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Division Algebras and Extended $N = 2, 4, 8$ SuperKdVs

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The first example of an $N = 8$ supersymmetric extension of the KdV equation is here explicitly constructed. It involves 8 bosonic and 8 fermionic fields. It corresponds to the unique $N = 8$ solution based on a generalized hamiltonian dynamics with (generalized) Poisson brackets given by the Non-associative $N = 8$ Superconformal Algebra. The complete list of inequivalent classes of parametric-dependent $N = 3$ and $N = 4$ superKdVs obtained from the "Non-associative $N = 8$ SCA" is also furnished. Furthermore, a fundamental domain characterizing the class of inequivalent $N = 4$ superKdVs based on the "minimal $N = 4$ SCA" is given.

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1 Introduction.

In the last several years integrable hierarchies of non-linear differential equations in $1+1$ dimensions have been intensely explored, mainly in connection with the discretization of the two-dimensional gravity (see [1]).

Supersymmetric extensions of such equations have also been largely investigated [2]-[7] using a variety of different methods. Unlike the bosonic theory, many questions have not yet been answered in the supersymmetric case.

In this paper we construct the first example of a global $N=8$ supersymmetric extension of the KdV equation. The strategy used is based on the derivation of the supersymmetric non-linear equations from a generalized hamiltonian system admitting the "Non-Associative $N=8$ Superconformal Algebra" of Englert et al. [8] as a generalized Poisson bracket. The non-associativity of such an algebra (i.e. the failure in fulfilling the Jacobi identities) allows to overcome a no-go theorem based on strict mathematical results. The higher-derivative term in the KdV equation can be seen as induced by the central extension of the Virasoro algebra. However, the complete list of allowed central charges for (ordinary) N-extended superconformal algebras has been produced in the mathematical literature [9]. Central charges can be introduced for $N \leq 4$ only. Indeed, supersymmetric generalizations of KdV up to $N=4$ have been constructed [3, 6]. In order to construct supersymmetric generalizations of KdV for $N>4$ one is therefore led to relax some condition on the nature of the superconformal algebras of Poisson brackets. Allowing non-associativity as in the $N=8$ SCA of reference [8] makes possible to introduce a central extension. It is therefore worth investigating whether this superconformal algebra can be related to the construction of N-extended superKdVs beyond the $N=4$ barrier. This is the purpose of the present paper.

The "Non-Associative $N=8$ SCA" involves 8 bosonic and 8 fermionic fields and is constructed in terms of octonionic structure constants. Its restriction to its real, complex or quaternionic subalgebras leads, respectively, to the ordinary $N=1,2,4$ Superconformal Algebras (in the last case it is the so-called "minimal $N=4$ SCA").

In this paper at first we revisit the $N=2,4$ KdV equations in the language of division algebras. We construct a fundamental domain for the parametric space of the inequivalent $N=4$ KdVs (our results complete and complement the work of [6]) and discuss the issue of integrability.

Later we apply the same techniques to investigate the most general globally $N=8$ invariant generalized hamiltonian for superextended KdV. It turns out that, if we further assume invariance under octonionic involutions, the hamiltonian is unique up to the normalization factor, giving rise to a unique set of $N=8$ KdV equations. Such equations, consistently reduced to the quaternionic subspace, produce the most symmetric (global $SU(2)$-invariant) $N=4$ KdV set of equations. This $N=4$ KdV system, despite being the most symmetric one, does not correspond to the integrable point of $N=4$ KdV. This result therefore suggests that the unique $N=8$ KdV is not an integrable system.

On the other hand the authors of [10] pointed out that global $N=2$ supersymmetric systems can be obtained from the "minimal $N=4$ SCA" Poisson brackets. We extended here such analysis by investigating the class of global $N=3$ and $N=4$ supersymmetric extensions of KdV which can be constructed via the "Non-Associative $N=8$ SCA" generalized Poisson brackets. The complete solution is reported. In the $N = 4$ case two inequivalent classes (both parametric-dependent) of solutions, are found. The existence of two $N = 4$ classes is in consequence of the two inequivalent ways of associating three invariant supersymmetry charges with imaginary octonions (i.e. either producing, or not, an $su(2)$ subalgebra), while the extra supersymmetry charge is always associated with the octonionic identity. In the $N = 3$ case just a single class of parametric solutions is found since any given pair of imaginary octonions is equivalent to any other pair.

