

(17) $$L = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi$$

which can, equivalently, be written

(17)' $$L = \frac{i}{6}\,\bar{\psi}\,\Sigma^{\mu\nu}\gamma^{\lambda}(\eta_{\nu\lambda}\partial_{\mu}-\eta_{\mu\lambda}\partial_{\nu})\psi$$

in which $\Sigma_{\mu\nu} = \frac{1}{2}\,(\gamma_{\mu}\gamma_{\nu}-\gamma_{\nu}\gamma_{\mu})$.

The proof of this is straightforward if one uses the identity

(18) $$\Sigma^{\mu\nu}\gamma^{\lambda} = 2\,\varepsilon^{\mu\nu\lambda\rho}\gamma_{\rho}\gamma_5 - \gamma^{[\nu}\eta^{\mu]\lambda}$$

The minimal coupling principle, applied to our case, yields for the coupling of ψ with $A_{\mu\nu\lambda}$ the following substitution:

$$\eta_{\nu\lambda}\partial_{\mu} - \eta_{\mu\lambda}\partial_{\nu}$$

goes into

(19) $$\eta_{\nu\lambda}\partial_{\mu} - \eta_{\mu\lambda}\partial_{\nu} + f\,A_{\mu\nu\lambda} \quad .$$

Remark that if dim $A_{\mu\nu\lambda}$ = (length)$^{-1}$ $\equiv L^{-1}$ then the constant f is adimensional. Using (19) into (17)'

$$L = \frac{i}{6}\,\overline{\psi}\,\Sigma^{\mu\nu}\gamma^{\lambda}(\eta_{\nu\lambda}\partial_{\mu}-\eta_{\mu\lambda}\partial_{\nu}+fA_{\mu\nu\lambda})\psi = i\overline{\psi}\gamma^{\mu}(\partial_{\mu} - \frac{1}{3}A_{\mu})\psi$$

in which $A_{\mu} \equiv A_{\mu}{}^{\alpha}{}_{\alpha}$.

We arrive at the result that the spinor field ψ does not interact minimally with the traceless part of $A_{\mu\nu\lambda}$, but only with its trace $A_{\mu}{}^{\sigma}{}_{\sigma}$.

Generalized Interaction

In the standard representation the field $\psi_{\mu\nu}$ couples to the energy-momentum tensor of the matter (represented by $T^{\mu\nu}_{(m)}$) through the universal term

(20) $$L_{int} = f\,\psi_{\mu\nu}\,T^{\mu\nu}_{(m)}$$

What can we say about this form of describing the interaction in terms of Fierz variables ? A simple direct way to do this is by the generalization of the current-field coupling of electrodynamics $J^{\mu}A_{\mu}$ via the term $J^{\mu\nu\lambda}A_{\mu\nu\lambda}$. We saw precedently an example of this. However, the form (20) leads us to argue that for both schemes to be coherent $J^{\mu\nu\lambda}$ should be constructed in terms of the energy-momentum tensor $T_{\mu\nu(m)}$. Although the presence of matter spoils the complete symmetry of the bridge formula (11a,b) we will take (11b) in its original form. This is a choice for the assymetry displayed by the bridge formuli when there is interaction, exhibiting the distinct roles of $A_{\mu\nu\lambda}$ and $\psi_{\mu\nu}$ when the sources cannot be neglected.

Using (11b) into (20) we obtain

(21) $$L_{int} = f[-\frac{1}{m^2} A_{\mu}{}^{\varepsilon}{}_{\nu,\varepsilon} T^{\mu\nu}_{(m)} + \frac{Q}{m^2} A_{\mu}{}^{\varepsilon}{}_{\varepsilon,\nu} T^{\mu\nu}_{(m)} - \frac{1}{3m^2} A^{\alpha\beta}{}_{\beta,\alpha} T_m]$$

in which $T_{(m)} \equiv T^{\mu}{}_{\mu(m)}$. The action, up to surface terms, takes the form

(22) $$S = \int L_{int}\, d_4x = \frac{f}{m^2} \int A^{\mu\varepsilon\lambda} \Big(\frac{1}{2} T_{\lambda[\mu,\varepsilon]} - \frac{1}{6} \eta_{\lambda[\mu} T_{,\varepsilon]} - Q\, \eta_{\varepsilon\lambda} T_{\mu}{}^{\alpha}{}_{,\alpha}\Big)\, d_4x \quad .$$

We can use the conservation of $T^{\mu\nu}{}_{(m)}$ to eliminate the last term of (22) (alternatively, we could choose B = 1 in eq. (11) and consequently Q = 0, without loss of generality, and postpone the discussion of the conservation of $T^{(m)}_{\mu\nu}$). From (22) we obtain the current

(22) $$J_{\mu\varepsilon\lambda} = \frac{1}{2} T_{\lambda[\mu,\varepsilon]} - \frac{1}{6} \eta_{\lambda[\mu} T_{,\varepsilon]}$$

The coupling constant f of (20) is adimensional. We set it to be given by $f = km^2$ (in natural units) in order to conform later on to Einstein's theory of gravity.k is Einstein's constant.

Let us pause for a while and examine more carefully such current. In order to understand its presence and of (the derivatives which appears in it) we will introduce a geometric language which suggests a (future) contact with Einstein's General Relativity.

Define, from the "metric" $\psi_{\mu\nu}$ a tensor $\hat{R}_{\mu\nu}$ ("curvature") and its trace $\hat{R}$ by the expressions:

(24) $$\hat{R}_{\mu\nu} = 2\big(\Box\, \psi_{\mu\nu} - \psi_{(\mu}{}^{\alpha}{}_{,\nu),\alpha} + \psi_{,\mu,\nu}\big)$$

(25) $\hat{R} = 4 \left(\Box \psi - \psi^{\alpha\beta}{}_{,\alpha,\beta} \right)$

Setting

$$A_{\mu\nu\lambda} = \psi_{\lambda\mu,\nu} - \psi_{\lambda\nu,\mu}$$

we arrive at an identity (Bianchi-type) involving the $C_{\alpha\beta\mu\nu}$ field (defined primarily in terms of $A_{\alpha\beta\mu}$) and the associated "curvature" tensors $\hat{R}_{\mu\nu}$ and $\hat{R}$

(26) $C^{\alpha\beta\mu\nu}{}_{,\nu} = \frac{1}{2} \hat{R}^{\mu[\alpha,\beta]} - \frac{1}{12} \eta^{\mu[\alpha}\hat{R}^{,\beta]}$

Such identity has a counter-part in curved space-time. Indeed, making the transformation which brings $C_{\alpha\beta\mu\nu}$ into the Weyl conformal tensor $W_{\alpha\beta\mu\nu}$, and $\hat{R}_{\mu\nu}$ into the contracted curvature tensor, $R_{\mu\nu}$ we arrive at the well-known Bianchi identity (eq.96b). We postpone this extension for a later section. Let us here concentrate our analysis to flat space time only.

The total action S_T is given by

(27) $S_T = S_0 + S_{int} = \int d^4 \left(- \frac{1}{8} C^{\alpha\beta\mu\nu}C_{\alpha\beta\mu\nu} + \right.$

$\left. + \frac{m^2}{2} A^{\alpha\beta\mu}A_{\alpha\beta\mu} + \frac{k}{2} A^{\mu\varepsilon\lambda} T_{\lambda[\mu,\varepsilon]} - \frac{k}{6} \eta_{\lambda[\mu}T_{,\varepsilon]}\right)$

which yields the equation of motion

(28) $C^{\alpha\beta\mu\nu}{}_{,\nu} + m^2A^{\alpha\beta\mu} = - \frac{k}{2} T^{\mu[\alpha,\beta]} + \frac{k}{6} \eta^{\mu[\alpha}T^{,\beta]}$

Subtracting the identity (26) from eq. (28) we obtain

(29) $(\hat{R}_{\mu\alpha} - \frac{1}{6} \hat{R}\eta_{\mu\alpha} + kT_{\mu\alpha} - \frac{k}{3} T \eta_{\mu\alpha})_{,\beta} -$

$$- (\hat{R}_{\mu\beta} - \frac{1}{6} \hat{R} \eta_{\mu\beta} + kT_{\mu\beta} - \frac{k}{3} T \eta_{\mu\beta})_{,\alpha} =$$

$$= - 2m^2 A_{\alpha\beta\mu} .$$

Using the bridge formula, this can be transformed into:

$$(\hat{R}_{\mu\alpha} - \frac{1}{6} \hat{R} \eta_{\mu\alpha} + k T_{\mu\alpha} - \frac{k}{3} T \eta_{\mu\alpha} + 2m^2 \psi_{\mu\alpha})_{,\beta} - \tag{30}$$

$$- (\hat{R}_{\mu\beta} - \frac{1}{6} \hat{R} \eta_{\mu\beta} + k T_{\mu\beta} - \frac{k}{3} T \eta_{\mu\beta} + 2m^2 \psi_{\mu\beta})_{,\alpha} = 0$$

Equation (30) shows that the description of the theory in terms of the standard variables $\psi_{\mu\nu}$ can be made only if we impose as an initial condition an expression containing second order derivatives of $\psi_{\mu\nu}$. This is a necessary condition to obtain a non-ambiguous scheme for the Cauchy data.

We would like to emphasize that the presence of second order derivatives in the Cauchy data is a consequence of passing from the $A_{\alpha\beta\mu}$ representation (eq.(28)) to the standard one. If we remain in the Fierz variables, the initial conditions must be limited to specifications of $A_{\alpha\beta\mu}$ and its first derivatives on the Cauchy surface.

From eq. (30) it appears natural to impose, as initial condition, the "Einstein equation"

$$\hat{S}_{\mu\nu} \equiv \hat{R}_{\mu\nu} + k(T_{\mu\nu} - \frac{1}{2} T \eta_{\mu\nu}) + \tag{31}$$

$$+ 2m^2 (\psi_{\mu\nu} + \frac{\psi}{2} \eta_{\mu\nu}) = 0$$

to be valid at an initial hypersurface Σ. Indeed, from $\hat{S}_{\mu\nu}(\Sigma) = 0$ we obtain, on Σ ,

$$\hat{R}_{\alpha\beta} - \frac{1}{6}\hat{R}\eta_{\alpha\beta} + kT_{\alpha\beta} - \frac{k}{3}T\eta_{\alpha\beta} + 2m^2\psi_{\alpha\beta} = 0 \quad . \tag{32}$$

Equation (30) then guarantees the validity of the condition $\hat{S}_{\mu\nu}=0$ throughout the whole space-time in the future of Σ. Thus, the initial conditions (31) are propagated in the whole ST and the equations of motion, in the standard variables, become

$$\hat{R}_{\alpha\beta} = - k(T_{\alpha\beta} - \frac{1}{2}T\eta_{\alpha\beta}) - 2m^2(\psi_{\alpha\beta} + \frac{1}{2}\psi\eta_{\alpha\beta}) \quad . \tag{33}$$

Substituting $\hat{R}_{\alpha\beta}$ by its definition eq. (24) we realize that eq.(33) is precisely the equation of motion for the massive spin-two field as given by eq. (1).

Thus, in the context of the $A_{\mu\nu\lambda}$ variables, eq. (33) for $\psi_{\mu\nu}$ (considered as a dependent variable related to the fundamental one $A_{\mu\nu\lambda}$ through the bridge relation (11)) is nothing but an initial condition imposed to be satisfied in a given hypersurface Σ. This condition is maintained beyond Σ due to the dynamical equation satisfied by $A_{\mu\nu\lambda}$.

This fulfils our aim to provide a coherent theory of massive spin two field in both representations and the inter-relations among them. This alternative scheme, using Fierz variables, is the flat space-time version of Jordan-Lichnerowicz representation of Einstein's General Relativity, as we will see later on.

The Massless Limit (Einstein's Geometric Interpretation)

We have now to face a fundamental question: has the precedent theory a well-behaved massless limit ? And, assuming that the answer is yes, can we show the equivalence of such theory and Einstein's General Relativity ?

We will answer (affirmatively) these questions in two steps. First, we will analyse the case of a weak gravitational field in order to exhibit how smoothly the precedent theory fits into a geometrical scheme; after that we will present a resumé of the work of Jordan, Lanczos, Lichnerowicz and collaborators in order to enlighten the second inquiry.

