

Virtual Processes and Superradiance

in

Spin-Boson Models

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Abstract

We consider spin-boson models composed by a single bosonic mode and an ensemble of N identical two-level atoms. The situation where the coupling between the bosonic mode and the atoms generates real and virtual processes is studied, where the whole system is in thermal equilibrium with a reservoir at temperature β^{-1} . Phase transitions from ordinary fluorescence to superradiant phase in three different models is investigated. First a model where the coupling between the bosonic mode and the j -th atom is via the pseudo-spin operator $\sigma_{(j)}^z$ is studied. Second, we investigate the generalized Dicke model, introducing different coupling constants between the single mode bosonic field and the environment, g_1 and g_2 for rotating and counter-rotating terms, respectively. Finally it is considered a modified version of the generalized Dicke model with intensity-dependent coupling in the rotating terms. In the first model the zero mode contributes to render the canonical entropy a negative quantity for low temperatures. The last two models presents phase transitions, even when only Hamiltonian terms which generates virtual processes are considered.

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1 Introduction

The implementation of new techniques and ideas has led to increase the interest in spontaneous radiation and collective effects of spin-boson models in free space and cavities [1] [2] [3] [4]. This has been a field of very attractive research, where experimental and theoretical progress have emerged, which can be useful in implementing concepts of quantum information and quantum computation in real physical systems [5]. The purpose of the present paper is to investigate different spin-boson systems, and the physical consequences that follow from assuming first that in interaction Hamiltonian of the models we are also considering the counter-rotating terms. In the interaction picture these terms describe non-resonant processes in which the two-level system and the bosonic field are excited or de-excited simultaneously. Second we study a interaction Hamiltonian that generates quite particular virtual processes. We consider in different models the possibility that virtual processes generate a quantum phase transition and also a phase transition at finite temperature from ordinary fluorescence to a superradiant phase characterized by the presence of a condensate.

We consider three different models, in which we assume that a single mode of a bosonic field interacts with an ensemble of N identical two-level atoms. The whole system is in thermal equilibrium with a reservoir at temperature β^{-1} . First, we study a modified version of the model discussed by Chang and Chakravarty, Leggett and others [6] [7] [8], which has been used to analyze dissipation in quantum computers [9] [10]. Owing to the coupling between the two-level systems and a bosonic mode, the zero mode contributes to render the canonical entropy a negative quantity for low temperatures. An *ad hoc* procedure, disregarding the zero-mode can make the model admissible from the thermodynamical point of view. After this procedure it is shown that the partition function is analytic for all temperatures. Therefore there is no phase transition in the model. Second, we study the generalized Dicke model [11] [12] [13] [14]. We introduce different coupling constants between the single mode of the bosonic field and the ensemble of N atoms, g_1 and g_2 , for rotating and counter-rotating terms, respectively. In a situation where only virtual processes contribute, the generalized Dicke model present a second order phase transition from the ordinary fluorescent to the superradiant phase respectively, at some critical temperature β_c^{-1} [15] and also a superradiant phase transition at zero temperature [16] [17], i.e., a quantum phase transition. We finally study the generalized Dicke model where the coupling between the bosonic mode and N two-level atoms is intensity-dependent [18] [19] [20]. The intensity-dependent contribution appears only in the real processes. In the last model we are showing the presence of the quantum phase transition.

The physics of superradiance is well known [21] [22] [23] [24] [25]. Let us suppose N identical two-level atoms prepared in the excited state. For dilute atomic systems, where there is no coupling between the atoms, the N atoms radiate incoherently, where the radiation rate is proportional on the number of the atoms N . Since the non-decay probability is exponential in time, the intensity of emitted light has the time dependence $I(\tau) = I_0 e^{(-\frac{\tau}{\tau_e})}$, where τ_e is the characteristic time for the spontaneous emission. Other characteristic of the ordinary fluorescence is that the radiation pattern of the atoms is isotropic. For an ensemble of atoms in a volume small compared to the emission wavelength, they start to radiate spontaneously much faster and stronger than the ordinary fluorescence case, where the radiation rate becomes quadratic-dependent on the number N of atoms. Other important characteristic of this cooperative process is that the emission has a well defined direction depending upon the geometry of the sample.

We are using path integral approach with the functional integration method to investigate the thermodynamic of the models, which is given by the analytic properties of the partition function in the complex β plane [26] [27] [28]. To study the nonanalytic behavior of the partition function of the models using functional methods two steps are mandatory. First, it is necessary to change the atomic pseudo-spin operators of the models by a linear combination of Grassmann Fermi fields to define fermionic models. Second, the thermodynamic limit ($N \rightarrow \infty$), where N is the number of two-level atoms must be taken. For the first two models that we are interesting, the coupling

between the pseudo-spin operators used to describe the N two-level atoms and the bosonic mode is linear. Consequently the path integral describing the ensemble of N atoms is Gaussian and the integration over the degrees of freedom describing the atoms can be performed exactly.

A summary of results obtained in the present literature in the models is in order. The advantage of the first model is that it allows an exact analytic solution, and presents destruction of quantum coherence without decay of population. Concerning the thermodynamics of the second spin-boson model, i.e., the Dicke model, an important result was obtained by Hepp and Lieb [29]. These authors presented the free energy of the model in the thermodynamic limit. For a sufficiently large value for the coupling constant between the N two-level atoms and the single quantized mode of the bosonic field, there is a second order phase transition from the normal to the superradiant phase. Later, without assuming the rotating-wave approximation and by using a coherent state representation, the study of the stability of the model with an infinite number of bosonic modes was presented [30]. The study of the phase transitions in the Dicke model was presented also by Wang and Hioe [31], where some of the results obtained by Hepp and Lieb were reobtained. The generalized Dicke model, where the counter-rotating terms are also present in the interaction Hamiltonian, was investigated also by Hioe [12], Carmichael et al. [13], Duncan [14] and Li et al. [32]. Hioe studied the thermodynamics of the generalized Dicke model with two different coupling constants using also the coherent states. Li et al. pointed out that the A^2 term has been neglected in many papers, since the presence of such term leads to non occurrence of the quantum phase transition, as was proved by Rzazewski and Wodkiewicz [33] [34]. Carmichael et al. claim that in the limit $N \rightarrow \infty$, the thermodynamical properties of the model are obtained by simply using the expression obtained from the model with only the rotating terms and doubling the coupling constant. As we will see, the thermodynamical properties of the generalized Dicke model deserve a more careful analysis [15].