We did not investigate here the issue of integrability since our focus was in the construction of supersymmetric extensions. However, we can notice that in the first class of $N = 4$ superKdV extension obtained from the "Non-associative $N = 8$ SCA" the parameters can be conveniently chosen so that a consistent reduction to the integrable $N = 4$ KdV can be made. This leaves room to the possibility that the integrable $N = 4$ KdV can be embedded in such a larger $N = 4$ system which still preserves integrability.

Some further comments are in order. This work is partly a continuation of our previous one [11] concerning the relation between the "Non-Associative $N = 8$ SCA" and the superaffined octonionic algebra. Indeed, by reconstructing via Sugawara the $N = 8$ SCA fields with the affine fields, we can induce on the affine fields a global $N = 8$ set of equations, generalizing both the NLS and mKdV equations, as well as the $N = 4$ construction of reference [12].

We heavily relied on the Thielemans'package for computing classical OPE's with Mathematica [13], supported by our own package to deal with octonionic structure constants.

2 On Division Algebras and the "Non-Associative $N = 8$ SCA".

In this section we recall (see [14] and [11]) the basic properties of the division algebra of the octonions which will be used in the following and introduce the "Non-Associative $N = 8$ Superconformal Algebra" according to [8] (see also [11]).

A generic octonion x is expressed as $x = x_a\tau_a$ (throughout the text the convention over repeated indices, unless explicitly mentioned, is understood), where x_a are real numbers while τ_a denote the basic octonions, with $a = 0, 1, 2, ..., 7$.

$\tau_0 \equiv \mathbf{1}$ is the identity, while τ_α, for $\alpha = 1, 2, ..., 7$, denote the imaginary octonions. In the following a Greek index is employed for imaginary octonions, a Latin index for the whole set of octonions (identity included).

The octonionic multiplication can be introduced through

$$\tau_\alpha \cdot \tau_\beta \quad = \quad -\delta_{\alpha\beta}\tau_0 + C_{\alpha\beta\gamma}\tau_\gamma, \tag{1}$$

with $C_{\alpha\beta\gamma}$ a set of totally antisymmetric structure constants which, without loss of generality, can be taken to be

$$C_{123} = C_{147} = C_{165} = C_{246} = C_{257} = C_{354} = C_{367} = 1. \tag{2}$$

and vanishing otherwise.

It is also convenient to introduce, in the seven-dimensional imaginary octonions space, a 4-indices totally antisymmetric tensor $C_{\alpha\beta\gamma\delta}$, dual to $C_{\alpha\beta\gamma}$, through

$$C_{\alpha\beta\gamma\delta} = \frac{1}{6}\varepsilon_{\alpha\beta\gamma\delta\epsilon\zeta\eta}C_{\epsilon\zeta\eta} \tag{3}$$

(the totally antisymmetric tensor $\varepsilon_{\alpha\beta\gamma\delta\epsilon\zeta\eta}$ is normalized so that $\varepsilon_{1234567} = +1$).

The octonionic multiplication is not associative since for generic a, b, c we get$(\tau_a \cdot \tau_b) \cdot \tau_c \neq \tau_a \cdot (\tau_b \cdot \tau_c)$. However, the weaker condition of alternativity is satisfied. This means that, for $a = b$, the associator

$$[\tau_a, \tau_b, \tau_c] \equiv (\tau_a \cdot \tau_b) \cdot \tau_c - \tau_a \cdot (\tau_b \cdot \tau_c) \tag{4}$$

is vanishing.

The specialization of the octonionic indices to, let's say, $0, 1$ or $0, 1, 2, 3$ leads respectively to the complex number or to the division algebra of quaternions.

The octonionic algebra admits seven involutions, specified by the mappings

$$\tau_0 \mapsto \tau_0, \quad \tau_p \mapsto \tau_p, \quad \tau_q \mapsto -\tau_q, \tag{5}$$

where p takes value in one of the seven triples entering (2), while q specifies the four complementary values. The three involutions for the quaternions (with two generators) are recovered as the restrictions to the $0, 1, 2, 3$ subspace.

