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Renormalization as an Extension Problem on the Countour Ordered Formalism in FTFT

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Dedicated to Prof. Olivier Piguet

Abstract

From a distributional-theoretical framework, we make efforts in order to fill a gap in the series of studies which discuss the inheritance of the renormalization behaviour of a finite temperature field theory (FTFT) from the analogous version in quantum field theory (QFT) at $T = 0$. Renormalization is treated as a distributional extension problem having the mathematical structure disentangled as much as possible from the physical aspects. The purely technical details essential for the discussion are briefly reviewed in a handle manner for further theoretical physics applications. The analysis elucidates some qualitative and quantitative distinctions concerning the divergences in the perturbation series when it is considered the FTFT version associated to a given QFT. Despite the differences, it turns clear the reason why the leading ultraviolet behaviour keeps unaffected when it is considered the FTFT version associated to a given QFT. The study is model independent and the approach allows one to consider the FTFT both imaginary and real time formalism at once in a unified way in the contour ordered formalism.

Key-words: *Finite Temperature, Renormalization, Wavefront Set.*

1 Introduction

Statistical Field Theory or FTFT arose from the fact that there are many problems in QFT exhibit many bodies aspects. QCD deconfinement phase transition, [1], the role of the quark-gluon plasma in the formation of dark matter in the early universe [2], besides the problems concerning superstring cosmology at finite temperature [3] are some early examples of applications. The FTFT, in the general case, is a relativistic QFT in the grand canonical ensemble, *i.e.*, finite temperature and finite density [4, 5]. There are some classes of formalisms for FTFT, two of then, the so called imaginary time formalism (ITF) and the real time formalism (RTF) are associated to particular cases of chosen contours in the continuation of the time axis into a complex plane having the imaginary axis associated to the inverse temperature. An additional class is the thermo field dynamics (TFD) which is an operator formulation in the C^*-algebra context and whose equivalence to the RTF is established by the reconstruction theorem. The ITF [6] is characterized by Euclidean propagators resulting from the representation of the inverse temperature as an imaginary time argument and consequently by the representation of the energy as a discrete set of imaginary poles, the Matsubara frequency, in the complex plane. This formalism is suitable for the calculation of the thermodynamic potential while the calculation of dynamic properties, like correlation functions and effects of external disturbances can not be done without a cumbersome and arbitrary non unique process of analytic continuation from the discrete energies to the complex plane. The non uniqueness of the continuation can be interpreted as the doubling of degrees of freedom naturally nested in the RTF what turns it to be suitable for dynamic calculations. It is characterized by a Minkowski space-time representation for the real time dependent propagators with spectral representation being functions of a real continuous variable and the effective doubling of degrees of freedom representing time and anti-time ordering what causes the propagators to have a matrix 2×2 structure. Despite of the above distinction between the ITF and RTF, both can be interpreted as particular cases of the countour ordered formalism (COF), a complex contour dependent formalism in which the time component of the support of the fields lies on a complex plane. In this framework, lines in Feynman graphs are related to complex contour dependent propagators and time ordering is replaced by contour ordering [7] on the complex time plane. Under some conditions on analyticity of the propagators, each contour leads to a specific formulation of FTFT.

As it occurs to the QFT, the FTFT also exhibit ultraviolet divergences. The problem of how to make sense out of the physical meaning behind the divergences in a mathematically proper way was satisfactorily solved by the known renormalization procedure. There are some well established prescription currently used in QFT to attribute meaning to the initially divergent distribution terms of the perturbation series associated to the quantities of interest. The latter can however be defined only up to certain renormalization ambiguities which, in principle, can be determined from physical reasonings. In facing the distinctions between the FTFT and QFT propagators, some questions take place. Once the divergences are related to some ill-defined products of distributions, the FTFT propagator would imply in changes on the conditions for the existence of the products and would introduce a temperature dependent renormalization problem. The ambiguities of the renormalization procedure associated to the physical parameters would then exhibit qualitative changes due to the temperature dependence. Moreover, it can also change the asymptotic divergent behaviour and consequently the amount of arbitrariness involved. The FTFT propagator, being separable into temperature dependent and independent pieces, causes the shuffle of the divergences and temperature dependent terms in crossing products in the higher order terms of the perturbation expansion. Depending on the renormalization procedure adopted some of those facts can become not clear. ¿From physical reasonings, one can not expected that the differences would have fundamental consequences to the UV behaviour because it arises from the short distance limit which is unaffected by the temperature once the thermal part of the propagator has support on the shell mass and decays rapidly with growing momentum because of the Bose-Einstein (or Fermi-Dirac) distribution function. This question has been investigated by many authors using various techniques, each one putting emphasis on different aspects of the problem. Let us mention for instance the TFD proof [8], the RFT method [9], the BPHZ momentum space subtraction procedure [4, 10] besides the framework of axiomatic quantum field theories at finite temperature [11]. More recently, it has been given by C. Kopper *et al.* [12] a rigorous proof of the renormalizability of the massive φ_4^4 theory at finite temperature based in the framework of Wilson's flow equations, to all orders of the loop expansion.

It is our purpose to fill a gap in this series of studies by approaching the problem from a central aspect of the renormalization which lies on the lack of definition of the distributional product in some particular context present in the perturbation series and on the necessary validity extension. These questions turn to be somewhat transparent when one deeps into the ground of the problem of the divergences and picks up to be studied the basic divergences themselves. Such an approach allows us to investigate the issue, as much as possible, in a model independent manner and free of the technical difficulties of thermal loop calculations way common to the various conventional approaches both in ITFs and RTFs. Moreover, it is allowed to be adopted the generalized unified framework of the countour ordered formalism (COF) considering at a time both ITF and RTF. The analysis is done under the light of a systematic use of the ideas and notions of the distribution theory. The microlocal theory of distributions in x-space is used for the characterization of the singular spectrum in terms of wavefront set of the propagators in order to characterize a sufficient condition for the existence of the products. The analysis of the asymptotic behaviour of the products of distributions near the singular points is done by the calculation of the scaling degree and the singular order which govern the amount arbitrariness present in the renormalization procedure. The whole apparatus furnish the basis for us to formulate the renormalization as the well posed mathematical problem of the distributional products support extension. One shows that the number of ill-defined products to a given order in the perturbation series increases. Furthermore, the ill-defined products in FTFT have in general temperature dependent factors in addition they would contribute in principle, to the counting of ambiguities. Nevertheless, the divergences found in FTFT are showed to arise from that factors having the same nature as those ill-defined ones in QFT. From the point of view of the renormalization as the extension problem, these factors can however be treated separately order by order in the perturbation series. Order by order the problem of the extension in FTFT is showed to reduce to the analogous one of the ordinary QFT. As a consequence, it is proved that the amount of arbitrariness in the renormalization procedure, as well the type of the ambiguities remains the same when passing from a given QFT to the associated FTFT version. There follows that a given FTFT remains finite when it is considered a renormalized analogous QFT.

The discussion of the pure mathematical aspects is done in a handle manner for further theoretical physics applications and the outline of the paper is as follows. We shall begin in Sec. 2 by describing some basics on the microlocal analysis of singularities where the wavefront set of a distribution is introduced together with a sufficient condition for the existence of products of distributions based on its WFSs. The Sec. 3 is devoted to an elementary introduction to the basic ideas of renormalization theory viewed as the distributional product support extension. In Sec. 4, the comparative analysis of the propagators of FTFT and QFT as distributions, besides their products, the problem of the renormalization and the perturbation series consequences take place. The Sec. 5 contains the final considerations. The Appendix recalls some properties of the oscillatory integrals focusing them as a tool for the calculation of the WFS of distributions.