More recently a bosonization procedure was employed to study the phase transitions in the generalized Dicke model. Employing a Holstein-Primakoff mapping [35] [36] [37], Emary and Brandes [38] [39] were able to express the generalized Dicke model in terms of a two mode-bosonic field. These authors discussed the relation between the quantum phase transition and the chaotic behavior that appear in the model for finite N , where the energy level-spacing statistics changes from Poissonian to one described by the Gaussian ensemble of the random-matrix theory [40] [41]. This chaotic behavior was discussed also in the Jaynes-Cummings model [42] by Graham and Hoherbach [43] [44] and Lewenkopf et al. [45], in the situation where the counter-rotating terms are present in the interaction Hamiltonian [46] [47], since the seminal paper of Milonni et al. [48].

At this point, some remarks concerning the practical realization of the generalized Dicke model in the laboratory is in order. Dimer et al. [49] pointed out that it remains a challenge to provide a physical system where the counter-rotating terms are dominant. In the Jaynes-Cummings model, Cirac et al. [50] discussed the possibility of controlling the relative importance of the counter-rotating terms using a ion trap. The question that arises now is if there is some model where at some range of the physical parameters, the counter-rotating terms are dominant. It is possible to show that in the generalized Dicke model with intensity-dependent coupling in the rotating terms the relative importance of the contributions from the rotating and counter-rotating terms can be controlled by changing the temperature of the thermal bath. Concerning the intensity-dependent coupling model, Buck and Sukumar [18] [19] showed that the Heisenberg equations of motion can be solved exactly and the behavior at finite temperature was analyzed. Finally, Buzek [20] using the rotating-wave approximation also studied the same model, by presenting the time evolution of the model.

This paper is organized as follows. In section II the path integral with functional integral method is applied to study the thermodynamics of a modified version of the model discussed by Chang and Chakravarty, Legget and others. In the section III we study the thermodynamics of the generalized fermionic Dicke model. In section IV we repeat our analysis for a generalized Dicke model by including the intensity-dependent coupling in the terms describing resonant processes. Conclusions are given in section V. In this paper we use the terms environment and reservoir for

a system with a finite number of degrees of freedom and a system with an infinite denumerable or not degrees of freedom, respectively. In the paper we use $k_B = c = \hbar = 1$.

2 The functional integral for the modified fermionic Chang and Chakravarty model

Let us consider a bosonic system S , with Hilbert space $\mathcal{H}^{(S)}$ which is coupled with an ensemble of N two-level atoms, with Hilbert space $\mathcal{H}^{(B)}$. Let us assume that the complete system is in thermal equilibrium with a reservoir at temperature β^{-1} . Let us denote by H_S , H_B and H_I are the Hamiltonians of the bosonic field, the free N two-level atoms, and the interaction between both systems, respectively. The Hamiltonian for the total system can be written as

$$H = H_S \otimes I_B + I_S \otimes H_B + H_I, \quad (1)$$

where I_S and I_B denotes the identities in the Hilbert spaces of the bosonic field and the ensemble of N atoms.

The aim of this section is to study the analytic behavior of thermodynamical quantities, i.e., whether or not the system exhibits a phase transition from normal to superradiant phase at some critical temperature characterized by the presence of a condensate in a model similar to the one introduced by Chang and Chakravarty, Legget and others. Chang and others describes a system of one two-level atom coupled to a reservoir of harmonic oscillators, where the coupling between the reservoir and the atom is done via the pseudo-spin operator σ^z . They studied finite-time radiative processes in the model by assuming that the environment is in thermal equilibrium. A generalization of this model can be achieved introducing N identical two-level atoms and also an arbitrary mode-dependent coupling constant. Since $[H, \sigma_{(j)}^z] = 0$, this model allows for an exact analytic solution. The Hamiltonian of this generalized model reads

$$H = I_R \otimes \frac{\Omega}{2} \sum_{j=1}^N \sigma_{(j)}^z + \sum_k \omega_k a_k^\dagger a_k \otimes I_S + \frac{1}{\sqrt{N}} \sum_{j=1}^N \sum_k (g_{jk} a_k^\dagger + g_{jk}^* a_k) \otimes \sigma_{(j)}^z, \quad (2)$$

where g_{jk} describes the coupling between the j -th two-level atom with the reservoir. As usual, we are using the pseudo-spin operators $\sigma_{(j)}^+$, $\sigma_{(j)}^-$ and $\sigma_{(j)}^z$ which satisfy the standard angular momentum commutation relations corresponding to spin $\frac{1}{2}$ operators. We are also shifting the zero of energy $\frac{1}{2}(\omega_1 + \omega_2)$ for each atom and defining $\Omega = \omega_2 - \omega_1$.

There are different physical situations that can be investigated and they are characterized by the ratio of the distances between the atoms (r_{ij}), which we assume to take fixed positions, and the correlation length L_c of the reservoir, i.e., $\frac{r_{ij}}{L_c} \ll 1$ and $\frac{r_{ij}}{L_c} \gg 1$. First, let us consider that the minimal distance between the atoms is quite large if we compare it to the correlation length of the reservoir. All the terms $\frac{L_c}{r_{ij}}$ for all i, j are nearly vanishing and therefore each atom interacts with its own reservoir. Since the phase transition from the fluorescent to the superradiant emission is a cooperative process involving a collective mode of all the atoms we concentrate our investigations in the limit where all the terms $\frac{r_{ij}}{L_c}$ almost vanish. We consider the situation where the linear dimension of the total atomic system is small compared to the correlation length L_c of the reservoir, therefore the N atoms interact collectively with the reservoir. We can proceed by assuming that for a fixed k -mode of the reservoir all the coupling constants g_{ik} $i = 1, 2, \dots, N$ of Eq. (2) are equal. The interaction between the reservoir and the two-level atoms is given by the collective pseudo-spin operators $J_z = \sum_{j=1}^N \sigma_{(j)}^z$. A simplified model is achieved if we assume that the two-level atoms and the bosonic modes are in the interior of a high- Q cavity. Assuming that

the frequency ω_0 of one of the cavity modes is near-resonant with the energy gap Ω of the two-level atoms, such as situation generates the following physical model: the two-level atoms effectively interact only with that mode, and all the other bosonic modes do not couple with the two-level atoms. Under these circumstances the model is reduced to a single mode of the bosonic field with the creation and annihilation operators b^\dagger and b , respectively, interacting with an ensemble of atoms.