Let $\phi_{\mu\nu}(x)$ represent a small perturbation of the Minkowski metric:

$$g_{\mu\nu} \approx \eta_{\mu\nu} + \varepsilon\ \phi_{\mu\nu}(x) + O(\varepsilon^2) \tag{34}$$

in which ε is an adimensional small parameter: $\varepsilon^2 << \varepsilon$. Then it follows that the curvature tensor $R_{\alpha\mu\beta\lambda}$ in first order is given by

$$R_{\alpha\mu\beta\lambda} \approx \frac{\varepsilon}{2}\ \{\phi_{\alpha\lambda,\mu\beta} + \phi_{\mu\beta,\alpha\lambda} - \phi_{\alpha\beta,\mu\lambda} - \phi_{\mu\lambda,\alpha\beta}\} \tag{35}$$

and the Weyl conformal tensor $W_{\alpha\beta\mu\nu}$ takes the form

$$W_{\alpha\beta\mu\nu} = -\frac{\varepsilon}{2}\ \{\phi_{\mu\alpha,\beta\nu} + \phi_{\beta\nu,\mu\alpha} - \phi_{\mu\beta,\alpha\nu} - \phi_{\alpha\nu,\mu\beta}\} - \frac{\varepsilon}{4}\ \{\ \Box\ \phi_{\nu\alpha} - \phi_{\nu}{}^{\varepsilon}{}_{,\varepsilon\alpha} - \phi_{\alpha}{}^{\varepsilon}{}_{,\varepsilon\nu} + \tag{36}$$

$$+ \phi_{,\nu\alpha}\} \eta_{\beta\nu} - \frac{\varepsilon}{4} \{ \Box \phi_{\beta\mu} - \phi_{\beta}{}^{\varepsilon}{}_{,\varepsilon\mu} -$$

$$- \phi_{\mu}{}^{\varepsilon}{}_{,\varepsilon\beta} - \phi_{,\beta\mu}\}\eta_{\alpha\nu} + \frac{\varepsilon}{4} \{ \Box \phi_{\alpha\mu}$$

$$- \phi_{\alpha}{}^{\varepsilon}{}_{,\varepsilon\mu} - \phi_{\mu}{}^{\varepsilon}{}_{,\varepsilon\alpha} + \phi_{,\alpha\mu}\} \eta_{\beta\nu} +$$

$$+ \frac{\varepsilon}{4} \{ \Box \phi_{\beta\nu} - \phi_{\beta}{}^{\varepsilon}{}_{,\varepsilon\nu} - \phi_{\nu}{}^{\varepsilon}{}_{,\varepsilon\beta} + \phi_{,\beta\nu}\} \eta_{\alpha\mu} +$$

$$- \frac{\varepsilon}{6} (\Box \phi - \phi^{\lambda\sigma}{}_{,\lambda\sigma}) (\eta_{\alpha\mu}\eta_{\beta\nu} - \eta_{\alpha\nu}\eta_{\beta\mu}) \quad .$$

A simple inspection on (36) and some algebraic manipulations show that it is possible to write, in the first approximation, the Weyl tensor given by (36) in terms of a potential $B_{\alpha\beta\mu}$ such that

$$(37) \qquad W_{\alpha\beta\mu\nu} = B_{\alpha\beta[\mu,\nu]} + B_{\mu\nu[\alpha,\beta]} + \frac{1}{2} B_{(\alpha\nu)}\eta_{\beta\mu} + \frac{1}{2} B_{(\beta\mu)}\eta_{\alpha\nu} - \frac{1}{2} B_{(\beta\nu)}\eta_{\alpha\mu} - \frac{1}{2} B_{(\alpha\mu)}\eta_{\beta\nu} + \frac{2}{3} B^{\sigma\lambda}{}_{\sigma,\lambda} (\eta_{\alpha\mu}\eta_{\beta\nu} - \eta_{\alpha\nu}\eta_{\beta\mu}).$$

Comparison of (36) and (37) yields for $B_{\alpha\beta\mu}$ the expression

$$(38) \qquad B_{\alpha\beta\mu} = - \frac{\varepsilon}{4} (\phi_{\mu\alpha,\beta} - \phi_{\mu\beta,\alpha})$$

or, in terms of the Christoffel symbols $\Gamma^{\alpha}_{\mu\nu}$ in this approximation, equivalently by

$$(39) \qquad B_{\alpha\beta\mu} = - \frac{1}{2} (\eta_{\alpha\lambda}\Gamma^{\lambda}_{\beta\mu} - \eta_{\beta\lambda}\Gamma^{\lambda}_{\alpha\mu}) \quad .$$

We recognize then that (37) is an expression analogous to eq. (5) and (38) is the bridge formula (11a) with the choice B=0.

This suggests the identification of the Weyl tensor with the previous tensor $C_{\alpha\beta\mu\nu}$ in the limit of vanishing mass, at least in the realm of Minkowskii geometry. The extension of such result to an arbitrary (i.e., nonweak) gravitational field will be matter of a future section.

We can then apply to the present massless case all the results of the precedent section in order to arrive at the description in the Fierz variables of a (massless) spin two field.

Let us add one more remark.

The inverse bridge formula (eq. (11b)) cannot be applied in the limit m = 0 even for small perturbations. The reason for this is due to the fact that the field $\phi_{\mu\nu}$ is massless <u>and</u> the space time background is flat - which inhibits the construction of a natural quantity with mass dimension. The situation is somewhat distinct in a curved background. The reason is that the evolution of small fluctuations of the generic metric tensor $\Delta g_{\mu\nu}$ acquires a mass-term induced by the gravitational field (of the background) in two competitive ways: either directly through the geometry (e.g. $\Delta m \sim R^{-1/2}$) or by the existence of a force field in the background characterized by the newtonian constant $\Delta m \sim (\Delta m)_{Planck} \sim k^{-1/2}$.

Gauge Invariance

In the limit m = 0 some of the constraints of the theory become arbitrary, exhibiting the gauge freedom of massless fields (of any spin). We will analyse more carefully this freedom in the next section. Let us see here what are in the specific case of the Fierz variables, these symmetries.

Consider the map

$$(40) \qquad B_{\alpha\beta\mu} \to \tilde{B}_{\alpha\beta\mu} = B_{\alpha\beta\mu} + M_{\alpha}\eta_{\beta\mu} - M_{\beta}\eta_{\alpha\mu}$$

Then it follows from direct calculation upon eq. (37) the invariance

$$(41) \qquad \tilde{W}_{\alpha\beta\mu\nu}[\tilde{B}^{\rho\sigma\lambda}] = W_{\alpha\beta\mu\nu}[B^{\rho\sigma\lambda}]$$

which means that the trace $B^{\alpha\mu}{}_{\mu}$ is completely arbitrary. Another invariance is obtained by making the map

$$(42) \qquad \tilde{B}_{\alpha\beta\mu} = B_{\alpha\beta\mu} + (K_{\alpha,\beta} - k_{\beta,\alpha})_{,\mu}$$

which yields the same tensor $W_{\alpha\beta\mu\nu}$.

The Energy

Let us come back to the free Lagrangian (12). The energy-momentum of the field, by Noether's theorem, is given by

$$(43) \qquad t^{\alpha}{}_{\beta} = \frac{\partial \mathcal{L}_0}{\partial A^{\mu\nu\lambda}{}_{,\alpha}} A^{\mu\nu\lambda}{}_{,\beta} - \mathcal{L}_0 \, \delta^{\alpha}{}_{\beta} \; .$$

We then obtain

$$(44) \qquad t^{\alpha}{}_{\beta} = - C^{\mu\nu\lambda\alpha} A_{\mu\nu\lambda,\beta} - \mathcal{L}_0 \, \delta^{\alpha}{}_{\beta} .$$

Alternatively, we can use Einstein's formalism of GR to define the energy-momentum tensor $T^{\mu}{}_{\nu}$ by varying the geometry:

$$(45) \qquad \delta \int \sqrt{-g} \; \mathcal{L}_0 \, d_4k = \frac{1}{2} \int \sqrt{-g} \, T_{\mu\nu} \, \delta g^{\mu\nu} \, d_4x \qquad .$$

We arrive at the expression

$$(46) \qquad T^{\alpha}{}_{\beta} = - \frac{1}{8} C^{2} \delta^{\alpha}{}_{\beta} - C^{\alpha}{}_{\rho\beta\sigma} A^{(\rho\sigma)} - - \frac{m^2}{2} A_{\rho\sigma\varepsilon} A^{\rho\sigma\varepsilon} \delta^{\alpha}{}_{\beta} + 2m^2 A^{\alpha\rho\sigma} A_{\beta\rho\sigma} + + m^2 A_{\rho\sigma\alpha} A^{\rho\sigma}{}_{\beta} + U^{\alpha}{}_{\beta}{}^{\lambda}{}_{,\lambda} \qquad .$$

in which $U^{\alpha}{}_{\beta}{}^{\lambda}$ is given by

$$(47) \qquad U^{\alpha}{}_{\beta}{}^{\lambda} = C^{\alpha}{}_{\rho\sigma}{}^{\lambda} A_{\beta}{}^{\rho\sigma} + C_{\beta\rho\sigma}{}^{\lambda} A^{\alpha\rho\sigma} + + C^{\alpha}{}_{\varepsilon\rho}{}^{\lambda} A_{\beta}{}^{\rho\varepsilon} + C_{\beta\varepsilon\rho}{}^{\lambda} A^{\alpha\rho\varepsilon} - C^{\alpha}{}_{\rho\sigma\beta} A^{\lambda(\rho\sigma)} \qquad .$$

Actually, both expressions $t^{\alpha}{}_{\beta}$ and $T^{\alpha}{}_{\beta}$ gives the same physical content for the distribution of energy. The easiest way to demonstrate this is by evaluating both expressions (44) and (46) on mass-shell. Indeed, using eq. (7) for $A_{\alpha\beta\mu}$ into the formula (46) we obtain

$$(48) \qquad T^{\alpha}{}_{\beta} = -C_{\nu\lambda\mu}{}^{\alpha}A^{\nu\lambda\mu}{}_{,\beta} - \mathcal{L}_0\, \delta^{\alpha}{}_{\beta} + \text{divergence}$$

To arrive at this expression a lot of work is saved if one notes the identity

$$(49) \qquad C^{\alpha}{}_{\mu\nu\lambda}C_{\beta}{}^{\mu\nu\lambda} = \frac{1}{4}\, C_{\rho\mu\nu\lambda}C^{\rho\mu\nu\lambda}\, \delta^{\alpha}{}_{\beta}$$

which is a direct consequence of the definition of $C^{\alpha}{}_{\beta\mu\nu}$. From (49) we obtain the equivalent relation:

$$(50) \qquad 2C^{\alpha\mu\nu\lambda}A_{\beta\mu\nu,\lambda} + C^{\alpha\mu\nu\lambda}(A_{\nu\lambda\beta,\mu} - A_{\nu\lambda\mu,\beta}) + $$

$$+ A_{(\mu\nu)}\, C^{\alpha\mu\nu}{}_{\beta} = \frac{1}{4}\, C^2\, \delta^{\alpha}{}_{\beta}$$

in which $C^2 \equiv C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}$

which simplifies the calculation to obtain (48) from (46).

The Hamiltonian

We have presented in eq. (12) the Lagrangian function which enables us to obtain the dynamical equation (7) for $A_{\alpha\beta\mu}$. However, in order to eliminate undesirable degrees of freedom we were led to impose upon $A_{\alpha\beta\mu}$ a further constraint contained in eq. (9). One should like to introduce this condition explicitly in the Lagrangian. This can be done by adding to L_0 a Lagrange multiplier.

To simplify the interpretation of the various quantities which appears in the formalism and to make easy the correspondence with the standard variable formulation, let us abandon for a while the explicit covariant form of the theory, performing the conventional 3+1 space-time decomposition. Adding to Lagrangian (16) the constraints needed to eliminate the excedent degrees of freedom we set:

$$
\begin{aligned}
(51) \qquad L_t = {} & -C^{ioko}C_{ioko} - \frac{1}{2} C^{ijko}C_{ijko} + \\
& - m^2 \left[\xi_i\xi^i + \frac{1}{4}\alpha_{ij}\alpha^{ij} + \frac{3}{4}\beta_{ij}\beta^{ij} + \right.\\
& \left. + \frac{1}{2}\Delta^{ijk}\Delta_{ijk} + \frac{1}{2}\gamma^i\gamma_i + \frac{1}{3}\phi^2\right] + \\
& + \Omega^{ijk}\left[\dot{\Delta}_{ijk} - \frac{1}{2}\alpha_{k[i,j]} - \frac{1}{3}\beta_{ij,k} + \right.\\
& + \frac{1}{6}\beta_{ki,j} + \frac{1}{6}\beta_{jk,i} - \frac{1}{4}\eta_{ki}(\alpha^{\ell}{}_{j}-\beta^{\ell}{}_{j}), + \\
& \left. + \frac{1}{4}\eta_{kj}(\alpha^{\ell}{}_{i}-\beta^{\ell}{}_{i})_{,\theta}\right] + Q\Omega^{ijk}\Omega_{ijk}
\end{aligned}
$$

in which to simplify our notation we have defined the irreducible parts of the $A_{\alpha\beta\mu}$ field to be given by:

(52a) $\phi = A_{\ell o}{}^{\ell}$

(52b) $\xi_i = A_{oi}{}^{o}$

(52c) $\gamma_i = A_{\ell i}{}^{\ell}$

(52d) $\beta_{ij} = A_{ij}{}^{o}$

(52e) $\alpha_{ij} = A_i{}^o{}_j + A_j{}^o{}_i - \frac{2}{3}\phi\,\eta_{ij}$

(52f) $\Delta_{ijk} = A_{ijk} - \frac{1}{2}\eta_{k[i}A^{\ell}{}_{j]\ell}$

A dot means time derivative.