2 Microlocal Study of Singularities

The UV divergences are a QFT inherent problem, because the fields, as well its correlation functions, having distributional character are defined on the a continuous spacetime. The perturbation expansions in QFT are made of the product of such distributions. However, products of distributions with overlapping singularities are in general not well-defined. Hence, it becomes convenient to shed some light on the problem of the under which conditions one has or not a well-defined product of distributions. Among the distributional analysis techniques, the framework of the *microlocal analysis* [13] is fairly suitable for the study of the UV divergences. The term microlocal analysis refers to a set of techniques of relatively recent origin which have turned out to be particularly useful in analyzing partial differential equations with variable coefficients, including those of particular interest to quantum field theory. We shall following describe an analytical method which provides sufficient conditions for the existence of the product of distributions based on the concept of the *wavefront set* (WFS) of a distribuiton f, denoted by $WF(f)$. It is a refined description of the singularity spectrum. Similar notion was developed in some versions by Sato [14], Iagolnitzer [15] and Sjöstrand [16]. The present definition is due to Hörmander [13] who has made use of this terminology due to an existing analogy between the "propagation" of singularities of distributions and the classical construction of propagating waves by Huyghens.

Let f be a distribution on an open set $X \subset \mathbb{R}^d$; then the *singular support* W of f is the complement of the largest relatively open subset X^1 of X whereon f is *smooth* ($f|_{X^1} \in C_0^\infty$). A point x_0, it is said to be a *non-singular point* of a distribution f if there exists a cutoff function $\phi \in C_0^\infty(V)$, with support in some neighbourhood V of x_0 such that the Fourier transform

$$\widehat{f\phi}(k) = \int d^d x \; f(x)\phi(x)e^{ikx} \ ,$$

is of fast decrease for all directions $k \in \mathbb{R}^d$. By a fast decrease in the k direction of $\hat{u}(k)$, one must understand that, there is $C_N \in \mathbb{R}$, $N = 1, 2, 3 \ldots$ such that $(1 + |k|)^N |\hat{u}(k)| \leq C_N$ remains bounded. Notice that if x_0 is a singular point of distribution f, and $\phi \in C_0^\infty(V)$ is such that $\phi(x_0) \neq 0$; then ϕf is also of compact support and singular in x_0. In this case of an existing singularity x_0 in the support of f, there can still occur some directions in k-space over which $\widehat{\phi f}$ is asymptotically bounded. A direction k for which the Fourier transform $\hat{u}(k)$ of $u(x) \in \mathcal{D}'(V)$ shows to be of fast decrease is called to be a *regular direction* of $\hat{u}(k)$. It suggests that we can single out singular directions as well as singular point and that for the establishment of these concepts, only the behaviour of f and of $\hat{f}$ restricted to an arbitrarily small neighbourhood of the singular point x_0 is relevant.

Let $f(x)$ be an arbitrary distribution not necessarily of compact support on an open set $X \subset \mathbb{R}^d$. Then, the set of all pairs composed first by the its singular points $x \in X$ and second by the associated nonzero singular directions k,

$$WF(f) = \{(x_0, k) \in X \times (\mathbb{R}^d \backslash 0) \mid k \in \Sigma_x(f)\} \ , \tag{2.1}$$

is called *wavefront set* of f. The $\Sigma_x(f)$ is defined to be the complement of the set of all $k \in \mathbb{R}^d \backslash 0$ with respect to $\mathbb{R}^d \backslash 0$, for which there is an open conic neighbourhood M of k such that $\widehat{\phi f}$ is of fast decrease on M.

In short, to determine whether (x_0, k) is in WFS set of f one must first to localize f around x_0, next to obtain Fourier transform $\hat{f}$ and finally to look at the decay in the direction k. Hence, the WFS not only describes the set where a distribution is singular, but also localizes the frequencies that constitute these singularities.

Example 2.1. A small "point" scatterer on $\mathbb{R}$.

$$V(x) = \delta(x) \propto \int d^dx \ \mathbf{1} e^{-ikx} \ ,$$

i.e., $\widehat{V} = \mathbf{1}$ does not decay in any direction k: $WF(\delta) = \{(0, k) \mid k \neq 0\}$ has singularities in all directions. ▲

Remarks 2.1. We now collect some basic properties of the WFSs:

1. The $WF(f)$ is conic in the sense that it remains invariant under the action of dilatations, *i.e.* when one multiply the second variable by a positive scalar. If $(x, k) \in WF(f)$ then $(x, \lambda k) \in WF(f)$ for all $\lambda > 0$.

2. From the definition of $WF(f)$, it follows that the projection onto the first coordinate $\pi_1(WF(f)) \rightarrow x$, consists of those points that have no neighbourhood whereon u is a smooth function, and the projection onto the second coordinate $\pi_2(WF(f)) \rightarrow \Sigma_x(f)$, is the cone around k attached to a such point denoting the set of high-frequency directions responsible for the appearance of a singularity at this point.

3. The WFS of a smooth function is the empty set.

4. For all smooth function ϕ with compact suport $WF(\phi f) \subset WF(f)$.

5. For any partial linear differential operator P, with C^∞ coefficients, one has

$$WF(Pf) \subseteq WF(f) \ .$$

6. If f and g are two distributions belonging to $\mathcal{D}'(\mathbb{R}^d)$, with wavefront set $WF(f)$ and $WF(g)$, respectively; then the wavefront set of $(f+g) \in \mathcal{D}'(\mathbb{R}^d)$ is contained in $WF(f) \cup WF(g)$. $\triangledown$

In the perturbation scheme of quantum field theories, one finds formal operations on distributions which can be in general not well-defined. In order to give precise statements on the existence of the product of these distributions, we appeal a criterion based on the WFS of the distributional factors the so-called **Hörmander's Criterion**. Let u and v be distributions; if the WFS of u and v are such that the following direct sum

$$WF(u) \oplus WF(v) \stackrel{\text{def}}{=} \{(x, k_1 + k_2) \mid (x, k_1) \in WF(u), (x, k_2) \in WF(v)\} \ , \tag{2.2}$$

does not contain any element of the form $(x, 0)$, then the product uv there exists and $WF(uv) \subset WF(u) \cup WF(v) \cup (WF(u) \oplus WF(v))$. Hence, the product of the distributions u and v is well-defined around x, if u, or v, or both distributions are regular in x. Otherwise, if u and v are singular in x, the product can still exist if the sum of the second components from $WF(u)$ and $WF(v)$ related to x can be linearly combined with nonnegative coefficients to vanish only by a trivial manner.

Example 2.2. The distributions $u, v \in \mathcal{D}'(\mathbb{R})$, $u(x) = \frac{1}{x+i\epsilon}$ and $v(x) = \frac{1}{x-i\epsilon}$, with the Heavyside distributions $\widehat{u}(k) = 2\pi i\theta(-k)$ and $\widehat{v}(k) = -2\pi i\theta(k)$ as their Fourier transforms, have the following WFSs:

$$WF(u) = \{(0, k) \mid k \in \mathbb{R}^{-}\backslash 0\} \ , \quad WF(v) = \{(0, k) \mid k \in \mathbb{R}^{+}\backslash 0\} \ .$$

Thus, from the Hörmander's Criterion one finds that there exist the powers of u^n and v^n. The products between u and v do not match the above criterion and do not exist, indeed. The example clearly indicates that one can multiply distributions even if they have overlapping singularities, provided their WFSs are in favorable positions. Such an observation is significant because it makes clear that *the problem is not only where the support is, but in which directions the Fourier transform is not rapidly decreasing*! ▲

Example 2.3. The **Feynman propagator** for massive scalar field

$$\Delta_{\mathrm{F}}(x) \stackrel{\text{def}}{=} \theta(x^0)\Delta_{+}(x; m^2) - \theta(-x^0)\Delta_{-}(x; m^2) \ ,$$

can have its WFS constitution studied from the WFS of the Wightman functions,

$$\begin{aligned} WF(\Delta_{\pm}) =& \{((0, \mathbf{0}); (\pm\lambda|\mathbf{k}|, \mp\lambda\mathbf{k})) \mid (\mathbf{k} \neq \mathbf{0}) \in \mathbb{R}^{\mathbf{3}}, \lambda \in \mathbb{R}_{+}\} \\ & \cup \{((|\mathbf{x}|, \mathbf{x}); (\pm\lambda|\mathbf{k}|, \mp\lambda\mathbf{k})) \mid \mathbf{x}, (\mathbf{k} \neq \mathbf{0}) \in \mathbb{R}^{\mathbf{3}},\ \lambda \in \mathbb{R}_{+}\} \ , \\ & \cup \{((-|\mathbf{x}|, \mathbf{x}); (\pm\lambda|\mathbf{k}|, \mp\lambda\mathbf{k})) \mid \mathbf{x}, (\mathbf{k} \neq \mathbf{0}) \in \mathbb{R}^{\mathbf{3}},\ \lambda \in \mathbb{R}_{+}\} \ , \end{aligned} \tag{2.3}$$

and from the WFS of $\theta(\pm t \mp t') = \theta^{\pm}$,

$$WF(\,\theta\pm) = \{((0,\mathbf{x}); (\pm\lambda k_0, \mathbf{0})) \mid \mathbf{x} \in \mathbb{R}^{\mathbf{3}}, \mathbf{k_0} \in \mathbb{R},\ \lambda \in \mathbb{R}_+\}\ . \tag{2.4}$$