Let us define the Fermi raising and lowering operators α_i^\dagger , α_i , β_i^\dagger and β_i , that satisfy the anti-commutator relations $\alpha_i\alpha_j^\dagger + \alpha_j^\dagger\alpha_i = \delta_{ij}$ and $\beta_i\beta_j^\dagger + \beta_j^\dagger\beta_i = \delta_{ij}$. We can also define the following bilinear combination of Fermi operators, $\alpha_i^\dagger\alpha_i - \beta_i^\dagger\beta_i$, $\alpha_i^\dagger\beta_i$ and finally $\beta_i^\dagger\alpha_i$. Note that $\sigma_{(i)}^z$, $\sigma_{(i)}^+$ and $\sigma_{(i)}^-$ obey the same commutation relations as the above bilinear combination of Fermi operators. Therefore, we can change the pseudo-spin operators of the spin-boson models by using the bilinear combination of Grassmann Fermi fields

$$\sigma_{(i)}^z \longrightarrow (\alpha_i^\dagger\alpha_i - \beta_i^\dagger\beta_i), \quad (3)$$

$$\sigma_{(i)}^+ \longrightarrow \alpha_i^\dagger\beta_i, \quad (4)$$

and finally

$$\sigma_{(i)}^- \longrightarrow \beta_i^\dagger\alpha_i. \quad (5)$$

Using the above results, the Euclidean fermionic action for the model that we have defined above, can be written as

$$S = \int_0^\beta d\tau \left(b^*(\tau) \frac{\partial b(\tau)}{\partial \tau} + \sum_{i=1}^N \left(\alpha_i^*(\tau) \frac{\partial \alpha_i(\tau)}{\partial \tau} + \beta_i^*(\tau) \frac{\partial \beta_i(\tau)}{\partial \tau} \right) \right) - \int_0^\beta d\tau H_F(\tau), \quad (6)$$

where H_F is given by

$$\begin{aligned} H_F &= \omega_0 b^*(\tau)b(\tau) + \frac{\Omega}{2} \sum_{i=1}^N \left(\alpha_i^*(\tau)\alpha_i(\tau) - \beta_i^*(\tau)\beta_i(\tau) \right) \\ &+ \frac{g}{N} \sum_{i=1}^N \left(\left(\alpha_i^*(\tau)\alpha_i(\tau) - \beta_i^*(\tau)\beta_i(\tau) \right) \left(b(\tau) + b^*(\tau) \right) \right). \end{aligned} \quad (7)$$

Usually, in spin-boson models, the coupling goes with $\frac{1}{\sqrt{N}}$. Since this model has no crossed terms of α_i and β_i , we are using a $\frac{1}{N}$ dependence in the coupling constant. We interpret such change only as a coupling constant renormalization. We can also justify the change adopted using the Holstein-Primakoff mapping, where we obtain a divergent Hamiltonian if we use the original coupling $\frac{1}{\sqrt{N}}$. With the $\frac{1}{N}$ dependence in the coupling constant the Hamiltonian of the system is well behaved.

Let us define the formal quotient of two functional integrals, i.e., the partition function of the interacting model and the partition function of the free model. Therefore, we are interested in to calculate the following quantity

$$\frac{Z}{Z_0} = \frac{\int [d\eta] e^S}{\int [d\eta] e^{S_0}}, \quad (8)$$

where $S = S(b, b^*, \alpha, \alpha^\dagger, \beta, \beta^\dagger)$ is the Euclidean action given by Eq. (6), $S_0 = S_0(b, b^*, \alpha, \alpha^\dagger, \beta, \beta^\dagger)$ is the free Euclidean action for the free single bosonic mode and the free two-level atoms and finally $[d\eta]$ is the path integral measure. In Eq. (8) we have functional integrals with respect to the complex functions $b^*(\tau)$ and $b(\tau)$ and Grassmann Fermi fields $\alpha_i^*(\tau)$, $\alpha_i(\tau)$, $\beta_i^*(\tau)$ and $\beta_i(\tau)$. Since we are using thermal equilibrium boundary conditions in the imaginary time formalism [51] [52] [53] [54], the integration variables in Eq. (8) obey periodic boundary conditions for the Bose

field, i.e., $b(\beta) = b(0)$, and anti-periodic boundary conditions for the Grassmann Fermi fields i.e., $\alpha_i(\beta) = -\alpha_i(0)$ and $\beta_i(\beta) = -\beta_i(0)$.

In order to obtain the effective action of the bosonic mode we must integrate over the Grassmann Fermi fields. Therefore, let us define the free action of the bosonic field by

$$S_0(b) = \int_0^\beta d\tau \left(b^*(\tau) \frac{\partial b(\tau)}{\partial \tau} - \omega_0 b^*(\tau) b(\tau) \right). \quad (9)$$

The total action of the whole system can now be separated into this free action and a Gaussian fermionic part. This is written in the form

$$S = S_0(b) + \int_0^\beta d\tau \sum_{i=1}^N \rho_i^\dagger(\tau) M(b^*, b) \rho_i(\tau), \quad (10)$$

where $\rho_i(\tau)$ is a column matrix given in terms of Grassmann Fermi fields

$$\begin{aligned} \rho_i(\tau) &= \begin{pmatrix} \beta_i(\tau) \\ \alpha_i(\tau) \end{pmatrix}, \\ \rho_i^\dagger(\tau) &= \begin{pmatrix} \beta_i^*(\tau) & \alpha_i^*(\tau) \end{pmatrix} \end{aligned} \quad (11)$$

and the matrix $M(b^*, b)$ is given by

$$M(b^*(\tau), b(\tau)) = \begin{pmatrix} \partial_\tau + \frac{\Omega}{2} + \frac{g}{N}(b(\tau) + b^*(\tau)) & 0 \\ 0 & \partial_\tau - \frac{\Omega}{2} - \frac{g}{N}(b(\tau) + b^*(\tau)) \end{pmatrix}. \quad (12)$$

These complex functions and Grassmann Fermi fields $b(\tau)$, $\alpha_i(\tau)$ and $\beta_i(\tau)$ can be represented in terms of a Fourier expansion. Therefore, we have

$$b(\tau) = \beta^{-1/2} \sum_{\omega} b(\omega) e^{i\omega\tau} \quad (13)$$

and

$$\rho_i(\tau) = \beta^{-1/2} \sum_p \rho_i(p) e^{ip\tau}. \quad (14)$$

Since the complex function $b(\tau)$ obeys periodic boundary conditions, and the Grassmann Fermi fields $\alpha_i(\tau)$ and $\beta_i(\tau)$ obey anti-periodic boundary conditions, we have that $\omega = \frac{2\pi n}{\beta}$ and $p = \frac{(2n+1)\pi}{\beta}$, where they are the bosonic and fermionic Matsubara frequencies respectively. Substituting the Fourier expansions given by the Eq. (13) and Eq. (14) in the matrix $M_{pq}(b^*, b)$ given by Eq. (12) we get

$$M_{pq}(b^*, b) = \begin{pmatrix} (ip + \frac{\Omega}{2})\delta_{pq} + Q & 0 \\ 0 & (ip - \frac{\Omega}{2})\delta_{pq} - Q \end{pmatrix} \quad (15)$$

where

$$Q = g N^{-1} \beta^{-1/2} \left(b^*(q-p) + b(p-q) \right). \quad (16)$$

To simplify our calculations, let us change variables in the following way:

$$b(\omega) \rightarrow \left(\frac{\pi}{(\omega_0 - i\omega)} \right)^{1/2} b(\omega) \quad (17)$$

and

$$b^*(\omega) \rightarrow \left(\frac{\pi}{(\omega_0 - i\omega)} \right)^{1/2} b^*(\omega). \quad (18)$$