Note that in the Lagrange multiplier term we did not use the full condition (9) but only the restrained form $A_{ij}{}^{k}{}_{,o} + A_{jo}{}^{k}{}_{,i} + A_{oi}{}^{k}{}_{,j} = 0$. The reason for this is explained in the note following eq. (9)', which shows that the compatibility condition (eq. (8b)), which is a consequence of the eq. of motion, has to be subtracted from the constraint (9).

We note also that the extra term involving a new Lagrangian multiplier Q is just to eliminate from the theory the ghost function Ω^{ijk}, as we will see.

The extended set of variables contained in (51) are ϕ, ξ_i, γ_i, α_i, β_{ij}, Δ_{ijk}, Ω_{ijk} and Q. The corresponding momenta canonically conjugated are Π, Π^i, P^i, Π^{ij}, P^{ij}, Π_{ijk}, P_{ijk} and P.

From the definition of the momenta we obtain

(53a) $$\Pi_{ij} = \frac{\delta L}{\delta\dot{\alpha}^{ij}} = -C_{iojo}$$

(53b) $$\Pi^{ijk} = \frac{\delta L}{\delta\dot{\Delta}_{ijk}} = -C^{ijko} + \Omega^{ijk} \approx -C^{ijko}$$

and the primary constraints

(54a) $$P_{ij} = \frac{\delta L}{\delta \dot{\beta}^{ij}} \approx 0$$

(54b) $$\Pi_i = \frac{\delta L}{\delta \dot{\xi}^i} \approx 0$$

(54c) $$P_i = \frac{\delta L}{\delta \dot{\gamma}^i} \approx 0$$

(54d) $$\Pi = \frac{\delta L}{\delta \dot{\phi}} \approx 0$$

(54e) $$P^{ijk} = \frac{\delta L}{\delta \dot{\Omega}_{ijk}} \approx 0$$

(54f) $$P = \frac{\delta L}{\delta \dot{Q}} \approx 0$$

in which we are using Dirac's [4] notation of weak identities.

Let us use the standard decomposition of arbitrary antisymmetric tensors in its electric ($E_{\mu\nu}$) and magnetic ($H_{\mu\nu}$) parts. Define

$$E_{\mu\nu} = - C_{\mu\alpha\nu\beta} V^\alpha V^\beta$$

$$H \quad = - \overset{*}{C}_{\mu\alpha\nu\beta} V^\alpha V^\beta$$

for an arbitrary vector V^μ.

Then it follows the properties

$$E_{\mu\nu} = E_{\nu\mu}$$
$$E_{\mu\nu} V^\nu = 0$$
$$E_{\mu\nu} \eta^{\mu\nu} = 0$$
$$H_{\mu\nu} = H_{\nu\mu}$$
$$H_{\mu\nu} V^\nu = 0$$
$$H_{\mu\nu} \eta^{\mu\nu} = 0$$

which implies that $E_{\mu\nu}$ and $H_{\mu\nu}$ have both 5 independent components. Here we will choose $V^\mu = \delta^\mu{}_0$ to obtain

$$E_{ij} = -C_{oioj}$$

$$H_{ij} = -\overset{*}{C}_{oioj}$$

Substituting this decomposition in the definition of the momenta (53) we obtain

$$\Pi_{ij} = - E_{ij}$$

$$\Pi_{ijk} = \eta_{ijmo}H^{m}{}_{k} = \varepsilon_{ijm}H^{m}{}_{k}$$

Note that the fact that we are dealing with a double pair of antisymmetric indices makes the c.c. momenta to be associated not only with the electric part of the field (as in the case of spin-1 field) but also with the magnetic part. The determination of the Hamiltonian follows from the canonical variables thus defined. The total Hamiltonian (including the constraints) is given by:

$$(55) \qquad \mathcal{H}_T = \Pi^{ij}\dot{\alpha}_{ij} + \Pi^{ijk}\dot{\Delta}_{ijk} - L_T =$$

$$= - \Pi^{ij}\Pi_{ij} + \Pi^{ij}\left(\gamma_{i,j} - \frac{1}{3}\gamma^{k}{}_{,k}\,\eta_{ij}\right) +$$

$$+ 2\,\Pi^{ij}\Delta_{i}{}^{m}{}_{j,m} + 2\,\Pi^{i\ell}\left(-\xi_{i,\ell} + \frac{1}{3}\xi^{m}{}_{,m}\eta_{i\ell}\right) +$$

$$- \frac{1}{2}\,\Pi^{ijk}\Pi_{ijk} + \Pi^{ijk}\left(\beta_{ij,k} - \frac{1}{2}\beta_{ki,j} -\right.$$

$$\left.- \frac{1}{2}\beta_{jk,i} - \frac{3}{4}\eta_{k[i}\beta^{\ell}{}_{j],\ell}\right) - \Pi^{ijk}\left(\frac{1}{2}\alpha_{k[i,j]} +\right.$$

$$\left.+ \frac{1}{4}\eta_{k[i}\alpha_{j]}{}^{\ell}{}_{,\ell}\right) - m^2\left(\xi_k\xi^k +\right.$$

$$+ \frac{1}{4}\alpha_{ik}\alpha^{ik} + \frac{3}{4}\beta_{ik}\beta^{ik} + \frac{1}{2}\Delta^{ijk}\Delta_{ijk} +$$

$$+ \frac{1}{2} \gamma^{\ell}\gamma_{\ell} + \frac{1}{3} \phi^2) - \Omega^{ijk} (- \Pi_{ijk} -$$

$$- \alpha_{k[i,j]} + \frac{2}{3} \beta_{ij,k} + \frac{1}{3} \beta_{ik,j} + \frac{1}{3} \beta_{kj,i} -$$

$$- \frac{1}{2} (\alpha_j{}^{\ell}+\beta^{\ell}{}_j)_{,\ell}\, \eta_{ik} + \frac{1}{2} (\alpha_i{}^{\ell}+\beta^{\ell}{}_i)_{,\ell}\, \eta_{jk}) +$$

$$+ Q\, \Omega^{ijk}\Omega_{ijk} + Z_{ij}\, P^{ij} + Y_j \Pi^j +$$

$$+ M\, \Pi + X_j\, P^j + D^{ijk}\, P_{ijk} + N\, P \quad ,$$

in which Z^{ij}, Y^j, M, X^j, X^{ijk} and N are functions of the canonical variables that allows the introduction of the constraints (54) into the Hamiltonian, following Dirac's approach.

Conservation of the primary constraints (54) yields the secondary ones. Let us find them. We have:

$$\dot{P} = [P, \mathcal{H}_T] = \Omega^{ijk}\Omega_{ijk} \tag{56}$$

which gives the secondary constraint

$$\Omega_{ijk} \approx 0 \quad . \tag{57}$$

From (54a)

$$[P_{ij}, \mathcal{H}_T] = \frac{3}{2} \Pi_{ij}{}^{k}{}_{,k} + \frac{3}{2} m^2 \beta_{ij} \approx 0 \tag{58}$$

that can be written as:

$$\frac{3}{2} \varepsilon_{ija} H^{ak}{}_{,k} + \frac{3}{2} M^2 \beta_{ij} \approx 0 \quad .$$

This is nothing but the equation of motion (7) $C^{\rho\sigma\lambda\mu}{}_{,\mu} + m^2 A^{\rho\sigma\lambda} = 0$ for $\rho = i$, $\sigma = j$ and $\lambda = 0$.

From the conservation of the primary constraint (54b) we obtain:

$$\dot{\Pi}^j = [\Pi^j, \mathcal{H}_T] = -\Pi^{j\ell}{}_{,\ell} + m^2 \xi^j \approx 0 \tag{59}$$

which is the equation of motion (7) for $\rho = 0$, $\sigma = i$ and $\lambda = 0$ and can be written as

(59)' $$E^{j\ell}{}_{,\ell} - m^2\xi^j = 0$$

From the primary constraint (54c) we have:

(60) $$\dot{P}^j = [P^j, \mathcal{H}_T] = \Pi^{j\ell}{}_{;,\ell} + m^2\gamma^j \approx 0$$

which is obtained by taking the trace of the equation of motion (7) for $\rho = i$, $\sigma = j$, $\lambda = k$. From the primary constraint (54d):

(61) $$\dot{\Pi} = [\Pi, \mathcal{H}_T] = \frac{2}{3} m^2\phi \approx 0$$

which is the compatibility condition (8a) $A^{\rho\sigma}{}_{\sigma} = 0$ for $\rho = 0$.

From the primary constraint (54e)

(62) $$\dot{P}_{ijk} = [P_{ijk}, \mathcal{H}_T] = -\Pi_{ijk} - \alpha_{k[i,j]} + \frac{2}{3}\beta_{ij,k} + \frac{1}{3}\beta_{ik,j} + \frac{1}{3}\beta_{kj,i} - \frac{1}{2}(\alpha_j{}^{\ell}+\beta^{\ell}{}_j)_{,\ell}\eta_{ik} + \frac{1}{2}(\alpha_i{}^{\ell}+\beta^{\ell}{}_i)_{,\ell}\eta_{jk} = \Theta_{ijk} \approx 0$$

We have then obtained the secondary constraints given by eq. (57), (58), (59), (60), (61), (62).

Note that since N is completely arbitrary we can discard Q and P from the set of canonical variables and treat Q as mere arbitrary coefficients which have to satisfy no conditions.

We have now to look for consistency conditions of preservation of the secondary constraints throughout the evolution of the system.

From (57) we get the determination of the Lagrange multiplier D_{ijk}:

$$\dot{\Omega}_{ijk} = [\Omega_{ijk}, \mathcal{H}_T] = D_{ijk} \approx 0 \tag{63}$$

From (58) we determine the Lagrange multiplier Z^{ij}:

$$[\Pi_{ij}{}^{k}{}_{,k} + m^2\beta_{ij}, \mathcal{H}_T] = \frac{1}{2}\,\Pi^{k}{}_{[i,j],k} + m^2\Delta_{ij}{}^{k}{}_{,k} + m^2\, Z_{ij} \tag{64}$$

Using eq. (60) we determine Z_{ij}:

$$Z_{ij} = \frac{1}{2}\,\gamma_{[i,j]} - \Delta_{ij}{}^{k}{}_{,k} = - A_{ij}{}^{k}{}_{,k} \tag{65}$$

From the preservation of the constraint (59):

$$[\Pi^{j\ell}{}_{,\ell} - m^2\xi^j, \mathcal{H}_T] = -\frac{1}{2}\,\Pi^{jmk}{}_{,m,k} - m^2Y^j + \frac{m^2}{2}\,\alpha^{j\ell}{}_{,\ell}$$

which determines Y^j:

$$Y^j = \frac{1}{2}\,(\alpha^{j\ell}+\beta^{j\ell})_{,\ell} = A^{jo\ell}{}_{,\ell} \tag{66}$$

using eq. (61).

From constraint (60):

$$[\Pi^{j\ell}{}_{,\ell} + m^2\gamma^j, \mathcal{H}_T] \approx -\frac{1}{2}\,\Pi^{jmk}{}_{,m,k} + m^2X^j + \frac{m^2}{2}\,\alpha^{j\ell}{}_{,\ell} \tag{67}$$

which gives the value of X^j:

$$X^j \approx -\frac{1}{2}\,(\alpha^{j\ell}+\beta^{j\ell})_{,\ell} = - A^{jo\ell}{}_{,\ell} \tag{68}$$

using eq. (61).