One can easily conclude that it is not possible to form a non trivial linear combination with nonnegative coefficients in order to produce a vanishing second component in the direct sum of the above WFSs. So,

$$(x,0) \notin WF(\theta\pm) \oplus WF(\Delta_{\pm})\,. \tag{2.5}$$

Therefore, from the Hörmander's criterion, the Feynman propagator can be well-defined in terms of the above product and

$$WF(\theta^{\pm}\cdot\Delta_{\pm}) \subset\ WF(\theta^{\pm}) \cup WF(\Delta_{\pm}) \cup (WF(\theta^{\pm}) \oplus WF(\Delta_{\pm}))\,. \tag{2.6}$$

However, in the powers $(\Delta_{\mathrm{F}})^n$ there exist products like $\Delta_+\Delta_-$ and from (2.3), one can see that $(x,0) \in WF(\Delta_+) \oplus WF(\Delta_-)$ and it occurs for the singular point $x = 0$. In this sense, one must be careful when manipulating such products. In fact, they are known to exist anywhere but $x \neq 0$. Such an ill-definition, manifested as divergences, requires the treatment of the renormalization. Notice further that

$$(x,0) \notin WF(\Delta_{\pm}) \oplus WF(\Delta_{\pm}). \tag{2.7}$$

In particular, it can be used

$$\Delta_{\pm}(x;m^2) = \frac{\pm i}{(2\pi)^3}\int d^4k_1\ \theta(\pm k_1^0)\delta(k_1^2 - m^2)\, e^{-ik_1x}\ .$$

and $\widehat{\Delta}_{\pm}(k_1,k_2) = \pm i(2\pi)^4\delta(k_1+k_2)\theta(\pm k_1^0)\delta(k_1^2-m^2)$ as a representation of the Fourier transform, to verify that the wavefront set of Feynman propagator has the following covariant form [17]:

$$\begin{aligned} WF(\Delta_{\mathrm{F}}) =& \{(x_1,k_1);(x_2,k_2) \in (\mathbb{R}^{1,3} \times \mathbb{R}^{1,3} \setminus 0) \mid x_1 \neq x_2, (x_1 - x_2)^2 = 0, \\ & \qquad k_1 \parallel (x_1 - x_2), k_1 + k_2 = 0, k_1^2 = 0, \\ & \qquad k_1^0 > 0 \text{ if } x_1 \succ x_2 \text{ and } k_1^0 < 0 \text{ if } x_1 \prec x_2\} \\ & \cup \{(x_1,k_1);(x_2,k_2) \in (\mathbb{R}^{1,3} \times \mathbb{R}^{1,3} \setminus 0) \mid x_1 = x_2, k_1 + k_2 = 0, k_1^2 = 0\}\,, \end{aligned}$$

where have used the notation that $x_1 \succ x_2$ if $x_1 - x_2$ is in the convex hull of the forward lightcone and $x_2 \succ x_1$ if $x_1 - x_2$ is in the convex hull of the backward lightcone. Notice that the condition $k_1^0 > 0$ if $x_1 \succ x_2$ and $k_1^0 < 0$ if $x_1 \prec x_2$ in $WF(\Delta_{\mathrm{F}})$ ensures the existence of products of Feynman propagators at all points away from diagonal, while these products do not satisfy the Hörmander's criterion for multiplication of distributions over the points of the diagonal, since the sum of the second components of the WFS on the diagonal can add up to zero. ▲

In the appendix we present a method by Hörmander and Duistermaat [18] to compute the WFSs based on Oscillatory Integral.

3 Elementary Notion of Renormalization Theory

This section is devoted to a brief review on the basic ideas of renormalization theory. In the previous section, it was discussed a sufficient condition for the existence of the product of distributions. However, it is well known that in the perturbation expansion there arise formal products of distributions which do not satisfy the Hörmader's criterion, deserving then some careful to be treated. Such an ill-definition, manifested as divergences, must be identified and dealt via some renormalization prescription in order to ensure finite results to physical meaningful parameters. In particular, as we have pointed out, some products of distributions of the perturbation series are well-defined everywhere but on the the diagonal – the coinciding points. In the context of the perturbation series the renormalization consists in the problem of the extension of the ill-defined products of distributions to that coinciding points. This can be achieved by a Taylor subtraction on the corresponding test functions. However, the extensions preserving the continuous linear functional *are in general not unique*. The amount or degree of arbitrariness can be evaluated in accordance with the *scaling degree* and the *singular order* of the distribution, also called superficial degree of divergence. The **scaling degree** σ of a distribution u at the coinciding points of $\mathbb{R}^n$ is defined to be [19]

$$\sigma(u) = \inf\left\{\omega \in \mathbb{N} \mid \lim_{\lambda\to 0} \lambda^{\omega+\varepsilon} u_\lambda(\varphi) = 0;\ 0 < \varepsilon < 1; \lambda \in \mathbb{R}_+\right\} \ . \tag{3.1}$$

The existence of the above limit depends on the behaviour of the distribution u under the proper scaling transformation $x' = \lambda x$ for $\lambda \to 0$. Notice that the scaling degree can be viewed as a generalization of the notion of the degree of homogeneity of a distribution.[1] There follows in a straightforward manner from the definition that

$$\sigma(uv) = \sigma(u) + \sigma(v)\,, \qquad \sigma(u+v) = \max\{\sigma(u), \sigma(v)\}\,. \tag{3.2}$$

The **singular order** of a distribution u for $x_1 = \cdots = x_d$ is defined by $\Sigma(u) = \sigma(u) - d$, where $\sigma(u)$ is the scaling degree of u and n is the dimension of the space.

[1] Let $u(x)$ be an homogeneous function of degree $a \in \mathbb{R}$, *i.e.* $u(\lambda x) = \lambda^a u(x)$ for $\lambda > 0$. Hence, it is induced the following relation:

$$u_\lambda(\varphi) \equiv \int dx'\ u(\lambda x')\varphi(x') = \int dx'\ \lambda^a u(x')\varphi(x') = \int dx\ \lambda^{-d} u(x)\varphi(\lambda^{-1}x) = u(\varphi_\lambda)\ ,$$

where $\varphi_\lambda(x) = \lambda^{-d}\varphi(\lambda^{-1}x)$. A distribution u on $\mathbb{R}^d\backslash 0$ is called *homogeneous* of degree a if (1) is valid for $\varphi \in \mathcal{D}(\mathbb{R}^d\backslash 0)$. If u is a distribution on $\mathbb{R}^d$ and (1) is valid for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, then u is said homogeneous of degree a on $\mathbb{R}^d$.