Note that Eq. (18) is not the complex conjugate of Eq. (17). It is easy to see that after these changes of variables the denominator of the Eq. (8) turns out to be equal to unity

$$\int [d\eta(b)] \exp\left(-\pi \sum_{\omega} b^*(\omega) b(\omega)\right) = 1. \quad (19)$$

We can express the ratio $\frac{Z}{Z_0}$ by the integral

$$\frac{Z}{Z_0} = \int [d\eta(b)] \exp\left(S_{eff}(b)\right), \quad (20)$$

where $S_{eff}(b)$ is the effective action of the bosonic mode which is given by

$$S_{eff} = -\pi \sum_{\omega} b^*(\omega) b(\omega) + N \ln \det(I + A). \quad (21)$$

The matrix A in the determinant of the above equation is given by

$$(I + A) = \begin{pmatrix} I + E & 0 \\ 0 & I + D \end{pmatrix} \quad (22)$$

where the components of the matrices E and D are given by

$$E_{pq} = -\left(\frac{\pi g}{\beta N}\right)^{\frac{1}{2}} \left(ip - \frac{\Omega}{2}\right)^{-\frac{1}{2}} \left(\frac{b^*(q-p)}{\sqrt{\omega_0 - i(q-p)}} + \frac{b(p-q)}{\sqrt{\omega_0 - i(p-q)}}\right) \left(iq - \frac{\Omega}{2}\right)^{-\frac{1}{2}} \quad (23)$$

and

$$D_{pq} = \left(\frac{\pi g}{\beta N}\right)^{\frac{1}{2}} \left(ip + \frac{\Omega}{2}\right)^{-\frac{1}{2}} \left(\frac{b^*(q-p)}{\sqrt{\omega_0 - i(q-p)}} + \frac{b(p-q)}{\sqrt{\omega_0 - i(p-q)}}\right) \left(iq + \frac{\Omega}{2}\right)^{-\frac{1}{2}}. \quad (24)$$

In order to perform the functional integral given by Eq. (20) we must to find a manageable expression for $\det(I + A)$, so we use the following identity

$$\det(I + A) = \det(I + E + D + ED) \simeq e^{tr E + tr D - \frac{1}{2}E^2 - \frac{1}{2}D^2} \quad (25)$$

where the approximation in the exponent on the right-hand side of Eq. (25) is performed up to second order in the E and D matrices. Therefore, upon using the expressions of E and D given in Eq. (23) and Eq. (24), respectively, into the approximation given in Eq. (25), we can find an expression for the ratio $\frac{Z}{Z_0}$ defined in Eq. (20) of the following form

$$\begin{aligned} \frac{Z}{Z_0} &= \int [d\eta(b)] \exp\left(-\pi \sum_{\omega} b^*(\omega) b(\omega) + a_0 (b(0) + b^*(0))\right) \\ &\quad - \frac{1}{N} \sum_{\omega} c(\omega) (b(\omega) b(-\omega) + b^*(\omega) b^*(-\omega)) - \frac{1}{N} \sum_{\omega} d(\omega) (b(\omega) b^*(\omega)), \end{aligned} \quad (26)$$

where the terms a_0 , $c(\omega)$ and $d(\omega)$ are, respectively, defined in the form

$$a_0 = g \left(\frac{\pi \beta}{\omega_0} \right)^{\frac{1}{2}} \tanh \left(\frac{\Omega \beta}{4} \right), \quad (27)$$

$$c(\omega) = \frac{g^2 \pi}{\beta(\omega_0^2 + \omega^2)^{\frac{1}{2}}} \sum_p \frac{\frac{\Omega^2}{4} - p(p - \omega)}{\left(p^2 + \frac{\Omega^2}{4}\right) \left((p - \omega)^2 + \frac{\Omega^2}{4}\right)}, \quad (28)$$

$$d(\omega) = \frac{2 g^2 \pi}{\beta(\omega_0 - i\omega)} \sum_p \frac{\frac{\Omega^2}{4} - p(p - \omega)}{\left(p^2 + \frac{\Omega^2}{4}\right) \left((p - \omega)^2 + \frac{\Omega^2}{4}\right)}. \quad (29)$$

It is possible to show that the contributions $c(\omega)$ and $d(\omega)$ cancel out for $\omega \neq 0$ and $c(0)$ and $d(0)$ are different from zero, but their values do not matter in the discussion next because in the limit $N \rightarrow \infty$ they do not contribute to the partition function. Therefore in the thermodynamic limit $N \rightarrow \infty$ we have that $\frac{Z}{Z_0}$ can be written as

$$\frac{Z}{Z_0} = \int \prod_{\omega \neq 0} db(\omega) db^*(\omega) e^{-\pi \sum_{\omega \neq 0} b^*(\omega) b(\omega)} \int db(0) db^*(0) e^{-\pi b^*(0) b(0) + a_0 (b(0) + b^*(0))}. \quad (30)$$

The integrations for $\omega \neq 0$ are not dependent on thermodynamical parameters. Thus it can be considered as a normalization constant, i.e.,

$$\frac{Z}{Z_0} = C_0 \int db(0) db^*(0) e^{-\pi b^*(0) b(0) + a_0 (b(0) + b^*(0))}. \quad (31)$$

The partition function is therefore an entire function. From the $\ln Z$ quantity, which is given by

$$\ln Z = \ln Z_0 + \ln C_0 + \frac{g^2 \beta}{\omega_0} \tanh^2 \left(\frac{\Omega \beta}{4} \right), \quad (32)$$

we can find the free energy of the model. Also the mean energy E and canonical entropy S can be computed. We get

$$E = E_0 - \frac{g^2}{2\omega_0} \tanh \left(\frac{\Omega \beta}{4} \right) \frac{\sinh \left(\frac{\Omega \beta}{2} \right) + \Omega \beta}{\cosh^2 \left(\frac{\Omega \beta}{4} \right)}, \quad (33)$$

and

$$S = S_0 + \ln C_0 - \frac{g^2 \beta^2 \Omega}{2\omega_0} \frac{\tanh \left(\frac{\Omega \beta}{4} \right)}{\cosh^2 \left(\frac{\Omega \beta}{4} \right)}. \quad (34)$$

We can see that the contribution of the last term in the right-hand side of Eq. (32) is coming from the zero mode. Such quantity generates a negative term in the expressions for the energy and the entropy. It is known in the literature that the zero modes can lead to problematic results [55]. In the models discussed by Dowker, the entropy becomes negative and also a temperature-dependent pole appears in the free energy [56]. At this point, there are two different paths that we can follow concerning the zero-mode problem. One is to disregard the zero mode, thus solving the problem of the sign of the canonical entropy. Note that we can not identify the zero-mode as a Nambu-Goldstone mode [57] [58] since there is no spontaneous breaking of the continuous symmetry in the model. With this procedure we get a free theory for the discussed model. Another one is to follow Brandes [4], which claims that the model is unrealistic from the point of view of application to real physical situations because of the absence of the coupling via $\sigma_{(j)}^+$ and $\sigma_{(j)}^-$ to the bosonic mode. If the *ad hoc* procedure of throwing away the zero-mode is not admissible, the negative entropy could be a sign of the inconsistency of the model.