From constraint (61):

(69) $$[\phi, \mathcal{H}_T] = M$$

which determines that M vanishes weakly:

(70) $$M \approx 0$$

The conservation of the constraint (62) yields:

(71) $$[\Theta_{ijk}, \mathcal{H}_T] = \Pi_{k[i,j]} - \gamma_{i,j,k} + \gamma_{j,i,k} + \Delta_k{}^m{}_{j,i,m} - \Delta_k{}^m{}_{i,j,n} + \Delta_j{}^m{}_{k,i,n} - \Delta_i{}^m{}_{k,j,m} - \frac{2}{3}\Delta_{ij}{}^{\ell}{}_{,\ell,k} - \frac{1}{3}\Delta_{ik}{}^{\ell}{}_{,\ell} - \frac{1}{3}\Delta_{kj}{}^{\ell}{}_{,\ell,i} + \frac{1}{2}\eta_{ki}\,(-\gamma_{j,\ell m}\eta^{\ell m} + \gamma^{\ell}{}_{,\ell,j} + \Pi_j{}^{\ell}{}_{,\ell} + 2\Delta^{\ell}{}_j{}^m{}_{,\ell,m}) - \frac{1}{2}\eta_{kj}\,(-\gamma_{i,\ell m}\eta^{\ell m} + \gamma^{\ell}{}_{,\ell,i} + \Pi_i{}^{\ell}{}_{,\ell} + 2\Delta^{\ell}{}_i{}^m{}_{,\ell,k}) - m^2\Delta_{ijk} \equiv \phi_{ijk} \approx 0$$

In order to arrive at this expression we have used expression (65) and a combination of (59) and (60), that is,

$$Z_{ij} = -A_{ij}{}^{\ell}{}_{,\ell} \quad \text{and} \quad A_i{}^o{}_o = -A_i{}^{\ell}{}_{\ell}\ .$$

Finally conservation on time of ϕ_{ijk}

(72) $$[\phi_{ijk}, \mathcal{H}_T] \approx 0$$

is guaranteed by a manipulation of eq. (58) and (62).

We have exhausted all equations of the Hamiltonian scheme. There remains just as an exercise of compatibility of the previous set of equations to reproduce the remaining "true" dynamical equations.

We have

(73) $$\dot{\Pi}^{ij} \approx [\Pi^{ij},\mathcal{H}_T] \approx -\frac{1}{2}\,\Pi^{jki}{}_{,k} - \frac{1}{2}\,\Pi^{ikj}{}_{,k} + \frac{m^2}{2}\,\alpha^{ij}$$

which can be written

(74) $$C^{iojo}{}_{,o} + \frac{1}{2}\,C^{ioji}{}_{,k} + \frac{1}{2}\,C^{joik}{}_{,k} + \frac{m^2}{2}\,\alpha^{ij} = 0$$

(75) $$\dot{E}^{ij} - \frac{1}{2}\,\varepsilon^{k\ell(j}H^{i)}{}_{\ell,k} - M^2\alpha^{ij} = 0$$

Evolution of Π^{ijk}:

(76) $$\dot{\Pi}^{ijk} = [\Pi^{ijk},\mathcal{H}_T] = \Pi^{k[i,j]} - -\frac{1}{2}\,\Pi^{\ell[i}{}_{,\ell}\eta^{j]k} + m^2\,\Delta^{ijk}$$

or,

(77) $$\dot{\Pi}^{ijk} - \Pi^{k[i,j]} + \frac{1}{2}\,\Pi^{\ell[i}{}_{,\ell}\eta^{j]k} - m^2\Delta^{ijk} \approx 0$$

or using (53b)

(78) $$C^{ijko}{}_{,o} - C^{oko[i,j]} + \frac{1}{2}\,C^{o\ell o[i}{}_{,\ell}\eta^{j]k} + m^2\Delta^{iji} = 0$$

that is,

(79) $$\varepsilon^{ija}\,\dot{H}_a{}^k + E^{k[i,j]} - \frac{1}{2}\,E^{\ell[i}{}_{,\ell}\eta^{j]k} + m^2\Delta^{ijk} = 0$$

There remains to consider the evolution of α_{ij}, Δ_{ijk} and ξ_i, we have

(80) $$\dot{\alpha}_{ij} = [\alpha_{ij}, \mathcal{H}_T] \approx -2\,\Pi_{ij} - \xi_{(i,j)} + \frac{2}{3}\,\xi^k{}_{,k}\eta_{ij} + \frac{1}{2}\,\gamma_{(i,j)} + \Delta_{(i}{}^m{}_{j),m} - \frac{1}{3}\,\gamma^m{}_{,m}\,\eta_{ij}$$

which, by using (53a), is nothing but the definition of C_{iojo} in terms of the Fierz variables (52)

(81) $$C_{iojo} = \frac{1}{2}\,\dot{\alpha}_{ij} + \frac{1}{2}\,\xi_{(,j)} - \frac{1}{4}\,\gamma_{(i,j)} - \frac{1}{3}\,\xi^k{}_{,k}\,\eta_{ij} - \frac{1}{2}\,\Delta_{(i}{}^m{}_{j),m} + \frac{1}{6}\,\gamma^m{}_{,m}\eta_{ij}\ ,$$

or the substitution of (52) in the previous variables:

(81)' $$C_{iojo} = A_{(i}{}^o{}_{j),o} - A_{(i}{}^{oo}{}_{,j)} - \frac{1}{2}\,A_{(ij)} - A_{oo}\eta_{ij} + \frac{2}{3}\,A^{\sigma\lambda}{}_{\sigma,\lambda}\eta_{ij}$$

The evolution of Δ_{ijk} gives:

(82) $$\dot{\Delta}_{ijk} = [\Delta_{ijk}, \mathcal{H}_T] \approx -\,\Pi_{ijk} + \beta_{ij,k} - \frac{1}{2}\,\beta_{ki,j} - \frac{1}{2}\,\beta_{jk,i} - \frac{1}{2}\,\alpha_{k[i,j]} - \frac{1}{4}\,\eta_{kj}(3\,\beta_i{}^\ell - \alpha_i{}^\ell)_{,\ell} + \frac{1}{4}\,\eta_{ki}(3\,\beta_j{}^\ell - \alpha_j{}^\ell)_{,\ell}$$

which is nothing but the definition of C_{ijko} in terms of Fierz quantities.

From ξ_i:

$$(83)\qquad \dot{\xi}_i = [\xi_i, \mathcal{H}_T] = Y_i = A_{io}{}^{\ell}{}_{,\ell}$$

or using (52b)

$$(84)\qquad \dot{A}_{ojo} + A_{oj}{}^{\ell}{}_{,\ell} = 0$$

From β_{ij}:

$$(85)\qquad \dot{\beta}_{ij} = [\beta_{ij}, \mathcal{H}_T] = Z_{ij} = - A_{ij}{}^{\ell}{}_{,\ell}$$

or

$$(86)\qquad \dot{A}^{ijo} + A^{ij\ell}{}_{,\ell} = 0$$

and for γ^k:

$$(87)\qquad \dot{\gamma}^k = [\gamma^k, \mathcal{H}_T] \approx X^j = -A^{jo\ell}{}_{,\ell} = -\dot{A}^{oj}{}_o$$

which is (84).

Collecting (84), (86) we see that these equations are nothing but the compatibility condition

$$(88)\qquad A^{\alpha\beta\lambda}{}_{,\lambda} = 0$$

Finally, the conservation of the constraint (62) is identically satisfied - yielding no subsidiary condition. We note that $\phi_{ijk} \approx 0$ is a secondary constraint and as such can be obtained from a combination of the dynamical equations (37), (80) and (86) and the constraint (62).

Let us now end our Hamiltonian analysis of freedom by just counting the true degress of freedom of our theory. Eliminating variables P and Q by the reasons pointed out previously, we are left with 20 $A_{\alpha\beta\mu}$'s and 20 corresponding momenta which gives 40 quantities; besides, there are 5 Ω^{ijk} and its corresponding momenta (5) which gives 10. Thus the total number of variables are 50.

All constraints being second class, the number of degrees of freedom to be removed are:

3 for (54a)

3 for (54b)

1 for (54d)

3 for (54c)

5 for (57)

5 for (54e)

5 for (62)

5 for (71)

3 for (58)

1 for (61)

3 for (59)

3 for (60) .

Adding all these numbers gives 40 second class constraints yielding $\frac{50-40}{2} = 5$ degrees of freedom - as it should be to describe a massive spin two field. We have thus accomplished our task to prove that the present theory gives a complete Lagrangian-

-Hamiltonian formulation of spin 2 field in Fierz coordinates.

Canonical Variables in the Fierz-Representation		
Variable	Redefinition	Momenta C.C.
$A_{\ell o}{}^{\ell}$	ϕ	Π
$A_{oi}{}^{o}$	ξ_i	Π_i
$A_{\ell i}{}^{\ell}$	γ_i	P_i
$A_{ij}{}^{o}$	β_{ij}	P_{ij}
$A_i{}^o{}_j + A_j{}^o{}_i - \frac{2}{3}\phi\eta_{ij}$	α_{ij}	Π_{ij}
$A_{ijk} - \frac{1}{2}\eta_{k[i}A^{\ell}{}_{j]\ell}$	Δ_{ijk}	Π_{ijk}
Ω_{ijk}	Ω_{ijk}	P_{ijk}
Q	Q	P

TABLE 2. The fundamental quantities to construct the Hamiltonian formalism for the massive (massless) spin 2 field. $A_{\alpha\beta\mu}$ is represented by its irreducible components $\phi(1)$, $\xi_i(3)$, $\gamma_i(3)$, $\beta_{ij}(3)$, $\alpha_{ij}(5)$, $\Delta_{ijk}(5)$ and corresponding c.c. variables. We add Ω_{ijk} and Q for completion of the theory.

Primary Constraints
$\Pi \approx 0$
$\Pi^{i} \approx 0$
$P^{i} \approx 0$
$\Pi^{ij} \approx 0$
$P^{ijk} \approx 0$
$P \approx 0$

TABLE 3 - The set of primary constraints for massive spin two field.

Secondary Constraints
$\phi \approx 0$
$\Pi^{ik}{}_{,k} - m^2\xi^i \approx 0$
$\Pi^{ik}{}_{,k} + m^2\gamma^i \approx 0$
$\Pi^{ijk}{}_{,k} + m^2\beta^{ij} \approx 0$
$\Pi^{ijk} + \alpha_{k[i,j]} - \frac{2}{3}\beta_{ij,k} - \frac{1}{3}\beta_{ik,j} - \frac{1}{3}\beta_{kj,i} + \frac{1}{2}(\alpha_j{}^{\ell}+\beta^{\ell}{}_j)_{,\ell}\,\eta_{ik} - \frac{1}{2}(\alpha^{\ell}{}_i+\beta^{\ell}{}_i)_{,\ell}\,\eta_{jk} \approx 0$
$\Omega_{ijk} \approx 0$
$\Pi_{k[i,j]} - \gamma_{i,j,k} + \gamma_{j,i,k} + \Delta_k{}^m{}_{j,i,m} - \Delta_k{}^m{}_{i,j,m} + \Delta_j{}^m{}_{k,i,m} - \Delta_i{}^m{}_{k,j,m} - \frac{2}{3}\Delta_{ij}{}^{\ell}{}_{,\ell,k} - \frac{1}{3}\Delta_{ik}{}^{\ell}{}_{,\ell} - \frac{1}{3}\Delta_{kj}{}^{\ell}{}_{,\ell,i} + \frac{1}{2}\eta_{ki}(-\gamma_{j,\ell m}\eta^{\ell m} + \gamma^{\ell}{}_{,\ell,j} + \Pi_j{}^{\ell}{}_{,\ell} + 2\Delta^{\ell}{}_j{}^m{}_{,\ell,m}) - \frac{1}{2}\eta_{kj}(-\gamma_{i,\ell m}\eta^{\ell m}+\gamma^{\ell}{}_{,\ell,i}+\Pi_i{}^{\ell}{}_{,\ell}+2\Delta^{\ell}{}_i{}^m{}_{,\ell,k}) - m^2\Delta_{ijk} \equiv \phi_{ijk} \approx 0$

TABLE 4 - The set of secondary constraint for massive spin two field.

The Massless Case

We now turn to the case of the Hamiltonian formalism for the case of massless spin-2 field in Fierz variables.