Example 3.1. For the distribution $\delta \in \mathcal{D}'(\mathbb{R}^d)$ we have:

$$\delta_\lambda(\varphi) = \int d^dx\ \delta(\lambda x)\varphi(x) = \int d^dx\ \lambda^{-d}\delta(x)\varphi(x) = \lambda^{-d}\delta(\varphi)\,.$$

Thus, the condition

$$\lim_{\lambda\to 0} \lambda^{\omega+\varepsilon}\delta_\lambda(\varphi) = \lim_{\lambda\to 0} \lambda^{\omega-d+\varepsilon}\varphi(0) = 0 \quad \Longrightarrow \quad \omega \geq d\ .$$

The scaling degree is then $\sigma(\delta) = d$, and the singular order $\Sigma(\delta) = 0$. ▲

Example 3.2. The distribution $\theta \in \mathcal{D}'(\mathbb{R})$ is such that $\theta(\lambda(t-t')) = \theta(t-t')$ or $\theta_\lambda(\varphi) = \theta(\varphi)$, hence the condition $\lim_{\lambda\to 0}\lambda^{\omega+\varepsilon}\theta_\lambda(\varphi) = 0$ implies that $\omega \geq 0$, $\sigma(\theta) = 0$ and $\Sigma(\theta) = -d$. ▲

Example 3.3. In the case of the Wightman function Δ_+ we have:

$$\Delta_+(\lambda(x-x');m^2) = \int \frac{d^{(d-1)}\mathbf{k}}{(2\pi)^3\,2\omega_k} e^{-ik(\lambda(x-x'))} = \lambda^{2-d}\Delta_+(x-x';\lambda^2m^2).$$

Then, one has $\omega \geq d-2$, the scale degree is $\sigma(\Delta_+) = d-2$ and singular order is $\Sigma(\Delta_+) = -2$. ▲

The following concern the renormalization of divergent integrals. Let the distributions $f \in \mathcal{D}'(\mathbb{R})$ be represented in terms of continuous linear functionals,

$$f(\varphi) = \langle f, \varphi\rangle = \int dx\ f(x)\varphi(x)\ , \tag{3.3}$$

with the test functions $\varphi(x) \in \mathcal{D}(\mathbb{R})$. In order to illustrate the renormalization as the problem of the extension, let us set $f(x) = 1/x$. In this case, $f(x)$ is locally integrable only away from the point $x = 0$, causing the integral to be non convergent. So, (3.3) can not define a regular distribution for $f(x) = 1/x$, However, if $\varphi(x)$ is supported away from 0, *i.e.*, $\operatorname{supp}\varphi \subset \mathbb{R}\backslash 0$, then the integral (3.3) turns to make sense, with $f(\varphi)$ defining a linear functional on $\mathcal{D}(\mathbb{R}\backslash 0)$ which is continuous. In accord to the Hahn-Banach theorem [20] , this functional on $\mathcal{D}(\mathbb{R}\backslash 0)$ can be extended to the whole $\mathcal{D}(\mathbb{R})$. Let $\vartheta(x) \in \mathcal{D}(\mathbb{R})$ be such that $\vartheta(0) = 1$ and $D^\alpha\vartheta(0) = 0$. Thus, we can define

$$U(\varphi) = \int dx\, f(x)\left[\varphi(x) - \vartheta(x)\varphi(0)\right]\ . \tag{3.4}$$

The above integral, which converges for all $\varphi(x) \in \mathcal{D}(\mathbb{R})$, is a renormalization of the integral (3.3). This functional is continuous on $\mathcal{D}(\mathbb{R})$ and coincides with $f(\varphi)$ on $\mathcal{D}(\mathbb{R}\backslash 0)$. It is possible then to obtain an interpretation for the integral (3.3) to $\varphi(x) \in \mathcal{D}(\mathbb{R})$ such that $\varphi \mapsto U(\varphi)$ defines a distribution. The functional U differs of the original functional only in the neighbourhood of the point $x_0 = 0$, *i.e.*, $\|x - x_0\| < r$.

Unfortunately, the extensions are not unique. One can add to the extended U any distribution supported at 0 to obtain another extension. Once every distribution whose support is concentrated at the origin can be written as a finite combination of derivatives of the Dirac measure at 0, any another extension S singular of order $\Sigma(S) \geq \Sigma(U)$ at 0 can be obtained from U by following general expression:

$$S = U + \sum_{|\alpha| \leq \Sigma(S)} c_\alpha D^\alpha \delta(x) \ , \quad \omega = 0, 1, 2, \dots \ . \tag{3.5}$$

In general, if $f(x)$ is a locally integrable function everywhere except at the point $x_0 = 0$, and if $x^{m+\varepsilon} f(x)$, for some integer $m > 0$ and $0 < \varepsilon < 1$, is homogeneous of degree $s > 0$, then $x^{m+\varepsilon} f(x)$ is locally integrable in the neighbourhood of $x_0 = 0$, and a "renormalization" of $f(x)$ can be achieved by means of a Taylor's subtraction:

$$U(\varphi) = \int d^n x \ f(x) \left[\varphi(x) - \vartheta(x) \sum_{|\alpha| \leq \Sigma(f)} \frac{x^\alpha}{\alpha!} \, D^\alpha \varphi(x)|_{x=0} \right] \ , \tag{3.6}$$

Remark 3.1. Notice that from (3.1) one has $\omega \geq 1$, $\sigma(1/x) = 1$, and consequently the singular order $\Sigma(1/x) = 0$. There follows immediatly (3.4) from (3.6).

In short, the ideas above lead to the procedure of renormalization of Feynman amplitudes in quantum field theories. For instance, in the framework of causal perturbation theory by Bogoliubov-Shirkov-Epstein-Glaser [21, 22] the process of renormalization is precisely equivalent to the (Hahn-Banach) extension process of time-ordered off diagonal distributions to the diagonal of coalescent points. This is achieved by an appropriate subtraction on the corresponding test functions.

Remark 3.2. In the renormalization of Feynman graphs both in configuration and momentum space, the renormalization constants, for example, the renormalized mass and coupling constants, there appear as local finite counter-terms which are local polynomials in the fields and their derivatives, order by order in perturbative theory. In the case of a power-counting renormalizable theory, the dimensions of these counter-terms do not exceed the spacetime dimension. It must be emphasized that the arbitrariness contained in such a free counter-terms is the same to that contained in the constants c_α in (3.5). In addition, the free constants c_α are fixed by normalization conditions as well, which define the physical parameters of the theory, *i.e.*, masses and coupling constants. However, often additional physical conditions as the reality, symmetries, Lorentz covariance, causality and unitarity may impose further restrictions to the free constants, thus limiting the number of arbitrary of constants or, analogously, of counter-terms. ∇

4 Renormalization of Distributions in FTFT

We know take advantage on the adopted distributional approach to treat both ITF and RTF at once under the framework of the COF. The time component of the support of the fields lies on a complex plane, lines in Feynman graphs are related to complex contour dependent propagators. Time ordering is replaced by contour ordering [7] what do not harm the validity of Feynman-Mathews-Salam formula [4]. In general, each contour leads to a specific formulation of FTFT. However, under some conditions on analyticity on the propagators, the class of allowed contours are restricted to those which have non increasing imaginary part [23]. In particular, among the formulations, the Matsubara realization of ITF correspond to a contour on the imaginary-t axis and the closed-time-path, a RFT formalism, to a closed contour running around the real-t axis in both forward and backward real time.

In order to study the structure of the renormalization scheme in FTFT, we turn to the analysis of the distributions and their products present in the perturbation series. The distributions are checked out via the The Hörmander's criterion based on its WFSs. Keeping in mind the renormalization procedure as an extension problem together to the its inherent arbitrariness governed by singular order, the perturbation expansion is further discussed. Without loss in generality, let us consider the case of a single, scalar field $\phi(x)$ in FTFT associated to spinless particles with mass $m > 0$. The generalization of the present prescription to any field with arbitrary spin is straightforward.

$$G^c(x,x') = \theta_c(t-t')\langle\hat{\phi}(x)\hat{\phi}(x')\rangle + \theta_c(t'-t)\langle\hat{\phi}(x')\hat{\phi}(x)\rangle. \tag{4.1}$$