3 The functional integral for the generalized fermionic Dicke model

The generalized Dicke model, where an ensemble of identical N atoms interacts linearly with one mode of a bosonic field, is defined by the Hamiltonian

$$H = I_S \otimes \sum_{j=1}^N \frac{\Omega}{2} \sigma_{(j)}^z + \omega_0 b^\dagger b \otimes I_B + \frac{g}{\sqrt{N}} \sum_{j=1}^N (b + b^\dagger) \otimes (\sigma_{(j)}^+ + \sigma_{(j)}^-). \quad (35)$$

In the above equation g is the coupling constant between the atom and the single mode of the bosonic field. The b and b^\dagger are the boson annihilation and creation operators of mode excitations that satisfy the usual commutation relation rules.

The aim of this section is to prove that a model with an interaction Hamiltonian generating only virtual processes presents a phase transition from normal to superradiant state at some temperature with the presence of a condensate and also a quantum phase transition at some critical coupling. Similarly to the last section we find that the Euclidean action S is given by Eq. (6), where now H_F is the full Hamiltonian for the generalized fermionic Dicke model given by

$$\begin{aligned} H_F = & \omega_0 b^*(\tau) b(\tau) + \frac{\Omega}{2} \sum_{i=1}^N \left(\alpha_i^*(\tau) \alpha_i(\tau) - \beta_i^*(\tau) \beta_i(\tau) \right) + \\ & + \frac{g_1}{\sqrt{N}} \sum_{i=1}^N \left(\alpha_i^*(\tau) \beta_i(\tau) b(\tau) + \alpha_i(\tau) \beta_i^*(\tau) b^*(\tau) \right) + \\ & + \frac{g_2}{\sqrt{N}} \sum_{i=1}^N \left(\alpha_i(\tau) \beta_i^*(\tau) b(\tau) + \alpha_i^*(\tau) \beta_i(\tau) b^*(\tau) \right). \end{aligned} \quad (36)$$

Note we are introducing two coupling constants, g_1 and g_2 , for the rotating and counter-rotating terms, respectively. As we discussed before, the main reason for this is that we are interested in to identify the contribution of the real and virtual processes in the phase transition with the formation of the condensate. We are interested in calculating the formal quotient given by Eq. (8). Following the same calculation from last section we can arrive at Eq. (20) and Eq. (21), where the matrix A is now given by

$$A = \begin{pmatrix} 0 & B \\ -C & 0 \end{pmatrix}. \quad (37)$$

In the equation above the quantities B and C are matrices with components given by

$$B_{pq} = \left(\frac{\pi}{\beta N} \right)^{\frac{1}{2}} \left(ip + \frac{\Omega}{2} \right)^{-\frac{1}{2}} \left(\frac{g_1 b^*(q-p)}{\sqrt{\omega_0 - i(q-p)}} + \frac{g_2 b(p-q)}{\sqrt{\omega_0 - i(p-q)}} \right) \left(iq - \frac{\Omega}{2} \right)^{-\frac{1}{2}} \quad (38)$$

and

$$C_{pq} = - \left(\frac{\pi}{\beta N} \right)^{\frac{1}{2}} \left(ip - \frac{\Omega}{2} \right)^{-\frac{1}{2}} \left(\frac{g_1 b(p-q)}{\sqrt{\omega_0 - i(p-q)}} + \frac{g_2 b^*(q-p)}{\sqrt{\omega_0 - i(q-p)}} \right) \left(iq + \frac{\Omega}{2} \right)^{-\frac{1}{2}}. \quad (39)$$

We shall investigate the integral given by Eq. (20) for temperatures that satisfy $\beta^{-1} > \beta_c^{-1}$. First of all let us show that this integral converges. Using the estimate

$$|\det(I + A)| \leq \exp \left(\text{Re}(tr A) + \frac{1}{2} tr(AA^\dagger) \right), \quad (40)$$

we can show that the ratio $\frac{Z}{Z_0}$ obeys the following inequality

$$\frac{Z}{Z_0} \leq \left[\left(1 - a_0(0) + 2c_0(0)\right) \left(1 - a_0(0) - 2c_0(0)\right) \right]^{-1/2} \prod_{\omega > 0} \left[\left(1 - a_0(\omega) + 2c_0(\omega)\right) \left(1 - a_0(\omega) - 2c_0(\omega)\right) \right]^{-1}. \quad (41)$$

where the $a_0(\omega)$ and $c_0(\omega)$ are given respectively by

$$a_0(\omega) = \frac{g_1^2 + g_2^2}{\beta(\omega_0^2 + \omega^2)^{1/2}} \sum_{p-q=\omega} \frac{1}{\left(\frac{\Omega^2}{4} + q^2\right)^{1/2}} \frac{1}{\left(\frac{\Omega^2}{4} + p^2\right)^{1/2}}, \quad (42)$$

and

$$c_0(\omega) = \frac{\omega_0 g_1 g_2}{\beta(\omega_0^2 + \omega^2)} \sum_{p-q=\omega} \frac{1}{\left(\frac{\Omega^2}{4} + q^2\right)^{1/2}} \frac{1}{\left(\frac{\Omega^2}{4} + p^2\right)^{1/2}}. \quad (43)$$

In a similar way like Popov and Fedotov [28] proved, for the case of rotating wave approximation we have that $0 < a_0(\omega) + 2c_0(\omega) < a_0(0) + 2c_0(0)$ and $a_0(0) + 2c_0(0) = O(\omega^{-2} \ln \omega)$. Therefore if $a_0(0) + 2c_0(0) < 1$, then Eq. (41) guarantees convergence of the expression $\frac{Z}{Z_0}$. The condition $a_0(0) + 2c_0(0) = 1$ is the equation for the transition temperature, then we have

$$a_0(0) + 2c_0(0) = \frac{(g_1 + g_2)^2}{\Omega \omega_0} \tanh\left(\frac{\beta_c \Omega}{4}\right) = 1. \quad (44)$$