From Lagrangian (51) taking m = 0 we obtain the Hamiltonian (55) with m = 0. Then it is straightforward to obtain

$$[P^{ij}, \mathcal{H}_T] = \frac{3}{2}\, \Pi^{ijk}{}_{,k} \approx 0 \quad .$$

Conservation of this constraint gives

$$[\Pi^{ijk}{}_{,k}, \mathcal{H}_T] = \frac{1}{2}\, \Pi^{k[i,j]}{}_{,k} \approx 0 \quad .$$

For Π^i:

$$[\Pi^i, \mathcal{H}_T] = -\, \Pi^{i\ell}{}_{,\ell}$$

and conservation of this is guaranteed by

$$[\Pi^{i\ell}{}_{,\ell}, \mathcal{H}_T] = -\frac{1}{2}\, \Pi^{imk}{}_{,k,m} \approx 0 \quad .$$

For the remaining quantities we have, in the same way as for the massive case:

$$[\Pi, \mathcal{H}_T] = 0$$

$$[P^j, \mathcal{H}_T] = \Pi^{j\ell}{}_{,\ell} \approx 0$$

$$[\Lambda_{ijk}, \mathcal{H}_T] = D_{ijk} \approx 0$$

$$[P_{ij}, \mathcal{H}_T] = -\, \Pi_{ijk} - \alpha_{k[i,j]} + \frac{2}{3}\, \beta_{ij,k} + \frac{1}{3}\, \beta_{ik,j}$$

$$+ \frac{1}{3}\, \beta_{kj,i} - \frac{1}{2}\, (\alpha_j{}^{\ell} + \beta^{\ell}{}_j)_{,\ell}\, \eta_{ik} +$$

$$+ \frac{1}{2} (\alpha_i{}^{\ell}+\beta^{\ell}{}_i)_{,\ell}\, \eta_{jk} = \Theta_{ijk} \approx 0$$

Performing the Poisson bracket between Θ_{ijk} and $\mathcal{H}_T$ yields a differential equation to determine Z_{ij}. We remark that in the m = 0 case there appears first class constraints, which did not appeared in the massive case. This is nothing a surprise once we know that first class constraints is a material evidence of the existence of gauge freedom of the theory.

The second class constraints are

$$\Omega_{ijk} \approx 0$$
$$P_{ijk} \approx 0$$
$$P_{ij} \approx 0$$
$$\Pi_{ij} \approx 0$$
$$\Theta_{ijk} \approx 0$$

Some combination of those second class constraints with its derivative are of first class. This will change the counting of the degrees of freedom. How we arrive at these combinations ? First of all we note that the combination

$$P^{ij}{}_{,j} + \Pi^{ij}{}_{,j} \approx 0$$

is a first class constraint.

Note that P^{ij} and $\Pi^{ij}{}_{,j}$ commutes with all others constraints except with Θ_{ijk}. However the above combination $P^{ij}{}_{,j} + \Pi^{ij}{}_{,j}$ commute also with Θ_{ijk} - and thus with all constraints-showing its first class category. We then divide Θ_{ijk} into two parts.

a) its divergence, $\Theta_{ij}{}^{k}{}_{,k}$

b) a divergenceless part, $F_{ijk}(\Theta_{abc})$ with the following properties:

i) It is a combination of Θ_{ijk} and its derivatives

ii) $F_{ijk}\eta^{ik} = 0$

iii) $F_{ijk}\varepsilon^{ijk} = 0$

and, of course,

iv) $F_{ij}{}^{k}{}_{,k} = 0$

So we divide the 5 constraints $\Theta_{ijk} \approx 0$ in the 3 conditions $\Theta_{ij}{}^{k}{}_{,k} \approx 0$ and the 2 other $F_{ijk} \approx 0$. Our next task is to construct an explicit form for the tensor F_{ijk}.

Let us use the operator

$$\bar{O}^i_j \equiv \delta^i{}_j - \Delta^{-1}\,\partial^i\partial_j$$

where Δ^{-1} is the inverse of the Laplacian operator. The properties of this operator are well known from the theory of electromagnetism. The most general tensor F_{ijk} with properties (i) (ii) and (iii) is

$$F_{ijk} = \Theta_{ijk} + b(\Theta_{ij\ell}\bar{O}^\ell_k - \tfrac{1}{2}\,\Theta_{ki\ell}\bar{O}^\ell_j + \tfrac{1}{2}\,\Theta_{kj\ell}\bar{O}^\ell_i) + c(\Theta_{i\ell k}\bar{O}^\ell_j + \Theta_{k\ell i}\bar{O}^\ell_j - \Theta_{j\ell k}\bar{O}^\ell_i - \Theta_{k\ell j}\bar{O}^\ell_i) - (\tfrac{3b+2c}{4})\,\eta_{k[i}\Theta^m{}_{j]\ell}\,\bar{O}^\ell_m$$

Property (iv) determine the coefficients (b) and (c).

So, F_{ijk} is given by:

$$F_{ijk} = -2\Theta_{ijk} + \Theta_{ij\ell}\bar{O}^\ell{}_k + \Theta_{i\ell k}\bar{O}^\ell{}_j - \Theta_{j\ell k}\bar{O}^\ell{}_i - \eta_{a[i}\Theta^m{}_{j]\ell}\,\bar{O}^\ell{}_m$$

or,

$$F_{ijk} = \Theta_{ijk} + \Delta^{-1}(-\Theta_{ij}{}^{\ell}{}_{,\ell,k} - \Theta_{i}{}^{\ell}{}_{k,\ell,j} + \Theta_{j}{}^{\ell}{}_{k,\ell,i} + \eta_{k[i}\Theta^{m}{}_{j]}{}^{\ell}{}_{,m,\ell})$$

Obviously F_{ijk} has zero Poisson bracket with $\Pi^{ij}{}_{,j}+P^{ij}{}_{,j}$ and with $\Theta_{ij}{}^{k}{}_{,k}$. If we calculate the Poisson bracket of F_{ijk} with P^{ij} we obtain

$$[F_{ijk}, P^{ab}] = 0$$

We then conclude that F_{ijk} is a first class constraint. Note furthermore that F_{ijk} commutes also with H_T

$$[F_{ijk}, H_T] = 0 \qquad .$$

Let us now finally count the degrees of freedom which are removed by the constraints:

$\Pi \approx 0$	2
$\Pi^{i} \approx 0$	6
$P^{i} \approx 0$	6
$\Pi^{ijk}{}_{,k} \approx 0$	6
$F_{ijk} \approx 0$	4
$\Theta_{ij}{}^{k}{}_{,k} \approx 0$	3
$\Pi^{ij}{}_{,j}+P^{ij}{}_{,j}$	6
$P^{ij} \approx 0$	3
$\lambda_{ijk} \approx 0$	5
$P_{ijk} \approx 0$	5

Adding all these numbers we obtain 46. The true degrees of freedom are then $\frac{50-46}{2} = 2$

as it should be for a massless spin two field.

We note finally that the conservation in time of $\Theta_{ijk}{}^{k}{}_{,k}$ gives rise to three differential equations for the three unknown Z_{ij}.

III - THE EQUATION OF MOTION OF SPIN-TWO FIELD IN CURVED (RIEMANNIAN) SPACE-TIME

One of the attractive feature of the theory we are presenting here is the property that its generalization to arbitrary Riemannian space-time is straightforward. In the standard representation the generalization of eq. (19) - when passing to curved space-time - introduces ambiguities, even if one adheres to the minimal coupling principle (MCP)[5,6,7,8]. The simplest way to see this is just by noting that eq. (1) contains second order derivatives (which appears also in the Lagrangian function see Table I). This causes no problem in Minkowskii space-time. However non-commutativity of covariant derivatives, in an arbitrary curved space introduces a factor ordering ambiguity. This does not occur in the Fierz representation, as we will now show. Indeed, from the potential $A_{\alpha\beta\mu}(x)$ we define the field $C_{\alpha\beta\mu\nu}$ by the expression suggested by the application of the MCP:

$$
(89)\qquad C^{\alpha\beta\mu\nu} = A_{\alpha\beta[\mu;\nu]} + A_{\mu\nu[\alpha;\beta]} + \frac{1}{2} A_{(\alpha\nu)}\, g_{\beta\mu} + \frac{1}{2} A_{(\beta\mu)} g_{\alpha\nu} - \frac{1}{2} A_{(\beta\nu)} g_{\alpha\mu} - \frac{1}{2} A_{(\alpha\mu)} g_{\beta\nu} + \frac{2}{3} A^{\sigma\lambda}{}_{\sigma;\lambda}\, (g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu})
$$

in which a semi comma (;) represents covariant differentiation.

Use of MCP in eq. (7) and (12) gives the eq. of motion

$$
(90)\qquad C^{\alpha\beta\mu\nu}{}_{;\nu} + m^2 A^{\alpha\beta\mu} = 0
$$

and for the Lagrangian

$$L = \sqrt{-g}\ \{-\tfrac{1}{8}\ C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu} + \tfrac{M^2}{2}\ A_{\alpha\beta\mu}A^{\alpha\beta\mu}\ \}\ . \tag{91}$$

We emphasize that there is no possibility of ambiguity in these expressions. We recover the same situation as in the theory of spin-one field. From the equations of motion we arrive at the modified compatibility conditions (see eq. (9)):

$$A^{\alpha\beta\mu}g_{\beta\mu} = 0 \tag{92a}$$

$$A^{\alpha\beta\mu}{}_{;\mu} = -\frac{1}{2M^2}\ W^{[\alpha\mu\nu\lambda}C^{\beta]}{}_{\mu\nu\lambda} \tag{92b}$$

in which $W^{\alpha\mu\nu\lambda}$ is the conformal Weyl tensor of the curved background geometry.

We know from our previous flat space-time analysis that in order to impose extra condition on $A_{\alpha\beta\mu}$ we must look into $\overset{*}{A}{}^{\alpha\beta\mu}{}_{;\beta}$. Beware however of the non-comutativeness of this covariant derivative.

From the tensorial identity

$$\overset{*}{A}{}^{\alpha\beta\mu}{}_{;\beta} = \frac{1}{6}\ \eta^{\rho\varepsilon\alpha\beta}\ \{A_{\rho\varepsilon}{}^{\mu}{}_{;\beta} + A_{\varepsilon\beta}{}^{\mu}{}_{;\rho} + A_{\beta\rho}{}^{\mu}{}_{;\varepsilon}\ \} \tag{93}$$

in which $\eta^{\rho\varepsilon\alpha\beta} = -\frac{1}{\sqrt{-g}}\ \varepsilon^{\rho\varepsilon\alpha\beta}$ and $g = \det g_{\mu\nu}$ we conclude that we cannot transport Fierz condition (9) to curved space-time. Indeed, calling $M^{\alpha\mu}$ the divergence

$$M^{\alpha\mu} \equiv \overset{*}{A}{}^{\alpha\beta\mu}{}_{;\beta} \tag{93,1}$$

we obtain

$$A_{\tau\lambda}{}^{\mu}{}_{;\varepsilon} + A_{\lambda\varepsilon}{}^{\mu}{}_{;\tau} + A_{\varepsilon\tau}{}^{\mu}{}_{;\lambda} = M^{\alpha\mu}\eta_{\alpha\varepsilon\lambda\tau}\quad . \tag{93,2}$$

Taking the contraction $\mu = \lambda$ and using (92,a)

(93,3) $$A^{\varepsilon\tau\mu}{}_{;\mu} = -\overset{*}{M}{}^{\varepsilon\tau} + \overset{*}{M}{}^{\tau\varepsilon}$$

using (92b) we obtain

(93,4) $$M_{[\varepsilon\tau]} = -\frac{1}{4M^2}\,\eta_{\varepsilon\tau\rho\sigma}\,W^{[\rho\alpha\beta\lambda}C^{\sigma]}{}_{\alpha\beta\lambda}$$

Just as in the flat space-time case, the relation between $M_{[\varepsilon\tau]}$ and $A^{\varepsilon\tau\mu}{}_{;\mu}$ implies that only the symmetric part $M_{(\varepsilon\tau)}$ can be counted independently (of $A^{\varepsilon\tau\mu}{}_{;\mu}$). However this symmetric part gives not 10 arbitrary extra condition but only 5. The counting is simple: from the 10 components that the symmetric $M_{(\varepsilon\tau)}$ has we must extract the trace(10-1=9)- which has been already counted, and the divergence identity

(93,5) $$A^{\overset{*}{\alpha}\beta\mu}{}_{;\beta;\mu} = -\frac{1}{2M^2}\,\eta^{\alpha\beta\rho\sigma}(W_\rho{}^{\mu\nu\lambda}C_{\sigma\mu\nu\lambda})_{;\beta} + \frac{1}{2}\,\overset{*}{W}_{\varepsilon\mu\rho\alpha}A^{\varepsilon\mu\rho}$$

which takes 4 conditions, leaving 9-4=5 as in flat space-time. The proof of (93,5) is straightforward:

From the definition $J^\alpha \equiv A^{\overset{*}{\alpha}\beta\mu}{}_{;\beta;\mu}$ we have

(93,6) $$J^\alpha = \frac{1}{2}\,\eta^{\alpha\beta}{}_{\rho\sigma}[A^{\rho\sigma\mu}{}_{;\mu;\beta} + R_{\rho\varepsilon\beta\mu}A^{\varepsilon\sigma\mu} + R^{\sigma\varepsilon}{}_{\beta\mu}A^{\rho\varepsilon\mu} + $$

$$- R_{\varepsilon\beta}\,A^{\rho\sigma\varepsilon}]$$

using (92b)

(93,7) $$J^\alpha = -\frac{1}{2M^2}\,\eta^{\alpha\beta\rho\sigma}(W_{\rho\mu\nu\lambda}C_\sigma{}^{\mu\nu\lambda})_{;\beta} + 2R^{\varepsilon\overset{*}{\alpha}\sigma\mu}A_{\varepsilon\sigma\mu} - R_{\varepsilon\beta}A^{\overset{*}{\alpha}\beta\varepsilon}$$

From the properties of the dual operador, we have the identity

(93,8) $$R_{\mu\overset{*}{\alpha}\beta\nu} = -\frac{1}{2}\,R_{\mu\nu\overset{*}{\alpha}\beta} \quad ,$$

and using the decomposition of $R_{\alpha\beta\mu\nu}$ into its irreducible parts we can write

$$R_{\varepsilon\mu\overset{*}{\sigma\alpha}}A^{\varepsilon\mu\sigma} = \overset{*}{W}_{\varepsilon\mu\sigma\alpha}A^{\varepsilon\mu\sigma} + \frac{1}{4}\,\eta_{\sigma\alpha}{}^{\tau\rho}M_{\varepsilon\mu\tau\rho}A^{\varepsilon\mu\sigma} - \frac{1}{6}\,R\,\eta_{\varepsilon\mu\sigma\alpha}A^{\varepsilon\mu\sigma}$$

with

$$M_{\varepsilon\mu\tau\sigma} = R_{\varepsilon\tau}g_{\mu\sigma} + R_{\mu\sigma}g_{\varepsilon\tau} - R_{\varepsilon\sigma}g_{\mu\tau} - R_{\mu\tau}g_{\varepsilon\sigma}$$

Finally, collecting all this we arrive at (93,5)

$$J^{\alpha} = -\frac{1}{2M^2}\,\eta^{\alpha\beta\rho\sigma}(W_{\rho\mu\nu\lambda}C_{\sigma}{}^{\mu\nu\lambda})_{;\beta} + \frac{1}{2}\,\overset{*}{W}_{\varepsilon\mu\sigma}{}^{\alpha}A^{\varepsilon\mu\sigma}$$

This accomplishes the task of providing a non-ambiguous coherent set of equations for massive spin-two field in curved space-time.

Let us now turn to the description of the present theory in terms of the ancient (standard) variables.

The Equivalence

The simplest way to obtain the equivalence formulas is just to note that eq. (11a,b) can be interpreted as general expressions in coordinates adapted to arbitrary frames. Then, we can set

(94a) $$A_{\mu\varepsilon\nu} = \psi_{\nu[\mu;\varepsilon]} + B\,\psi_{,[\mu}g_{\varepsilon]\nu} - B\psi_{[\mu}{}^{\alpha}{}_{;\alpha}g_{\varepsilon]\nu}$$

and, conversely,

(94b) $$\psi_{\mu\nu} = -\frac{1}{2M^2}A_{(\mu}{}^{\varepsilon}{}_{\nu);\varepsilon} + \frac{1}{2M^2}\left(\frac{1-B}{1-3B}\right)A_{(\mu}{}^{\varepsilon}{}_{\varepsilon;\nu)} - \frac{1}{3M^2}A^{\alpha\beta}{}_{\beta;\alpha}\,g_{\mu\nu} \quad .$$

Compatibility of these two expressions gives the eq. of motion in the $\psi_{\mu\nu}$ variable:

(95) $$\Box\,\psi_{\mu\nu} - \psi_{(\mu}{}^{\varepsilon}{}_{;\nu);\varepsilon} + \psi_{,\mu;\nu} + g_{\mu\nu}(\psi^{\alpha\beta}{}_{;\alpha;\beta} - \Box\,\psi) + R_{\mu\alpha\nu\beta}\psi^{\alpha\beta} - \frac{1}{2}R_{\alpha(\mu}\psi^{\alpha}{}_{\nu)} + M^2(\psi_{\mu\nu}-\psi g_{\mu\nu}) = 0$$

or, equivalently,

(95)' $$\Box\,\psi_{\mu\nu} - \psi_{(\mu}{}^{\varepsilon}{}_{;\varepsilon;\nu)} + \psi_{,\mu;\nu} + g_{\mu\nu}(\psi^{\alpha\beta}{}_{;\alpha;\beta} - \Box\,\psi) - R_{\mu\alpha\nu\beta}\,\psi^{\alpha\beta} + \frac{1}{2}R_{\alpha(\mu}\psi^{\alpha}{}_{\nu)} + M^2(\psi_{\mu\nu}-\psi g_{\mu\nu}) = 0 \quad .$$

Einstein's Theory in Jordan-Lanczos-Lichnerowicz Framework

Let us pause for a while and summarize what we have achieved. We have shown that in flat space-time it is completely equivalent to use standard $\psi_{\mu\nu}$ or Fierz $A_{\alpha\beta\mu}$ variables to describe spin two fields,massive and massless. We have extended this for curved space-time in the case $m \neq 0$. Now it is time to answer the question: what about the $m = 0$ case ? Or, in other words, what about non-weak gravity in Fierz variables ?

The road to a complete answer to this question pass through two independent stages : (i) The Jordan formulation of General Relativity; (ii) The Lanczos extension of the Fierz variables.

Let us examine both in this order.

The starting point of Jordan formulation of Einstein's GR [9] is the existence of Bianchi identities for the riemannian geometry.

These identities can be written in two equivalent ways [10] , either in the form

(96a) $$R^{\alpha\beta\mu\nu}{}_{;\nu} = R^{\mu[\alpha;\beta]}$$

or using Weyl conformal tensor

(96b) $$W^{\alpha\beta\mu\nu}{}_{;\nu} = \frac{1}{2} R^{\mu[\alpha;\beta]} - \frac{1}{12} g^{\mu[\alpha}R^{,\beta]}$$

The Weyl conformal tensor $W_{\alpha\beta\mu\nu}$ is given by the traceless part of the curvature, that is

$$(97) \qquad W_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} + \frac{1}{2} R_{\beta\nu}g_{\alpha\mu} + \frac{1}{2} R_{\alpha\mu}g_{\beta\nu} - \frac{1}{2} R_{\alpha\nu}g_{\beta\mu} - \frac{1}{2} R_{\beta\mu}g_{\alpha\nu} - \frac{1}{6} R(g_{\alpha\mu}g_{\beta\nu}-g_{\alpha\nu}g_{\beta\mu}) \quad .$$

The idea of Jordan and his collaborators was to search for a new set of equations - similar in its form to (96b) - such that the right-hand side is identified to a current:

$$(98) \qquad W^{\alpha\beta\mu\nu}{}_{;\nu} = J^{\alpha\beta\mu} \quad .$$

If we intend to make an ulterior contact with Einstein's GR then the current should be constructed only in terms of the energy-momentum tensor $T_{\mu\nu}$.

Then, the hypothesis has been made that $J_{\alpha\beta\mu}$ admits a decomposition in the very convenient form

$$(99) \qquad J_{\alpha\beta\mu} = -\frac{k}{2} T_{\mu[\alpha;\beta]} + \frac{k}{6} g_{\mu[\alpha}T_{,\beta]}$$

in which constant k of dimension $(\text{energy})^{-1}(\text{length})$ will be identified, a posteriori, to Newton's constant (in natural units).

Although the leitmotiv that legitimated Jordan et al. to propose the current (99) was not exhibited in their work, one is led to suspect, after the precedent sections of our paper that it is indeed the most natural extension, for curved space-time, of the coupling of matter with gravity by the intermediation of a (then hidden) third order potential $L_{\alpha\beta\mu}$.

Let us now show how this suggestion can become real.

The argument, due originally to Lichnerowicz, goes in the following way.

Consider the simultaneous validity of (96b) and (98,99). If we note that in passing from quasi-Minkowskian space-time to a curved one the standard variable $\psi_{\mu\nu}$ become the metric $g_{\mu\nu}$, this eq. needs as Cauchy data a set of second order derivatives of $g_{\mu\nu}$.

We take Einstein's equation to be valid in a given initial hypersurface Σ [the reader should compare our reasoning here with that in flat space-time presented after eq. (21)], that is,

$$(100) \qquad (R^{\mu}{}_{\nu} - \frac{1}{2} R\delta^{\mu}{}_{\nu} + kT^{\mu}{}_{\nu})_{\Sigma} = 0$$

It then follows that the set of equations (96,98) propagates this condition beyond the hypersurface Σ, showing the validity of Einstein's equations throughout the whole space-time [11].

After what we have shown up to now we can state the following: the identification of the theory of gravity as mediated by a massless spin two field can be described either in standard variables $\psi_{\mu\nu}$ or in Fierz variables $A_{\mu\nu\lambda}$. The first choice yields the Einstein description; and the second one leads to Jordan's scheme of General Relativity.

The final piece of demonstration of this assertion will be done next by the identification of Fierz variables in Jordan's description.

Lanczos Potential

We saw previsously how to relate Fierz variables with the standard ones in the case of weak gravity (see eq.(38)). However the crucial question on know how this could be extrapolated for strong fields has been a challenge for scientists for a long period. It was only in 1962, with the work of C. Lanczos that this problem was settled in a definite and unsuspected way [12,13, 14,15].

The knowledge obtained in the previous sections - dealing with flat space-time - makes us suspect that in the massless limit the bridge formuli (which allows a complete algorithm of passage from one representation to the other) do not exist. That this is indeed true was shown only recently by the work of Lanczos [12] and later on by Bampi and Caviglia [14]. They showed that a potential $L_{\alpha\beta\mu}$ can be constructed, in any Riemannian geometry, for the Weyl conformal tensor (but not for the full curvature tensor except in some special case). Thus, besides formula (97), the Weyl tenser admits a representation in terms of a third order tensor $L_{\alpha\beta\mu}$ given by

(101) $$W_{\alpha\beta\mu\nu} = L_{\alpha\beta[\mu;\nu]} + L_{\mu\nu[\alpha;\beta]} + \frac{1}{2} L_{(\alpha\nu)} g_{\beta\mu} + \frac{1}{2} L_{(\beta\mu)} g_{\alpha\nu} - \frac{1}{2} L_{(\beta\nu)} g_{\alpha\mu} - \frac{1}{2} L_{(\alpha\mu)} g_{\beta\nu} + \frac{2}{3} L^{\sigma\lambda}{}_{\sigma;\lambda} (g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu})$$

in which

(102a) $$L_{\alpha\beta\mu} = - L_{\beta\alpha\mu}$$

(102b) $$L_{\alpha\beta\mu} + L_{\beta\mu\alpha} + L_{\mu\alpha\beta} = 0 \quad .$$

Although $L_{\alpha\beta\mu}$ has 20 degrees of freedom $W_{\alpha\beta\mu\nu}$ has only 10. This shows that there are 10 gauge symmetries hideen in the new variable. This is analogous to the case in Minkowskii space--time (eq. (40,42)) - generalized for curved space-time.

The Lanczos potential has been considered since its introduction in 1962 until today, a very special and isolated property of the traceless part of the curvature tensor (Weyl tensor). From what we have learned in the previous sections we can recognize now its real meaning and its natural historical context: it is nothing but the form Fierz variable takes in the Einstein's geometric description of massless spin-two field.

The Dynamics

[Choosing the Fluctuations of the Geometric Quantities]

Once decided to work with Lanczos potential we are guided almost uniquely to set up a theory in which the dynamics is provided by the action

$$(103)\qquad \delta S_E = \int \sqrt{-g}\; W_{\alpha\beta\mu\nu} W^{\alpha\beta\mu\nu}\, d_4x \quad .$$

We then face a new problem. As a consequence of the description of the gravitational field in terms of geometric quantities, not only $L_{\alpha\beta\mu}$ but also $g_{\mu\nu}$ are to be varied in the action S_E. We cannot left $L_{\alpha\beta\mu}$ to undergo arbitrary fluctuations and keep $g_{\mu\nu}$ frozen - unless we restrict strongly ourselves to a very particular model. We will not explore this restricted case here.