The brakets $\langle\cdots\rangle$ stand for statistical average related to states of a complete orthogonal basis of the Fock space. The index (c) accounts for the contour ordering in the complex time plane-t $(t = x_0 + ix_4)$ whose the imaginary and real parts are interpreted to be the inverse temperature and actual time respectively. For the contour ordering prescription given by $\theta_c(t-t')$, it is supposed that the contour c is monotonously increasing and regular parameterized by a real $\tau \in \mathbb{R}$ parameter, $C = \{t \in \mathbb{C} \mid \mathrm{R}e\, t = x_0(\tau), \mathrm{I}m\, t = x_4(\tau), \tau \in \mathbb{R}\}$ and $\theta_c(t-t') = \theta(\tau-\tau')$. The spectral decomposition of $\hat{\phi}$ in terms of plane waves has the ordinary form,

$$\hat{\phi}(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_k}\left[a_k e^{-ikx} + a_k^\dagger e^{ikx}\right], \tag{4.2}$$

though the normalization of the Fock space states for the thermal case include the statistical distribution of the particles associated, in the present case, the Bose-Einstein statistic given by $N(k_0) = \frac{1}{e^{\beta k_0 - 1}}$. Hence,

$$\langle a_k^\dagger a_k\rangle = (2\pi)^3 2\omega_k N(w_k)\delta(\mathbf{k}-\mathbf{k}') \tag{4.3}$$

$$\langle a_k a_k^\dagger\rangle = (2\pi)^3 2\omega_k \left[N(w_k)+1\right]\delta(\mathbf{k}-\mathbf{k}') \tag{4.4}$$

The correlation functions $C^>(x,x') = \langle\hat\phi(x)\hat\phi(x')\rangle = C^<(x',x)$ turns to have the following spectral expansion

$$\langle\hat\phi(x)\hat\phi(x')\rangle = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-x')}\rho(k)\,[1+N(k_0)]\,, \tag{4.5}$$

where $\rho(k) = 2\pi\,[\theta(k_0)-\theta(-k_0)]\,\delta(k^2-m^2)$. Their Fourier transforms, related by $\tilde C^<(k) = \rho(k)\,[N(k_0)+1] = e^{\beta k_0}\tilde C^>(k)$, can be used in order to write the contour ordered propagator in the form

$$G^c(x,x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-x')}\rho(k)\,[\theta_c(t-t')+N(k_0)]\,. \tag{4.6}$$

Another useful form is obtained after integration on k_0,

$$\begin{aligned} G^c(x,x') =&\theta_c(t-t')\int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_k}\left\{[N(\omega_k)+1]\,e^{-ik(x-x')} + N(\omega_k)e^{ik(x-x')}\right\} \\ &+ \theta_c(t'-t)\int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_k}\left\{[N(\omega_k)+1]\,e^{ik(x-x')} + N(\omega_k)e^{-ik(x-x')}\right\}. \end{aligned} \tag{4.7}$$

Although each possible contour would correspond to a specific formalism of FTFT, there are restrictions on the contours due to the necessary analyticity of the correlation functions and the KMS condition [4]. These conditions cause the support of the two point function to be analytic on the strip given by $-\beta \le \mathrm{I}m\,(t-t') \le \beta$, which on the closure the distributional character takes place. Furthermore, for the analyticity of $C^>(x,x')$, which because of the factor $\theta_c(t-t')$ has vanishing contributions to the propagator (4.1) if t' succeeds t on C, it is required that $-\beta \le \mathrm{I}m\,(t-t') \le 0$. Conversely, for the analyticity of $C^<(x,x')$, with factor $\theta_c(t'-t)$, it is required that $0 \le \mathrm{I}m\,(t-t') \le \beta$. Combining both relations one can conclude that if the complex time t_1 succeeds t_2 on C, then there follows that $\mathrm{I}m\,t_2 \ge \mathrm{I}m\,t_1$ what imposes that C must have a non increasing imaginary part. In other words C must have constant or decreasing imaginary part. This is called *monotonousness condition*.

At this point, once the adopted approach does not depend on Feynman graphics calculations, we can proceed the analysis without the need in specializing to Minkowiskian RTF or Euclidean ITF parameterizations of the contour. ¿From (4.7) we select two typical distributions a temperature dependent piece, $G^{(c)\pm}_{mat}$ and a temperature independent piece, $G^{(c)\pm}_{vac}$ whose labels refer to *matter piece* and *vacuum piece* due to their origin in (4.4).

$$G^{(c)\pm}_{mat}(x,x') = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_k} N(\omega_k)e^{\mp ik(x-x')}, \tag{4.8}$$

$$G^{(c)\pm}_{vac}(x,x') = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_k} e^{\mp ik(x-x')}. \tag{4.9}$$

In terms of these distributions, the general contour propagator turns to be

$$G^c(x,x') = \theta_c(t-t')\left[G^{(c)+}_{mat}(x,x') + G^{(c)+}_{vac}(x,x') + G^{(c)-}_{mat}(x,x')\right] + \theta_c(t'-t)\left[G^{(c)-}_{mat}(x,x') + G^{(c)-}_{vac}(x,x') + G^{(c)+}_{mat}(x,x')\right]. \tag{4.10}$$

The structure of the propagators of FTFT suggests that, at a given order in perturbation series, the crossing products between matter and vaccum pieces would produce qualitatively different divergences. Furthermore, one could expect it to have also a proliferation of divergent terms. Another possible distinction between FTFT and QFT version would be on the amount of arbitrariness through the contribution to the singular order besides the establishment of a temperature dependent renormalization extension problem. We are going to verify that in some sense the above fact do occur. The perception of one or other of these points and the consequences would be difficulted or not depending on the renormalization procedure adopted.

We now turn to investigate the divergent content of the distributions $G^{(c)\pm}_{mat(vac)}$ by calculating their WFSs. We then proceed the study by using the stationary phase method discussed in the Appendix. The phases of the above distributions have all the same form $\mp ik(x-y)$. Then, it is usefull to unify the notation as much as possible and represent them all by defining the following integral

$$G^{(c)\pm}_{mat(vac)} = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{\tilde{f}_{mat(vac)}(\mathbf{k};m^2,\beta)}{2\omega_k}\; e^{\mp i[\omega(t-t')-\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')]}\;, \tag{4.11}$$

where $\tilde{f}_{mat}(\mathbf{k};m^2,\beta) = N(\omega_k)$ and $\tilde{f}_{vac}(\mathbf{k};m^2,\beta) = 1$ and from now on, for simplicity just $\tilde{f}(\mathbf{k};m^2,\beta)$ except where the distinction turn to be necessary. One can define the phase function $\varphi_\pm$,

$$\varphi_\pm(\mathbf{k},x-x') = \pm\left[(t-t')|\mathbf{k}| - (\mathbf{x}-\mathbf{x}')\cdot\mathbf{k}\right]\;, \tag{4.12}$$

to obtain the following oscilatory integrals for the distributions:

$$G^{(c)\pm} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} a_{\pm mat(vac)}(t-t',|\mathbf{k}|;m^2)e^{-i\varphi_\pm(\mathbf{k},x-x')}\,. \tag{4.13}$$

where

$$a_{\pm mat(vac)}(t-t',|\mathbf{k}|;m^2) = \frac{\tilde{f}_{mat(vac)}(\mathbf{k};m^2,\beta)}{2\omega_k}e^{\mp i[(\omega-|\mathbf{k}|)(t-t')]} \tag{4.14}$$

is the asymptotic symbol. From the definition of the phase function (4.12) and from the discussion in the Appendix one can easily see that it must be such that Im $(t-t') \leq 0$. Then, had the monotonousness condition not previously selected the possible contours, the $\varphi_\pm$ would be ill-defined. Both are in fact manifestations of the necessary analyticity of the Green functions. The directions along which the phase in the integrand do not vary satisfying $\partial_{\mathbf{k}}\varphi_{\pm} = 0$ give us the following critical set,

$$\begin{aligned}
\mathcal{C}_{\varphi_\pm} = & \big\{ (x - x' = (0, \mathbf{0}), k) \mid (k \neq 0) \in \mathbb{R}^4 \big\} \\
& \cup \big\{ (x - x', k) \mid \ (\ \mathbf{x} - \mathbf{x}' \parallel \mathbf{k} \neq \mathbf{0} \ ) \in \mathbb{R}^3, \ (t - t') \in \mathbb{C}, \ \mathbf{k} \cdot (\mathbf{x} - \mathbf{x}') > 0, \\
& \qquad \mathrm{Re}\,(t - t') = |\mathbf{x} - \mathbf{x}'|, \ \mathrm{Im}\,(t - t') = 0 \ \big\} \qquad (4.15) \\
& \cup \big\{ (x - x', k) \mid \ (\ \mathbf{x} - \mathbf{x}' \parallel \mathbf{k} \neq \mathbf{0} \ ) \in \mathbb{R}^3, \ (t - t') \in \mathbb{C}, \ \mathbf{k} \cdot (\mathbf{x} - \mathbf{x}') < 0, \\
& \qquad \mathrm{Re}\,(t - t') = |\mathbf{x} - \mathbf{x}'|, \ \mathrm{Im}\,(t - t') = 0 \ \big\} \ .
\end{aligned}$$