The inverse of the critical temperature β_c is given by

$$\beta_c = \frac{4}{\Omega} \tanh^{-1}\left(\frac{\Omega \omega_0}{(g_1 + g_2)^2}\right). \quad (45)$$

Note that there is a quantum phase transition where the coupling constants g_1 and g_2 satisfy $g_1 + g_2 = (\omega_0 \Omega)^{\frac{1}{2}}$. For $g_1 + g_2 \neq (\omega_0 \Omega)^{\frac{1}{2}}$ the partition function is no more an entire function in the positive half of the complex plane for the temperature β_c^{-1} given by Eq. (45). The system enters in a superradiant phase. To calculate the asymptotic behavior of the functional integrals at temperatures that satisfy $\beta^{-1} > \beta_c^{-1}$, we can do the following approximation

$$\det^N(I + A) = \det^N(I + BC) \rightarrow \exp\left(N \operatorname{tr}(BC)\right). \quad (46)$$

After some calculations $\frac{Z}{Z_0}$ can be written as

$$\begin{aligned} \frac{Z}{Z_0} &= \left[\left(1 - a(0) + 2c(0)\right) \left(1 - a(0) - 2c(0)\right) \right]^{-1/2} \\ &\quad \prod_{\omega > 0} \left[\left(1 - a(\omega)\right) \left(1 - a(-\omega)\right) - c^2(\omega) \right]^{-1} + \\ &\quad + O(N^{-1}), \end{aligned} \quad (47)$$

where $a(\omega)$ and $c(\omega)$ in the above equation are given, respectively, by

$$a(\omega) = \left(\frac{g_1^2 (\Omega - i\omega)^{-1} + g_2^2 (\Omega + i\omega)^{-1}}{(\omega_0 - i\omega)} \right) \tanh\left(\frac{\beta \Omega}{4}\right) \quad (48)$$

and

$$c(\omega) = \left(\frac{g_1 g_2 \Omega}{(\omega_0^2 + \omega^2)^{1/2} (\Omega^2 + \omega^2)} \right) \tanh \left(\frac{\beta \Omega}{4} \right). \quad (49)$$

Taking the limit ($N \rightarrow \infty$) in Eq. (47) we are in the thermodynamic limit. We turn out to the discussion concerning the local elementary excitation of the ground state. To find the collective excitation energy level spectrum we have to use the equation

$$c^2(\omega) - \left(1 - a(\omega)\right) \left(1 - a(-\omega)\right) = 0, \quad (50)$$

and making the analytic continuation ($i\omega \rightarrow E$), we obtain the following equation

$$\begin{aligned} 1 = & - \left[\frac{g_1^4 + g_2^4}{(\omega_0^2 - E^2)(\Omega^2 - E^2)} \right] \tanh^2 \left(\frac{\beta \Omega}{4} \right) + \\ & - \left[\frac{g_1^2 g_2^2}{(\omega_0^2 - E^2)} \left(\frac{1}{(\Omega - E)^2} + \frac{1}{(\Omega + E)^2} - \frac{4\Omega^2}{(\Omega^2 - E^2)^2} \right) \right] \tanh^2 \left(\frac{\beta \Omega}{4} \right) + \\ & + \left[\frac{g_1^2 (\Omega - E)^{-1} + g_2^2 (\Omega + E)^{-1}}{(\omega_0 - E)} + \frac{g_1^2 (\Omega + E)^{-1} + g_2^2 (\Omega - E)^{-1}}{(\omega_0 + E)} \right] \tanh \left(\frac{\beta \Omega}{4} \right). \end{aligned} \quad (51)$$

Solving the above equation for the case $\beta^{-1} = \beta_c^{-1}$ we find the following roots

$$E_1 = 0 \quad (52)$$

and

$$E_2 = \left(\frac{g_1 (\Omega + \omega_0)^2 + g_2 (\Omega - \omega_0)^2}{(g_1 + g_2)} \right)^{1/2}. \quad (53)$$

Its low energy state of excitation is a Nambu-Goldstone mode. Now, let us present the critical temperature and the energy level spectrum of the collective bosonic excitations of the model with the rotating-wave approximation, where $g_1 \neq 0$ and $g_2 = 0$. The result obtained by Popov and Fedotov is recovered, where the equation

$$a(0) = 1 \quad (54)$$

and

$$\frac{g_1^2}{\omega_0 \Omega} \tanh \left(\frac{\beta_c \Omega}{4} \right) = 1, \quad (55)$$

give the inverse of the critical temperature, β_c . It is given by

$$\beta_c = \frac{4}{\Omega} \operatorname{arctanh} \left(\frac{\omega_0 \Omega}{g_1^2} \right). \quad (56)$$

The order parameter of the transition is the expectation value of the number of excitation associated to the bosonic mode per atom, i.e., $\lim_{N \rightarrow \infty} \frac{\langle \beta^\dagger \beta \rangle}{N} \neq 0$. Note that again ω_0 , Ω and g_1 define also a non-zero critical temperature where the partition function is no more an entire function in the positive half of the complex β -plane. We may expect a superradiant phase for the temperature β_c^{-1} given by Eq. (56). The energy level spectrum of the collective Bose excitations in this case is

$$E_1 = 0, \quad (57)$$

and

$$E_2 = \Omega + \omega_0. \quad (58)$$

In this case, there is also a quantum phase transition, i.e., a zero temperature phase transition when $g_1 = (\omega_0 \Omega)^{\frac{1}{2}}$. Now we will show that is possible to have a condensate with superradiance in a system of N two-level atoms coupled with one mode of a Bose field where only virtual processes contribute. In the pure counter-rotating wave case, i.e., $g_1 = 0$ and $g_2 \neq 0$, the inverse of the critical temperature, β_c is given by

$$\beta_c = \frac{4}{\Omega} \operatorname{arctanh}\left(\frac{\omega_0 \Omega}{g_2^2}\right), \quad (59)$$

and the spectrum of the collective Bose excitations given by

$$E_1 = 0, \quad (60)$$

and

$$E_2 = |\Omega - \omega_0|. \quad (61)$$

A comment is in order concerning the spectrum of the Bose excitations. In both of the cases: working with the pure counter-rotating or the rotating-wave terms, there is a phase transition. In the case of the rotating-wave approximation $g_1 \neq 0$ and $g_2 = 0$, there is a Nambu-Goldstone mode ($E = 0$). In the pure counter-rotating case $g_1 = 0$ and $g_2 \neq 0$ also there is a Nambu-Goldstone (gapless) mode. Thence we show that it is possible to have a condensate with superradiance in a system of N two-level atoms coupled with one mode of a Bose field where only virtual processes contribute. Since the energy $E_2 \approx 0$, local elementary excitations of the ground state with low energy can easily be created causing a significant fluctuation effect. Unfortunately we are not able to evaluate these effect in the systems.