Besides, as pointed out by Lanczos [12], "$L_{\mu\nu\lambda}$ is reducible to the metric tensor $g_{\mu\nu}$ only by an integral operation, i.e., the value of $L_{\mu\nu\lambda}$ depends globally on the geometry of the menifold, and yet the tensor $L_{\mu\nu\lambda}$ participates locally in the formation of the field equations."

Thus we should have

$$(104)\qquad \delta S_E = \int \sqrt{-g}\; [M^{\alpha\beta\mu}\delta L_{\alpha\beta\mu} + N^{\mu\nu}\delta g_{\mu\nu}]\; d_4x \quad .$$

In order to develop a definite form of the variational principle to yield a coherent general theory we must provide a correlation between the perturbations $\delta L_{\mu\nu\lambda}$ and $\delta g_{\mu\nu}$. Each choice of s h dependence entails a given model for the evolutionary equ ons of the field.

In the search of such relation we will be guided by the principle - whose roots are to be found in the bridge formuli (94) - which states that the evolution of perturbation $\delta g_{\mu\nu}$ is to be identified with the equation of motion of a massive spin-two field undulating in an arbitrary background Riemannian space-time, provided by eq. (95)(*). There remains to fix the value of the mass, which could depend either on the background geometry (e.g. $m \sim \sqrt{R}$) or on some independent universal constant of gravity theory (e.g. $m^2 \sim \frac{1}{k}$).

Let us examine here the consequences of making the second choice and set $m^2 = \frac{1}{2k}$.

Following eq. (94) we set:

(105) $$\delta g_{\mu\nu} = -k\ [\delta L_{(\mu\alpha\nu);\lambda}]g^{\alpha\lambda} - \frac{2}{3}\ k\ [\delta L_{\alpha\beta\lambda;\rho}]g_{\mu\nu}g^{\beta\lambda}g^{\alpha\rho}$$

The total Lagrangian

(106) $$\mathcal{L}_T = \sqrt{-g}\ [-\frac{1}{8}\ W^{\alpha\beta\mu\nu}W_{\alpha\beta\mu\nu} + \mathcal{L}_M]$$

with

$$\delta \int \sqrt{-g}\,\mathcal{L}_M = \int \frac{1}{2}\sqrt{-g}\ T^{(m)}_{\mu\nu}\,\delta g^{\mu\nu} \quad .$$

(*) This proposal is not a new one. Some years ago V.L. Ginzburg et al. have examined the problem of describing dynamically arbitrary fluctuations (not necessarilly small ones) $\Delta g_{\mu\nu}$ over an average field $\langle g_{\mu\nu}\rangle$. Non-linearity of Einstein's equations is responsible for the possibility that $\langle g_{\mu\nu}\rangle$ and the total (observed) metric $g_{\mu\nu} = \langle g_{\mu\nu}\rangle + \Delta g_{\mu\nu}$ do not satisfy the same equation. Ginzburg et al. [16], Novello [17], and others have proposed alternative set of equations to describe the average geometry $\langle g_{\mu\nu}\rangle$, when the total metric $g_{\mu\nu}$ satisfies Einstein's equations.

Then for the total action:

$$S_T = \int \mathcal{L}_T \, d_4x$$

(107) $$\delta S_T = \int \sqrt{-g} \, \{W^{\alpha\beta\mu\nu}{}_{;\nu} \, \delta L_{\alpha\beta\mu} + \frac{1}{2} T_{\mu\nu} \, \delta g^{\mu\nu}\}$$

in which $T_{\mu\nu}$ contains the contribution of $T^m_{\mu\nu}$(matter) and $\chi_{\mu\nu}$ comes from the quadratic Lagrangian of gravity

(108) $$T_{\mu\nu} = T^{(m)}_{\mu\nu} + \chi_{\mu\nu}$$

In this way we can interpret $\chi_{\mu\nu}$ as the energy of the gravitatio nal field, given by

(109) $$\chi_{\mu\nu} = U_{\mu\nu}{}^{\lambda}{}_{;\lambda} + W_{\mu\varepsilon\rho\nu} \, L^{(\varepsilon\rho)} - \frac{1}{8} W^2 \, g_{\mu\nu}$$

in which (compare with (46))

$$U_{\mu\nu}{}^{\lambda} = W_{(\mu}{}^{\varepsilon\rho\lambda} L_{\nu)\varepsilon\rho} + W_{(\mu}{}^{\varepsilon\rho\lambda} L_{\nu)\rho\varepsilon} - W_{(\mu}{}^{\rho\sigma}{}_{\nu)} \, L^{\lambda}{}_{\rho\sigma}$$

Thus, finally, from $\mathcal{L}_T$ using (105) into (107) we obtain:

(110) $$W^{\alpha\beta\mu\nu}{}_{;\nu} = - \frac{k}{2} \, T^{\mu[\alpha;\beta]} + \frac{k}{6} \, g^{\mu[\alpha}T^{,\beta]}$$

with $T_{\mu\nu}$ given by eq. (108).

This system of equations is just the Jordan-Lichnerowicz system with the peculiarity that here there is an extra term for the source $T_{\mu\nu}$ which comes from the geometry.

IV - USES OF THE NEW VARIABLES $A_{\mu\nu\lambda}$ BEYOND GENERAL RELATIVITY

In the early seventies many scientists have analysed the properties of a theory which contains a massive graviton [18, 19, 20, 21,22]. This so called finite-range gravitation suffered for either being non-covariant, dealing with two "metrical" fields or by the addition of a scalar field ϕ which should not be incorporated in the standard variable $g_{\mu\nu}(\lambda)$. These difficulties can be overcame if we abandon the standard variable $\psi_{\mu\nu}$ and use the Fierz variable $A_{\mu\nu\lambda}$. Indeed, we have been analysing in the previous section the equation of motion of a massive spin two field given by

$$C^{\alpha\beta\mu\nu}{}_{,\nu} + m^2 A^{\alpha\beta\mu} = 0 \tag{111}$$

or, in a general curved background,

$$C^{\alpha\beta\mu\nu}{}_{;\nu} + m^2 A^{\alpha\beta\mu} = 0 \tag{112}$$

In either case, these equations are self-consistent and make no appeal to extra fields or to two metric variables. Besides this two simple trivial uses, Fierz variables appears of much more natural use in different contexts. For instance consider the so called f-dominance of gravity, a theory which has been invented by Isham, Salam and Strathdee in the early seventies [23,24,25]. This theory proposed to translate to the gravitational world a model of hadron electrodynamics. The idea developed was that the photon interacts directly with leptons but only indirectly with hadrons via a simple ρ^0-γ mixing — in which ρ^0 is a vector field (actually a ρ-ω-ϕ complex). The success of such theory led Isham et al. to propose a similar model for gravity: this was the f-g theory which treated $f_{\mu\nu}$ and $g_{\mu\nu}$ as two metric fields obeying equations of motion of Einstein's type corresponding formally to the strong gravity metric $f_{\mu\nu}$ and the general metric $g_{\mu\nu}$.

It is a simple but elucidative exercise of the advantages of the use of Fierz variables to translate this f-g theory in our new formalism. We do not present this model here but proceed instead to a new application of the Fierz variables in a somewhat distinct but much more comprising model which can be synthesized in the search of a definite answer to the single question:

<u>Does There Exists a Gravitational Analogue of the Electro-Weak Unification ?</u>

We have seen in the precedent sections that the use of Fierz variable $A_{\mu\nu\lambda}$ (instead of the standard one $\psi_{\mu\nu}$) to describe spin-2 field gives much more transparency into the similitude that exists between electrodynamics and gravity. If we follow this road of analogies further on we will be guided to the suggestive conclusion that there should exists a new short range force. How this could occur ?

Algebraic Structure

In the seventies a theory gained an important role in elementary particle physics, which proposed the modification of weak and electromagnetic interactions [26,27]. This successful scheme aimed the unification of a set of three massive vector $(W^{+}_{\mu}, W^{-}_{\mu}, W^{0}_{\mu}) \equiv W^{(i)}$ which mediate weak processes, and the photon (A_μ) which mediate electromagnetic processes. We could roughly say that electromagnetic forces acquired a short range counterpart described in an unique algebraic scheme by a gauge theory. The establishment of these extra short range fields as companions of the photon provokes naturally the question: should the gravitational interaction have a short-range counterpart too ? We will contemplate here this hypothesis and examine the consequences of a theory in which gravity has a distinguished local counter-part represented by a new force mediated by massive spin 2 fields - which we will call, just for future reference , the RIO-Force.

To simplify our presentation we consider the case of weak gravity that is, when the space-time is in the quasi--Minkowskian regime. The extension for arbitrary curved space time was done in Novello-Heintzmann (Proceed. IV Marcel Grossmann Meeting (ed. Ruffini) 1986). Thus, as we saw previously, the gravitational field can be coherently and completely described in terms of Fierz variable, i.e., a third order tensor. Besides this - and by analogy to the case of electrodynamics - we introduce a set of three objects $A^{(i)}_{\alpha\beta\mu}$ in which the index (i) is an SU-2 index. The theory thus contains 8 fundamental fields, separated in two sectors: the vector sector $A^{(i)}_\alpha$, B_μ and the tensorial sector $A^{(i)}_{\alpha\beta\mu}$, $B_{\alpha\beta\mu}$.

The electro-weak unification scheme of Salam-Glashow--Weinberg assumes a local SU(2) symmetry which means that any group transformation becomes a space-time dependent function. The theory then follows the standard rules of the Yang-Mills gauge model, introducing for consistency an internal covariant derivative in which the vectors $A_\mu^{(i)}$ are to be treated as SU(2) connections. This means that an infinitesimal map generated by the space-time dependent function $\eta^{(i)}(x)$ induces on $A_\mu^{(i)}$ a corresponding inhomogeneous change given by the formula [28]

$$\delta A_\mu^{(i)} = \tilde{A}_\mu^{(i)} - A_\mu^{(i)} = -g\varepsilon^{ijk}\,\eta_{(j)}A_{(k)\mu} + \eta^{(i)}_{,\mu} \tag{113}$$

in which g is a constant. Thus for any SU(2) object, say ψ, we define the corresponding covariant derivative by setting

$$\psi_{||\mu} = \psi_{,\mu} - \Gamma_\mu\,\psi \tag{114}$$

in which

$$\Gamma_\mu \equiv \frac{ig}{2}\,A_\mu{}^i\tau_i \tag{115}$$

and τ_i are the Pauli matrices.

The SGW theory deals with spin zero doublets (scalars) ϕ^a and spinors (ψ). From what we have learned, in the preceding sections on the coupling of $A_{\alpha\beta\mu}$ with scalars and spinors, we are conducted to treat the tensorial sector as true vectors of SU(2) and not as connection, leaving this function only to the electro-weak fields. This means that $A^{(i)}_{\mu\nu\lambda}$ changes under the form

$$\delta A^{(i)}_{\mu\nu\lambda} = \tilde{A}^{(i)}_{\mu\nu\lambda} - A^{(i)}_{\mu\nu\lambda} = -g\varepsilon^{ijk}\,\eta_j A_{\mu\nu\lambda(k)} \tag{116}$$

We can then use the definition of the covariant derivative to write

(117) $$A^{(i)}_{\alpha\beta\mu||\nu} = A^{(i)}_{\alpha\beta\mu,\nu} + g\ \varepsilon^{ijk}\ A_{\nu(j)}A_{\alpha\beta\mu(k)}$$

The quantities $A^{(i)}_{\mu}$ and $A^{(i)}_{\alpha\beta\mu}$ are the potentials of the fields $F^{(i)}_{\mu\nu}$ and $C^{(i)}_{\alpha\beta\mu\nu}$ thus defined:

(118) $$F^{(i)}_{\mu\nu} = A^{(i)}_{[\mu,\nu]} - g\ \varepsilon^{ijk}\ A_{\mu(j)}A_{\nu(k)}$$

(119) $$C^{(i)}_{\alpha\beta\mu\nu} = A^{(i)}_{\alpha\beta[\mu||\nu]} + A^{(i)}_{\mu\nu[\alpha||\beta]} + \frac{1}{2}\ A^{(i)}_{(\beta\mu)}\eta_{\alpha\nu} + \frac{1}{2}\ A^{(i)}_{(\alpha\nu)}\eta_{\beta\mu} - \frac{1}{2}\ A^{(i)}_{(\alpha\mu)}\eta_{\beta\nu} - \frac{1}{2}\ A^{(i)}_{(\beta\nu)}\eta_{\alpha\mu} + \frac{2}{3}\ A^{(i)\sigma\lambda}{}_{\sigma||\lambda}\ (\eta_{\alpha\mu}\eta_{\beta\nu}-\eta_{\alpha\nu}\eta_{\beta\mu}) \quad .$$

This acomplishes the task of defining of the algebraic structure of the extended unified theory of the tensorial sector. Let us now go into the dynamics .