Though there is the restriction to those terms in (4.10) which satisfies the monotonousness condition, from the additional condition $\mathrm{Im}\,(t - t') = 0$, one can see that there are no contributions coming from the pieces of the contour with non vanishing imaginary part. It has important consequences in the analysis of the WFS for the ITFs. Because the set of singular points of the WFS is a subset of $\mathcal{C}_{\varphi_\pm}$ (see Appendix), because ITF-like pieces of the contour are such that $\mathrm{Im}\,(t - t') > 0$, they are free of those singular points and one can conclude since now that the WFS associated to ITF correlation functions is empty. It can be now obtained the stationary phase manifold Λ_φ to be the set of points of critical set having the non vanishing four momentum component given by the gradients $\partial_\mu\varphi_+ = (|\mathbf{k}|, -\mathbf{k})$ and $\partial_\mu\varphi_- = (-|\mathbf{k}|, \mathbf{k})$. Then,

$$\begin{aligned}
\Lambda_{\varphi_\pm} = & \big\{ (x - x' = (0, \mathbf{0}), (\pm\lambda|\mathbf{k}|, \mp\lambda\mathbf{k})) \ \mid \ (\ \mathbf{x} - \mathbf{x}' \parallel \mathbf{k} \neq \mathbf{0} \ ) \in \mathbb{R}^3, \ \lambda \in \mathbb{R}_+ \big\} \\
& \cup \ \big\{ (x - x', (\pm\lambda|\mathbf{k}|, \mp\lambda\mathbf{k})) \ \mid \ (\ \mathbf{x} - \mathbf{x}' \parallel \mathbf{k} \neq \mathbf{0} \ ) \in \mathbb{R}^3, \ (t - t') \in \mathbb{C}, \\
& \qquad \lambda \in \mathbb{R}_+, \ \mathbf{k} \cdot (\mathbf{x} - \mathbf{x}') > 0, \ \mathrm{Re}\,(t - t') = |\mathbf{x} - \mathbf{x}'|, \ \mathrm{Im}\,(t - t') = 0 \ \big\} \quad (4.16) \\
& \cup \ \big\{ (x - x', (\pm\lambda|\mathbf{k}|, \mp\lambda\mathbf{k})) \ \mid \ (\ \mathbf{x} - \mathbf{x}' \parallel \mathbf{k} \neq \mathbf{0} \ ) \in \mathbb{R}^3, \ (t - t') \in \mathbb{C}, \\
& \qquad \lambda \in \mathbb{R}_+, \ \mathbf{k} \cdot (\mathbf{x} - \mathbf{x}') < 0, \ \mathrm{Re}\,(t - t') = |\mathbf{x} - \mathbf{x}'|, \ \mathrm{Im}\,(t - t') = 0 \ \big\} \ .
\end{aligned}$$

The above result can be interpreted as the set of pairs of which the critical character of the phase is such that it brakes certain natural tendency of the integrals to converge due to its oscillatory character (*See* Riemman-Lebesgue Lemma [24]). Such pairs are, therefore, suspect to be responsible to some bad behaviour of the oscillatory integral. As it is discussed in the Appendix, one has $WF(G^{(c)\pm}) \subseteq \Lambda_{\varphi\pm}$. Because they are still able to save the convergence in some or even in all those critical directions, there remains to be studied the contributions of the asymptotic symbols, $a_{\pm mat(vac)}$ and, in particular, $\tilde{f}_{mat(vac)}$ to the convergence of the integrals. For the temperature dependent part, to every possible contribution considered in the stationary phase manifold (4.16), the exponential factor $e^{\beta\omega_k}$ in the denominator of the integrand $\tilde{f}_{mat}(\mathbf{k}; m^2, \beta) = N(\omega_k)$ assures the condition for a fast decreasing function (see Sec. 2) to be fulfilled in every of those critical directions. This guarantees the existence of the oscillatory integral and characterizes $G^{(c)\pm}_{mat}$ to be a smooth function. Its WFS contribution is then empty. However, in the case of the vacuum piece, $\tilde{f}_{vac}(\mathbf{k}; m^2, \beta) = 1$, the factor $\frac{1}{\omega_k}$ does not suffice to assure the asymptotic fast decrease in none of those critical directions. So, every pair in $\Lambda_{\varphi\pm}$ turns to be an element of the WFS. Therefore we have

$$WF(G^{(c)\pm}_{vac}) = \Lambda_{\varphi\pm}\,, \qquad WF(G^{(c)\pm}_{mat}) = \emptyset\ . \tag{4.17}$$

Hence, there are no contributions coming from the matter piece temperature dependent part to the WFS $WF(G^{(c)\pm})$. It is necessary to emphasize that the $G^{(c)\pm}_{vac}$, which was at the start a contour ordered propagator on the complex t-plane, due to the restriction $\text{Im}\,(t - t') = 0$ has exactly the same singular spectrum as the Wightman function $\Delta_\pm$, (2.3), for of the ordinary QFT. Thus, we have settled that

$$WF(G^{(c)\pm}_{vac}) = WF(\Delta_\pm). \tag{4.18}$$

There follows then the same rules discussed for $\Delta_\pm$, in particular, for the product $\theta^\pm \cdot G^{(c)\pm}_{vac}$ one has

$$WF(\theta^\pm \cdot G^{(c)\pm}_{vac}) = WF(\theta^\pm \cdot \Delta_\pm)\,, \tag{4.19}$$

$$(x, 0) \notin (WF(\theta^\pm) \oplus WF(G^{(c)\pm}_{vac})), \tag{4.20}$$

what characterize it as well-defined and consequently, from the results of the condition of the Hörmander's criterion (2.2),

$$\begin{aligned} WF(\theta^\pm \cdot G^{(c)\pm}_{vac}) &\subset WF(\theta^\pm) \cup WF(G^{(c)\pm}_{vac}) \cup (WF(\theta^\pm) \oplus WF(G^{(c)\pm}_{vac})) \\ &= WF(\theta^\pm) \cup WF(\Delta_\pm) \cup (WF(\theta^\pm) \oplus WF(\Delta_\pm))\,. \end{aligned} \tag{4.21}$$

Because $G^{(c)\pm}_{mat}$ is a smooth function from the property (4) in Remark (2.1), the product $\theta^\pm \cdot G^{(c)\pm}_{mat}$ is such that

$$WF(\theta^\pm \cdot G^{(c)\pm}_{mat}) \subset WF(\theta^\pm), \qquad (x, 0) \notin WF(\theta^\pm)\,. \tag{4.22}$$

For this reason, in view of (4.20), (4.21) and (4.22), the FTFT contour propagator $G^{(c)}$, (4.10), is well-defined as sum of well-defined products. From the property 6 in Remarks 2.1 and (2.6)

$$\begin{aligned} WF(G^{(c)}) \subset\ &[\ WF(\theta^+) \cup WF(\Delta_+) \cup (WF(\theta^+) \oplus WF(\Delta_+)) \cup \\ &\cup WF(\theta^-) \cup WF(\Delta_-) \cup (WF(\theta^-) \oplus WF(\Delta_-))\ ] \supset WF(\Delta_F)\,. \end{aligned} \tag{4.23}$$

By other hand, in the higher orders of the perturbative calculations there arise products of propagators. In special, let us consider those terms in which there are products like $G^{(c)+}_{vac} \cdot G^{(c)-}_{vac}$. From (4.17) one can see that in the same way of the ordinary QFT for Δ_+,

$$(x, 0) \in WF(G^{(c)+}_{vac}) \oplus WF(G^{(c)-}_{vac})\,. \tag{4.24}$$

It does not match the condition for the Hörmander's criterion. Indeed, this is also a not well-defined product if the support of the distributions include $x = 0$ what turns it to be a problem to be treated through the renormalization procedure. But products like $G^{(c)s}_{mat} \cdot G^{(c)s'}_{mat}$ and $G^{(c)s}_{mat} \cdot G^{(c)s'}_{vac}$, where $s, s' = +, -$ are sign indexes, because $G^{(c)s}_{mat}$ are smooth functions, they are well-defined. Therefore, when considered products of propagators in the FTFT, both in RTF and ITF, one can expect that the presence of the matter piece does not contribute to generate ill-defined terms beside those yet found in the ordinary QFT. Nevertheless, in the higher orders in the perturbation expansion, it appears as temperature dependent factors to the ordinary divergences. Roughly speaking, although the ill-defined products are the same as the QFT ones, they appear with temperature dependent factors.