An important question is the way of practical realization of the second model, i.e., the generalized Dicke model in the laboratory. As was stressed by Dimer et al [49] it remains a challenge to provide a physical system where the counter-rotating terms are dominant. Experimental observation of the superradiant phase in a situation where is possible to control the importance of the counter-rotating terms in the generalized Dicke model ($g_1 \approx 0$, $g_2 \neq 0$) could improve our understanding of this phenomenon.

4 The functional integral for the fermionic Dicke model with intensity-dependent coupling

A model where the behavior is quite interesting from the physical point of view is the one where the coupling between N identical two-level atoms and one mode of a bosonic field in a lossless cavity is intensity-dependent. The generalization for this model introducing the counter-rotating terms is straightforward. Therefore, in this paper we discuss the model with rotating and counter-rotating terms, where the intensity-dependent contribution appears only in the real processes. The necessity for disregard the intensity dependent coupling contribution in the part of the interaction Hamiltonian that generates the virtual processes is easy to justify. It is counterintuitive that the virtual processes are amplified when the number of excitations in the bosonic sector is enhanced. We will show that at low temperatures the contribution from the counter-rotating terms dominate over the rotating ones and the system presents a quantum phase transition.

After this discussion, the total Hamiltonian of the atoms and the bosonic mode takes the form

$$H = I_S \otimes \sum_{j=1}^N \frac{\Omega}{2} \sigma_{(j)}^z + \omega_0 b^\dagger b \otimes I_B +$$

$$\frac{g}{\sqrt{N}} \sum_{j=1}^N \left(b (b^\dagger b)^{\frac{1}{2}} \otimes \sigma_{(j)}^+ + b^\dagger (b^\dagger b)^{\frac{1}{2}} \otimes \sigma_{(j)}^- + b^\dagger \otimes \sigma_{(j)}^+ + b \otimes \sigma_{(j)}^- \right). \quad (62)$$

At this point, let us make a parallel between the model given by Eq. (62) assuming the rotating-wave approximation, and the Jaynes-Cummings model [42], where one two-level atom is coupled to a single mode quantized electromagnetic field. It is possible to generalize this model, by using the rotating-wave approximation, known as the two-photon Jaynes-Cummings model [59] [60]. The introduction of the counter-rotating terms is straightforward. It has been discussed in the literature the possibility of controlling the relative importance of them in laboratory using an ion trap [50].

Going back to the model we have also non-Gaussian functional integrals to solve. The main point we wish to demonstrate in this section is that the model defined by the equation bellow at low temperatures is exactly soluble and the partition function can be presented in a closed form. Changing the atomic pseudo-spin operators by a linear combination of Grassmann Fermi field yields the fermionic generalized Dicke model with intensity-dependent coupling defined by the Hamiltonian

$$H = \sum_{j=1}^N \frac{\Omega}{2} \sigma_{(j)}^z + \omega_0 b^\dagger b + \frac{g_1}{\sqrt{N}} \sum_{j=1}^N \left(b (b^\dagger b)^{\frac{1}{2}} \alpha_i^\dagger \beta_i + b^\dagger (b^\dagger b)^{\frac{1}{2}} \beta_i^\dagger \alpha_i \right) + \frac{g_2}{\sqrt{N}} \sum_{j=1}^N \left(b \beta_i^\dagger \alpha_i + b^\dagger \alpha_i^\dagger \beta_i \right), \quad (63)$$

where again we are introducing different coupling between the single mode bosonic field and the reservoir, g_1 and g_2 for rotating and counter-rotating terms, respectively. The difference of the Hamiltonian for this model, given by Eq. (63), when contrasted with the Hamiltonian of the generalized Dicke model, Eq. (36), is the term $(b^\dagger b)^{\frac{1}{2}}$ in the part of the interaction Hamiltonian which generates the virtual processes. The fermion integration on the generalized Dicke model is repeated for this case, so we can take expressions given by Eq. (20) and Eq. (21) where the matrix A is given by

$$A = \begin{pmatrix} 0 & B \\ -C & 0 \end{pmatrix}. \quad (64)$$

In the equation above the quantities B and C are matrices with components given now by

$$B_{pq} = -\left(\frac{\pi}{N\beta} \right)^{\frac{1}{2}} \left(ip + \frac{\Omega}{2} \right)^{-\frac{1}{2}} \left(iq - \frac{\Omega}{2} \right)^{-\frac{1}{2}} \times \left(g_1 \left(\frac{\pi}{\beta\omega_0} \right)^{\frac{1}{2}} (b^*(0) b(0))^{\frac{1}{2}} \frac{b^*(q-p)}{\sqrt{\omega_0 - i(q-p)}} + g_2 \frac{b(p-q)}{\sqrt{\omega_0 - i(p-q)}} \right) \quad (65)$$

and

$$C_{pq} = \left(\frac{\pi}{N\beta} \right)^{\frac{1}{2}} \left(ip - \frac{\Omega}{2} \right)^{-\frac{1}{2}} \left(iq + \frac{\Omega}{2} \right)^{-\frac{1}{2}} \times \left(g_1 \left(\frac{\pi}{\beta\omega_0} \right)^{\frac{1}{2}} (b^*(0) b(0))^{\frac{1}{2}} \frac{b(p-q)}{\sqrt{\omega_0 - i(p-q)}} + g_2 \frac{b^*(q-p)}{\sqrt{\omega_0 - i(q-p)}} \right). \quad (66)$$

For this case, in order to find an approximate expression for the partition function in the thermodynamic limit ($N \rightarrow \infty$), we use the approximation $\det(I + A) = \det(I + BC) \simeq \exp(\text{tr} BC)$.

Similar procedure was used for the generalized Dicke model case. Then we can arrive at the following approximate expression

$$\begin{aligned} \frac{Z}{Z_0} &= \int [d\eta(b)] \exp\left(-\pi \sum_{\omega} (1 - c(\omega)) b^*(\omega) b(\omega) \right. \\ &\quad \left. + \pi \sum_{\omega} a(\omega) b^*(0) b(0) b^*(\omega) b(\omega) + \pi \sum_{\omega} d(\omega) (b^*(0) b(0))^{\frac{1}{2}} \right. \\ &\quad \left. \times (b(\omega) b(-\omega) + b^*(\omega) b^*(-\omega)) \right), \end{aligned} \quad (67)$$

where $a(\omega)$ and $c(\omega)$ of above equation are given, respectively, by

$$a(\omega) = \frac{\pi g_1^2}{\beta \omega_0} \tanh\left(\frac{\beta \Omega}{4}\right) \frac{1}{(\Omega - i\omega)(\omega_0 - i\omega)}, \quad (68)$$

$$c(\omega) = g_2^2 \tanh\left(\frac{\beta \Omega}{4}\right) \frac{1}{(\Omega + i\omega)(\omega_0 - i\omega)}, \quad (69)$$

and

$$d(\omega) = g_1 g_2 \left(\frac{\pi}{\beta \omega_0}\right)^{\frac{1}{2}} \tanh\left(\frac{\beta \Omega}{4}\right) \frac{1}{(\Omega + i\omega)(\omega_0^2 + \omega^2)^{\frac{1}{2}}}. \quad (70)$$