Dynamical Structure

We take Glashow-Salam-Weinberg theory as a paradigm and construct an analogous dynamics for the tensorial sector.

For the free part, GSW sets

(120) $$L_I = -\ \frac{1}{4}\ F^{(i)}_{\mu\nu}F^{\mu\nu}_{(i)} - \frac{1}{4}\ B_{\mu\nu}B^{\mu\nu}$$

in which $B_{\mu\nu} \equiv B_{[\mu,\nu]}$.

Correspondingly

(121) $$L_{II} = -\ \frac{1}{8}\ C^{(i)}_{\alpha\mu\nu\beta}\ C^{\alpha\beta\mu\nu}_{(i)} - \frac{1}{8}\ D^{\alpha\beta\mu\nu}D_{\alpha\beta\mu\nu}$$

in which $D_{\alpha\beta\mu\nu}$ is constructed in the standard way, with the potential $B_{\alpha\beta\mu}$. The leptonic part in GSW scheme is given by

(122) $$L_{III} = \frac{g}{2}\,\overline{L}\,\gamma_\mu\,\vec{\tau}L.\vec{A}_\mu - g'(\frac{1}{2}\,\overline{L}\,\gamma_\mu\,L + \overline{R}\,\gamma_\mu R)B^\mu$$

in which $L = \frac{1-\gamma_5}{2}\begin{pmatrix}\nu_e\\ e\end{pmatrix}$ and R is the singlet $R = (\frac{1+\gamma_5}{2})e$.
This part has no correspondent in the tensor sector. This is due to the transparence of the spinor field to the pure tensor part (see the spinorial coupling with $A_{\mu\nu\lambda}$ in section II). Then comes the interaction with a scalar doublet which is the responsible for giving mass to some of the fundamental fields. The GSW model sets

(123). $$L_{IV} = |\partial_\mu\phi - igA^i_\mu\tau_i\phi - \frac{1}{2}\,g'B_\mu\phi|^2 - 2M^2\phi^\dagger\phi + 2h(\phi^\dagger\phi)^2$$

in which $\phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$. The quartic them on ϕ is responsible for allowing the spontaneous symmetry breaking generated by the vacuum expectation value $\langle\phi\rangle = \lambda\binom{0}{1}$.

Then we get for the tensorial sector, in an analogous way,

(124) $$L_V = \frac{1}{6}\,|\partial_\mu\phi\eta_{\nu\lambda} - \partial_\nu\phi\eta_{\mu\lambda} - 3i\tau^i A_{\mu\nu\lambda(i)}\phi + 3iB_{\mu\nu\lambda}\phi|^2$$

We see that in the absence of the tensors, L_V contributes with a term $\overline{\phi}_{,\mu}\phi_{,\nu}\eta^{\mu\nu}$ to the dynamics. This is the origin of the factors 2 in front of M and h in L_{IV}. In the SGW scheme, to recognize the massless photon field, one redefines the vector fields by rotating $A^{(3)}_\mu$ and B_μ through an angle θ such that

$$e = g\,\sin\theta = g'\,\cos\theta$$

in which e is the electric charge. The fact that in the tensorial sector we are dealing with quantities whose dimensionality is $[B_{\alpha\beta\mu}] = [A^{(i)}_{\alpha\beta\mu}] = (\text{length})^{-1}$ (which we assumed in order to introduce no new constants in the theory) associated to the specificity of the theory we are developping imply that the corresponding rotation of the tensorial sector (which allows the characterization of the massless gravitational field) is 45^{o}. Indeed, by setting

$$(125) \qquad \begin{pmatrix} A_{\mu\nu\lambda} \\ Z_{\mu\nu\lambda} \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} B_{\mu\nu\lambda} \\ A^{3}_{\mu\nu\lambda} \end{pmatrix}$$

one obtains from L_V, after the spontaneous symmetry breaking mechanism enters in the action (which has the net result to take in the Lagrangian the scalar doublet to be given by $\phi = \lambda \binom{0}{1}$)

$$(126) \qquad \begin{aligned} m\,(A^{\pm}_{\mu\nu\lambda}) &= \frac{3}{2}\,\lambda \\ m\,(Z_{\mu\nu\lambda}) &= 3\,\frac{\sqrt{2}}{2}\,\lambda \\ m\,(A_{\mu\nu\lambda}) &= 0 \end{aligned}$$

Remark that we are using the minimal hypothesis of universality which states that the scalar field which gives mass for the vector sector is the same which provides masses for the tensors. This enables us to obtain from the vectors the value of the vacuum expectation value of ϕ, that is, λ - yielding the corresponding numerical predictions of the existence of three massive tensors:

$$m\ (A^{\pm}_{\alpha\mu\beta}) = 369 \text{ GeV}$$

$$m\ (Z_{\alpha\beta\mu}) = 520{,}3 \text{ GeV} \quad ,$$

and a massless one $A_{\alpha\beta\mu}$ which we identify with the (linear) graviton . The massive particles ($A^{\pm}_{\alpha\beta\mu}$, $Z_{\alpha\beta\mu}$) are the carriers of the new short range RIO force.

V - CONCLUSION

We have presented in this work a complete theoretical framework to describe spin-two field in terms of Fierz variables $A_{\alpha\beta\mu}$. One of the main results of this is the exhibition of the great formal similitude between the equations of motion of gravity and electrodynamic. Then, one may be conducted to the question: why there was so long an exile, why physicists have abandoned for near fourty years almost completely such Fierz scheme of representation ?

We will find an answer to this question by examining the geometrical interpretation of gravitational phenomena contained in Einstein's conception.

Indeed, if one deals with standard variable $\psi_{\mu\nu}$, then some of the known properties of gravity (e.g., its universality, its coupling with the stress-energy tensor, etc) provokes naturally the transformation of $\psi_{\mu\nu}$ into a geometric quantity, e.g. the metric tensor $g_{\mu\nu}$.

The same does not occur for $A_{\mu\nu\lambda}$, which in the realm of General Relativity is reduced to the Lanczos potential, a quantity that is restrained to be part of the geometry and which can be used to define the metric $g_{\mu\nu}$ only through some non-local operations of rather complicated nature. There were, however, throughout all this period, since the advent of General Relativity some disconfort about this metric description. We can quote, for instance, the inexistence of a true tensor which represents the energy distribution of the gravitational field. Besides this, the peculiar role of gravity in determining the properties of the metric evolution, gives to this theory a very unique status, making difficult the generation of an unified scheme containing, besides gravity, other forces.

Anyone who deals with quantization aspects of gravity has in some way faced the above difficulties.

We have decided to pass to $A_{\mu\nu\lambda}$ variables in a tentative to overcome all those troubles. Although some work has been left for the future, we think that many advantages of our new formalism were put into evidence in this paper. For instance, we can quote the Hamiltonian treatment of section II. As is well known, one of the biggest problem of quantizing Einstein's theory of gravity comes from the very complicated structure of the constraints arising from General Relativity. In fact, those constraints generate the dynamics of Einstein's theory making it hardly treatable. This difficulty led some authors (Ashtekar [29]) to modifications of the basic set of geometric variables, introducing quantities which are combinations of the extrinsic curvature and the spin connection. Curiously, these quantities appears as a sort of potentials for the Weyl conformal tensor. Thus, one should suspect that Ashtekar's proposal may be related intimately with the one presented here.

In our present scheme, as the Hamiltonian analysis in flat Minkowskii space-time shows, the structure of the constraints are easier to solve. Indeed, they are very similar to gauge theories and they do not generate the dynamics of the theory - which becomes a crucial point of simplification.

Note that although we have limited our analysis to flat space-time it seems that such formal simplicity can be transported for more general cases. The proof of this assertion seems to be one of the main tasks for the near future.

Finally, it is worth to mention the role of Fierz variable in the unification program of all forces in Nature. We present such model here for the purpose to exhibit the power of our new treatment.

Let us finish this conclusion with one more remark. As we saw in this paper, the whole scheme generated using $A_{\mu\nu\lambda}$ variables could be related to Einstein's General Relativity only after the work of three scientists: Lanczos (who has shown, although implicitly, how Fierz variables could be related to the standard ones), Jordan (who described the dynamics of gravity in terms of the divergence of Weyl tensor) and Lichnerowicz (who has shown that Jordan's description is nothing but General Relativity displayed in a distinct formalism). We have thus learned from the work of these scientists that in terms of Fierz variables the Einstein's equations become nothing but an initial condition, propagated in an invariant form throughout the space-time.

In the present paper we have shown how those pieces of information from Lanczos, Jordan and Lichnerowicz, when assembled together, can be used to describe a theory of gravity in the realm of Einstein's General Relativity.

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APPENDIX 1

Variational Principle (Alternative Approaches)

Let us consider the Einstein-Hilbert Lagrangian

$$L_{EH} = \sqrt{-g}\ R$$

If we vary $g_{\mu\nu}$ we obtain

$$\delta \int \sqrt{-g}\ R = \int \sqrt{-g}\ (R_{\mu\nu} - \frac{1}{2} Rg_{\mu\nu})\, \delta g^{\mu\nu} d_4 x$$

In the conventional scheme one considers that all variation of the geometry are permissable , restrained by the only condition that $g_{\mu\nu}$ remains riemannian. Then one arrives at Einstein's equation

$$R_{\mu\nu} - \frac{1}{2} Rg_{\mu\nu} = 0 \quad .$$

However, as Lanczos has emphasized some years ago, the sort of the fluctuations are not contained in the Lagrangian but must be added to the variational principle. Indeed, the variational procedure is an algorithm that allows one to obtain a given set of equations from a functional once the variational procedure is given. Different forms of fluctuation yields distinct equations. In the above example, we used Einstein-Hilbert scheme which allows complete freedom of $\delta g_{\mu\nu}$. Let us now consider an alternative procedure.

We will assume that fluctuations of the geometry are restricted to be such that $\delta g_{\mu\nu}$ behaves as a massive spin two field propagating in a curved (arbitrary) geometry of metric $g_{\mu\nu}$, the mass being equal to the Planck mass. This means the restriction the fluctuations to very short range, of the order of the Planck length. In general, it is believed that for a region of dimension L "those virtual histories of field evolution which contribute most to the propagation function in Feynman's path integral are those for which $\delta g \sim \frac{L_{Planck}}{L}$" (J. Wheeler [30]). If we accept this hypothesis, we can employ the precedent bridge formula and write

$$\delta g_{\mu\nu} = \frac{k}{2}\, \delta L_{\alpha(\mu\nu);\varepsilon}\, g^{\alpha\varepsilon} - \frac{k}{3}\, \delta L_{\alpha\beta\lambda;\rho} g_{\mu\nu} g^{\beta\lambda} g^{\alpha\rho}$$

Using this value into the variational formula, we have:

$$\int \sqrt{-g}\,(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu})\, \delta g_{\mu\nu} d_4x =$$

$$= \frac{k}{2} \int \sqrt{-g} \left\{ R^{\nu[\mu,\alpha]} - \frac{1}{6}\, g^{\nu[\mu}R^{,\alpha]} \right\} \delta L_{\mu\alpha\nu} \quad .$$

Assuming that the variation $\delta L_{\alpha\mu\nu}$ are the true independent ones, we obtain

$$\frac{1}{2} R_{\nu[\mu;\alpha]} - \frac{1}{12}\, g_{\nu[\mu}R_{,\alpha]} = 0$$

which , using Bianchi identity (96b) transforms into

$$W^{\alpha\beta\mu\nu}{}_{;\nu} = 0$$

If instead of the free field L_{EH} we take the total Lagrangian with a source term:

$$L = \sqrt{-g}\ [\frac{1}{2k} R + L_M]$$

with $\int \delta \sqrt{-g}\ L_M = \frac{1}{2}\int \sqrt{-g}\ T_{\mu\nu}\delta g^{\mu\nu} d_4x$, we obtain after some algebraic manipulation

$$W^{\alpha\beta\mu\nu}{}_{;\nu} = - \frac{k}{2} T^{\mu[\alpha;\beta]} + \frac{k}{6} g^{\mu[\alpha}T^{,\beta]}$$

which is the set of equations assumed in Jordan's version of GR.

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