Lets us focus another character of the renormalization concerning the arbitrariness or ambiguity of the process and its relation to physical symmetries. As we have discussed in the section 3, the amount of arbitrariness is governed by the singular order and scale degree of the distributions involved. Once for $G^{(c)\pm}_{mat}$,

$$\begin{aligned}(G^{(c)\pm}_{mat})_\lambda = G^{(c)\pm}_{mat}(\lambda(x-x');\ m^2,\ \beta) &= \int \frac{d^{(d-1)}\mathbf{k}'}{(2\pi)^3\, 2\omega_{k'}} N(\omega_{k'}) e^{\mp ik'(x-x')} \\ &= \lambda^{2-d} G^{(c)\pm}_{mat}(x-x'; \lambda^2 m^2, \lambda^{-1}\beta)\,,\end{aligned}$$

then, one has $\omega \geq d-2$, the scale degree is $\sigma(G^{(c)\pm}_{mat}) = d-2$ and singular order is $\Sigma(G^{(c)\pm}_{mat}) = -2$. Notice further that

$$\Sigma(G^{c\pm}_{mat}) = \Sigma(\Delta_\pm) = \Sigma(G^{(c)\pm}_{vac})\,. \tag{4.25}$$

For the FTFT propagator $G^{(c)}$ in (4.10), Δ_F from the correlation functions and by using (3.2), the example (3.2) and (4.25) that

$$\sigma(\theta^\pm G^{c\pm}_{mat(vac)}) = \sigma(G^{(c)}) = \sigma(\Delta_F) = d-2\,, \tag{4.26}$$

$$\Sigma(\theta^\pm G^{c\pm}_{mat(vac)}) = \Sigma(G^{(c)}) = \Sigma(\Delta_F) = -2\,. \tag{4.27}$$

Keep in mind that the ITF matter piece correlation functions are free of singularities and the RTF ones, despite of the existence of stationary phase points indicating singularities are of fast decrease and in the spite of avoid the complexity introduced by the doubling of degrees of freedom unnecessary to the present analysis of the distributions involved in the RFT, we turn to analyze as a representative case of the higher order product in the perturbation series the square of the propagator associated to the branch of the contour which is parameterized forward in the real time only. We consider again the products of propagators arising in the perturbative series. For the products like $G^{(c)s}_{mat} \cdot G^{(c)s'}_{mat}$, $G^{(c)s}_{mat} \cdot G^{(c)s'}_{vac}$ and $G^{(c)s}_{vac} \cdot G^{(c)s'}_{vac}$ we have

$$\begin{aligned}\sigma(G^{(c)s}_{mat(vac)} \cdot G^{(c)s'}_{mat(vac)}) &= 2(d-2)\,, \\ \Sigma(G^{(c)s}_{mat(vac)} \cdot G^{(c)s'}_{mat(vac)}) &= 2(d-2) - d\,.\end{aligned} \tag{4.28}$$

Notice that the scale degree and singular order is the same for both the matter or vacuum pieces products. In view of (3.5) one can see that the singular order determines the number of arbitrary coefficients (counter-terms) in the renormalization procedure. Therefore, in concerning the renormalization the treatment of the arbitrary constants the matter piece would require, in the worse case, the same number of counter terms necessary to fix the ill-defined products in the ordinary QFT. Before we step to the conclusions, let us examine the example of a overlapping 1-loop in $\frac{g}{4!}\phi^4$, a truncated 4-point diagram with two internal lines connecting two different vertex,

$$\Gamma^{(4)} \sim g^2[G^{(c)}(x-x')]^2 = g^2 \Bigg\{ \sum_{s=+,-} \theta^s\theta^s G^{(c)s}_{vac} G^{(c)s}_{vac} + \sum_{s=+,-} \theta^s\theta^{-s} G^{(c)s}_{vac} G^{(c)-s}_{vac} + \\ + 2\sum_{s\ s'} \theta^s\theta^s G^{(c)s}_{vac} G^{(c)s'}_{mat} + 2\sum_{s\ s'} \theta^s\theta^{-s} G^{(c)s}_{vac} G^{(c)s'}_{mat} + \\ + \sum_{s\ s' s''} \theta^s\theta^s G^{(c)s'}_{mat} G^{(c)s''}_{mat} + \sum_{s\ s'\ s''} \theta^s\theta^{-s} G^{(c)s'}_{mat} G^{(c)s''}_{mat} \Bigg\} . \tag{4.29}$$

The sum of products of distributions fall then into different categories. As it was showed in (4.17) and in the chains of reasoning about, the last term and the before the last are composed by smooth functions times the product of Heavyside functions. The last one vanishes because the product of $\theta^s\theta^{-s}$. As it showed, (4.17) and reasonings just after (4.24), the product of the correlations functions belonging to the third and forth terms are well-defined. The forth term vanishes too. The first term shows to be a well-defined product as well in accord to (4.18) and (2.7). The only term which exhibits a ill-defined product is the second one. That product is well-defined elsewhere but at $x - x' = 0$. This is the target of the renormalization procedure in the present case, the problem of the extension of the validity of that product. The degree of arbitrariness is governed by (4.28) and the related number of coefficients of (3.5) (counter-terms) associated to is to be determined by certain physical symmetries. Let us consider now a superior order overlapping loop, three internal lines connecting two vertex $\sim g^2[G^{(c)}(x-x')]^3$. One finds the proliferation of ill-defined terms if compared to the ordinary QFT case. There will arise the ill-defined product $\theta^s\theta^s\theta^{-s}G^{(c)s}_{vac}G^{(c)s}_{vac}G^{(c)-s}_{vac}$ beside others like $\theta^s\theta^s\theta^{-s}G^{(c)s}_{mat}G^{(c)s}_{vac}G^{(c)-s}_{vac}$. The former suffers of the same illness of the second term of (4.29), though a different degree and is to be treated in a temperature independent fashion. The latter, though the presence of a matter piece factor, what in view of (4.28), one could expect it to constitute a temperature dependent extension problem. However, from (4.17), (4.24) and the discussion immediately after, one can easily that the above ill-definition is only to the vacuum pieces product. Notice that this term was temperature independently treated in a inferior order of (4.29). This quantitative analysis has showed that despite of the increase in number ill-definition and the presence of the temperature dependent factors the extension problem and the amount of arbitrariness in the singular order dominant term are temperature independent.

The above example makes clear that the matter piece, being absent of a singular spectrum, can not include any new contribution to FTFTs concerning the category of ill-defined products yet found in the ordinary QFT. In other words, the set ill-defined terms is the same though the occurrence increases. Despite of the existence of a temperature dependent factors multiplying the ill-defined products, from the point of view of the extension problem, it can be treated order by order as a vacuum extension problem. Furthermore, the degree of arbitrariness in the process for a given order is limited by the singular order of the temperature independent piece and, from products of them, there arise order by order the ill-defined product leading in singular order and degree of arbitrariness.