In order to perform the functional integral given by Eq. (67) we have separated the zero and non-zero modes, so we can write

$$\begin{aligned} \frac{Z}{Z_0} &= \int db(0) db^*(0) e^{\pi (a(0) (b^*(0))^2 b^2(0) + (c(0)-1) b(0) b^*(0))} \\ &\quad \int \prod_{\omega \neq 0} db(\omega) db^*(\omega) e^{-\pi \sum_{\omega \neq 0} (1-c(\omega)) b^*(\omega) b(\omega)} \\ &\quad \times e^{\pi \sum_{\omega \neq 0} a(\omega) b^*(0) b(0) b^*(\omega) b(\omega) + d(\omega) (b^*(0) b(0))^{\frac{1}{2}} (b(\omega) b(-\omega) + b^*(\omega) b^*(-\omega))}. \end{aligned} \quad (71)$$

It is quite difficult to compute in a close form the quantity $\frac{Z}{Z_0}$ since the integrals that appear in Eq. (71) are not Gaussian. Nevertheless, at low temperature ($\beta \rightarrow \infty$), we can make it evident that the contribution coming from the counter-rotating terms dominates over the rotating ones. We can take this limit in the Eq. (68), Eq. (69) and Eq. (70) so we get

$$\lim_{\beta \rightarrow \infty} \frac{Z}{Z_0} = \int \prod_{\omega} db(\omega) db^*(\omega) e^{-\pi \sum_{\omega} (1-c(\omega)) b^*(\omega) b(\omega)}. \quad (72)$$

Performing this last integral we have

$$\lim_{\beta \rightarrow \infty} \frac{Z}{Z_0} = \prod_{\omega} (1 - c(\omega))^{-1}, \quad (73)$$

where the term $c(\omega)$ in the limit $\beta \rightarrow \infty$ is

$$\lim_{\beta \rightarrow \infty} c(\omega) = \frac{g_2^2}{(\Omega + i\omega)(\omega_0 - i\omega)}. \quad (74)$$

From this last equation we can see that the ratio Z/Z_0 is non-analytic for $g_2 = (\omega_0 \Omega)^{\frac{1}{2}}$, thus corresponding to a quantum phase transition. Therefore, in the model at some range of the physical

parameters, the counter-rotating terms are dominant, leading to a phase transition. Since we are not able to evaluate the ratio $\frac{Z}{Z_0}$ given by Eq. (71) for all temperatures, it is not possible to describe the phase diagram of the system near the quantum phase transition. The possibility that appears a line of second-order phase transition for $\beta^{-1} > 0$ terminating at the quantum critical point must be investigated.

5 Conclusions

In the present paper we are considering the issue of the formation of the condensate with a superradiant phase transition in spin-boson models with a single bosonic mode with quite particular couplings between the single bosonic mode and a environment of N two-level atoms. We have assumed that the whole system is in thermal equilibrium with a reservoir at temperature β^{-1} . The interaction Hamiltonian of the models also generate virtual processes. We first investigate a model based in one discussed by Chang and Chakravarty, Legget and others. Second, with the couplings g_1 and g_2 for rotating and counter-rotating terms, respectively, we has defined the generalized Dicke model. Finally we study the generalized Dicke model with an intensity-dependent coupling. In this last model it is necessary to use perturbation theory to investigate the thermodynamic of the model. Nevertheless in the low temperature limit the model is exactly soluble.

Studying the case where in identical two-level atoms act as a thermal reservoir ($N \rightarrow \infty$), we investigated the thermodynamics of the three models using the path integral approach with functional integration method. We found that the first model is unrealistic from the thermodynamical point of view, since the entropy becomes negative at some temperature. An *ad hoc* procedure, disregarding the zero-mode, can make the model admissible. After this procedure it is shown that the partition function is analytic for all temperatures. Consequently there is no phase transition in the model. In the second one the situation is more interesting. We studied the nonanalytic behavior of thermodynamical quantities of the generalized Dicke model by evaluating the critical transition temperature and presenting the spectrum of the collective bosonic excitations, for the case $g_1 \neq 0$ and $g_2 = 0$, $g_1 = 0$ and $g_2 \neq 0$ and also in the general case. Our result show that it is possible to have a condensate with superradiance in a system of N two-level atoms coupled with one mode of a bosonic field where only virtual processes contribute. It is important to realize that the energy of the mode, which is not the Nambu-Goldstone mode in Eq. (58), is always larger than the one in Eq. (61), i.e., in the system where the condensate appears due to the virtual processes. Therefore both processes, real and virtual ones give different contributions to generate the condensate. In the generalized Dicke model with the intensity-dependent coupling we found a superradiant phase transition at zero temperature, i.e., a quantum phase transition.

One interesting aspect of superradiance is the fact that initial quantum fluctuations that are able to cause spontaneous emission in a few atoms are amplified leading to fluctuations on a macroscopic scale. We would like to point out that another mechanism to enhance the importance of virtual processes, in quantum field theory in the presence of macroscopic boundaries, was proposed by Ford [61]. In a local calculation, Ford presented a mechanism based on the fact that the contribution of various part of the frequency spectrum of the Casimir effect is an oscillatory function. The contributions of different ranges of frequency almost, but not quite, cancel out one another, and there is the possibility of enhancing the magnitude of the effect by altering the reflectivity of the boundary in selected frequency ranges. A quite different mechanism for amplification of vacuum fluctuations was proposed by Ford and Svaiter [62] [63] using parabolic mirrors to produce large vacuum fluctuations near mirror's focus. These authors studied the renormalized vacuum fluctuations associated with a scalar and electromagnetic field near the focus of a parabolic mirror. Using the geometric optics approximation they found that the mirror geometry can produce large vacuum fluctuations near the focus, similar to what happens in the classical focusing effect by the parabolic mirror geometry.

With the development of quantum information and its application in computation and communication the entangled states have been attracting enormous interest, since several quantum protocols can be realized exclusively with the help of entangled states. In the spin-boson model with the dipole-dipole interaction included, one can address the issue of how does the dipole-dipole interaction change the critical temperature, where again the system exhibits a phase transition from the fluorescent to the superradiant phase. An important subject for the near future is to study the degree of entanglement between the N two-level atoms and the bosonic mode near the quantum phase transition in the generalized Dicke model [64] [65] [66]. This subject is under investigation by the authors.

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