5 Conclusions

¿From the Hörmanders criterion based on the WFSs of these distributions, one shows that the contribution to form ill-defined products comes from the temperature independent pieces only. Hence, the matter piece do not contribute to form divergent terms. The structure of the propagators of FTFT, being separable into vacuum and matter pieces turns easy the analysis of the ill-defined products. In fact, one shows that the separation also generates an increasing on the number of ill-defined products in the perturbation series due to the mixing of these factors in crossing products in the higher order terms of the perturbation expansion. The matter piece appears then as temperature dependent factors of ill-defined products vacuum pieces in the higher orders of the perturbation series. By other hand, the calculations of the scale degree and singular order which determines amount of arbitrariness also shows that one can consider in general the contributions coming from the matter piece. These facts together can mislead one to face the problem as an inherent temperature dependent renormalization problem. The ambiguities can also be viewed as inexorably temperature dependent. In some sense the above facts do occur. Focusing now the perturbation series, the degree of arbitrariness in the process for a given order is determined by the temperature independent ill-defined product leading in singular order. This leading in singular order ill-defined product is showed to be temperature independent order by order. Once it has the support extended to a given order, in the next order, it can appears only under a temperature dependent factor what do not harm its already temperature independent renormalized content. Hence, the problem of the extension reduces order by order to to the analogous one of the ordinary QFT. Consequently, it is proved that the amount of arbitrariness in the renormalization procedure, as well the type of the ambiguities, if conveniently treated, remains the same when passing from a given QFT to the associated FTFT version. The perception of one or other of these points and the consequences could be difficult or not depending on the renormalization procedure adopted. There follows that a given FTFT remains finite when it is considered a renormalized analogous QFT. It is sufficient for one to state that the theories remains finite in FTFT version when it was renormalized in QFT temperature independent version.

An important role is played by the microlocal techniques is to turn possible to investigate the matter in a free of the technical difficulties of thermal loop calculations way, common to the various conventional approaches both in ITFs and RTFs. For that purpose, the distributional point of view admits a generalized unified framework of the contour ordered formalism considering at a time without any particular parameterization for ITF or RTF contours. The Hörmanders criterion together to the wavefront set of the distributions show to provide powerful methods for one to make conclusions about the perturbation series. Once the approach is based in mathematical grounds, although we have appealed to the scalar ϕ^4_4-theory, the method of analysis is clearly model independent. However, the Hörmander's criterion furnishes only sufficient condition for the existence of the products. Therefore for the determination of whether some product which does not match the criterion is ill-defined, one must use another method. A useful method for physical applications is to check the convergence of the convolution of the Fourier transforms.

The results concerning the renormalization of FTFT are already known. The present contribution lies on the method which allows to clarify some points in the comparison between QFT and FTFT renormalization in the unified contour ordered formalism. The problem of the divergences were faced from the ground by the mathematical study of the basic ill-defined products distributions *i. e.* the lack of definition of the distributional product on the coinciding points. A central role was played by viewing renormalization procedure as an extension problem together to the inherent arbitrariness present in the process whose amount governed by singular order. It furnishes the basis for the study of the nature of the arbitrariness of the procedure. The technical aspects were briefly revised in order to make it handle for theoretical physics applications.

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A Oscillatory Integral: Some Basic Concepts

In this appendix, we review some results from the Fourier Integral Operator Theory, or Oscillatory Integral, that are used in the text. In particular it provides the stationary phase method [25] to compute the WFS of distributions. This matter is useful in the theory of pseudodifferential operators in order to find the asymptotic behaviour of an integral of the form $\int dk\ e^{-i\lambda\varphi(k)}a(k)$, when $\lambda \to \infty$ and φ has critical points.

Pseudodifferential operators generalize the concept linear differential operators with variable coefficients $a(x,D) = \sum_{|\alpha|\leq m} a_\alpha(x) D_x^\alpha$ with $D = i\partial/\partial x$. To the pseudodifferential operator $a(x,D)$, one can attribute meaning on $u(x) \in \mathcal{D}(\mathbb{R}^n)$ by the Fourier inverse transformation

$$a(x,D)u(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} d^n k\ a(x,k) e^{-ikx}\widehat{u}(k)\ , \tag{A.1}$$

where $\widehat{u}(k)$ is the Fourier transform. If $a(x,k) \in C^\infty(X)$ x is in the open set $X \subset \mathbb{R}^d$, then one defines the space of symbols, $S^m(X \times \mathbb{R}^s)$, such that

$$\left| D_k^\beta a(x,k)\right| \leq C_{\beta,\Omega}(1+|k|)^{m-|\beta|} \quad \forall\, x \in \Omega; k \in \mathbb{R}^s\ , \tag{A.2}$$

where Ω is any compact subset of X and $D_x^\alpha = D_{x^1}^{\alpha_1}\cdots D_{x^n}^{\alpha_n}$, The lower constants $C_{\alpha,\beta,\Omega}$, in (A.2) are semi-norms

$$\|a\|_{\beta,\Omega} = \sup_{x\in\Omega;\, k\in\mathbb{R}^s}(1+|k|)^{|\beta|-m}\left| D_k^\beta a(x,k)\right|\ . \tag{A.3}$$

There are more general classes of space of symbols [18], but for the reason that in general, in the most physical applications it is sufficient to deal with the above spaces, we restrict ourselves to them.

Let $a(x,k) \in S_{1,0}^m(X \times \mathbb{R}^s)$, then by the Fourier transform, we obtain

$$Au(x) = \frac{1}{(2\pi)^d}\int d^d k\ e^{-ikx}a(x,k)\widehat{u}(k) = \frac{1}{(2\pi)^d}\int d^d k d^d y\ e^{-ik(x-y)}a(x,k)u(y)\ . \tag{A.4}$$

The kernel of A can then be written by means of an oscillatory integral

$$K_A(x,y) = \frac{1}{(2\pi)^n}\int d^n k\ e^{-ik(x-y)}a(x,k)\ . \tag{A.5}$$

An oscillatory integral on $X \times \mathbb{R}^s$ is formally written as

$$I_\varphi(a) = \int dk\ e^{-i\varphi(x,k)}a(x,k)\ , \tag{A.6}$$

where $\varphi(x,k)$ is a **phase function** and $a(x,k)$ is an asymptotic symbol.

Let $X \subset \mathbb{R}^d$ be open and $\Gamma \subset \mathbb{R}^s \backslash 0$ be an open cone. Let us further consider the set of all pairs $(x, k) \in X \times \mathbb{R}^s \backslash 0$. The set is invariant by multiplying the second component of the pairs by real positive scalars. It happens because Γ is itself invariant under this operation. One defines $\varphi(x, k) \in C^\infty(\Gamma)$ as a phase function in $X \times \Gamma$ if

1. φ is homogeneous of first degree in k: $\varphi(x, \lambda k) = \lambda \varphi(x, k)$ if $(x, k) \in \Gamma, \ \forall \ \lambda > 0$.

2. $\operatorname{Im} \varphi(x, k) \leq 0$.

3. $d\varphi = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i} dx_i + \sum_{i=j}^{s} \frac{\partial \varphi}{\partial k_j} dk_j \neq 0$, *i.e.*, φ has no critical points in Γ. This means that at every point in Γ, some $\frac{\partial \varphi}{\partial x_i}$ or $\frac{\partial \varphi}{\partial k_j}$ is non-vanishing.

If $\varphi \in C^\infty(X \times \mathbb{R}^s \backslash 0)$ is a phase function, we call

$$\mathcal{C}_\varphi = \{(x, k) \in X \times \mathbb{R}^s \backslash 0 \mid \varphi'_k(x, k) = 0\} \ ,$$

the *critical set of* φ, where $\varphi'_k(x, k) = \left(\frac{\partial \varphi}{\partial k_1}, \ldots, \frac{\partial \varphi}{\partial k_s} \right)$. The condition $\varphi'_k(x, k) = 0$ sets the regions on $X \times \mathbb{R}^s \backslash 0$ over which the phase function has its second component kept constant. The *stationary phase manifold* is the point set

$$\Lambda_\varphi = \{(x, \varphi'_x(x, k)) \mid (x, k) \in \mathcal{C}_\varphi; k \neq 0\} \ ,$$

with, $\varphi'_x(x, k) = \left(\frac{\partial \varphi}{\partial x_1}, \ldots, \frac{\partial \varphi}{\partial x_n} \right)$. The second component is then settled to be the directions on $X \times \mathbb{R}^s \backslash 0$ restricted to $\mathcal{C}_\varphi$ along which the phase $\varphi(x, k)$ has the strongest sensibility on x. Hence, the behaviour of $\varphi(x, k)$ and $a(x, k)$ near Λ_φ determines the singularities of $I_\varphi(a)$. For that reason $WF(I_\varphi(a)) \subset \Lambda_\varphi$.

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