The $N = 8$ extension of the Virasoro algebra (Non-associative $N = 8$ SCA) involves 8 bosonic and 8 fermionic fields and is constructed in terms of the octonionic structure constants. Besides the spin-2 Virasoro field denoted as T, it contains eight fermionic spin-$\frac{3}{2}$ fields Q, Q_α and 7 spin-1 bosonic currents J_α. It is explicitly given by the following Poisson brackets

$$\begin{aligned}
\{T(x), T(y)\} &= -\frac{1}{2}{\partial_y}^3\delta(x-y) + 2T(y)\partial_y\delta(x-y) + T'(y)\delta(x-y), \\
\{T(x), Q(y)\} &= \frac{3}{2}Q(y)\partial_y\delta(x-y) + Q'(y)\delta(x-y), \\
\{T(x), Q_\alpha(y)\} &= \frac{3}{2}Q_\alpha(y)\partial_y\delta(x-y) + {Q_\alpha}'(y)\delta(x-y), \\
\{T(x), J_\alpha(y)\} &= J_\alpha(y)\partial_y\delta(x-y) + {J_\alpha}'(y)\delta(x-y), \\
\{Q(x), Q(y)\} &= -\frac{1}{2}{\partial_y}^2\delta(x-y) + +\frac{1}{2}T(y)\delta(x-y), \\
\{Q(x), Q_\alpha(y)\} &= -J_\alpha(y)\partial_y\delta(x-y) - \frac{1}{2}{J_\alpha}'(y)\delta(x-y), \\
\{Q(x), J_\alpha(y)\} &= -\frac{1}{2}Q_\alpha(y)\delta(x-y), \\
\{Q_\alpha(x), Q_\beta(y)\} &= -\frac{1}{2}\delta_{\alpha\beta}{\partial_y}^2\delta(x-y) + C_{\alpha\beta\gamma}J_\gamma(y)\partial_y\delta(x-y) + \\
&\quad +\frac{1}{2}(\delta_{\alpha\beta}T(y) + C_{\alpha\beta\gamma}{J_\gamma}'(y))\delta(x-y), \\
\{Q_\alpha(x), J_\beta(y)\} &= \frac{1}{2}(\delta_{\alpha\beta}Q(y) - C_{\alpha\beta\gamma}Q_\gamma(y))\delta(x-y), \\
\{J_\alpha(x), J_\beta(y)\} &= \frac{1}{2}\delta_{\alpha\beta}\partial_y\delta(x-y) - C_{\alpha\beta\gamma}J_\gamma(y)\delta(x-y).
\end{aligned} \tag{6}$$

Notice the presence of the central term, essential in order to obtain supersymmetric KdV equations. Due to the non-associativity of octonions the structure constants of (6) do not satisfy the Jacobi identity (see [11] for a detailed discussion).

3 The $N = 2$ and the $N = 4$ KdVs Revisited.

By restricting the Greek indices to take either the values 1 or $1, 2, 3$, we recover from (6) the $N = 2$ and the $N = 4$ Superconformal algebras respectively (in the case of $N = 4$ the corresponding algebra is known as the "minimal $N = 4$ SCA"). They can be regarded as one of the Poisson brackets for the $N = 2$ and the $N = 4$ KdVs [3, 6].

These non-linear equations can be constructed by looking for the most general hamiltonian with the right dimension (i.e. whose hamiltonian density has dimension 4) invariant under global supersymmetric charges given by $\int dxQ(x)$ and $\int dxQ_\alpha(x)$. This approach was used to construct the $N = 2$ KdV in [3], while the $N = 4$ KdV was obtained in terms of a harmonic superspace formalism in [6].

For what concerns the $N = 2$ case we summarize here the results of [3]. We avoid writing explicit formulas since they can be immediately recovered from a suitable reduction of the $N = 4$ KdV results as discussed later. Up to a normalization factor, the $N = 2$-invariant hamiltonians depend on a single real parameter, denoted as "a", which labels inequivalent $N = 2$ KdVs. Three special values for a, i.e. $a = -2, 1, 4$, correspond to the three inequivalent $N = 2$ KdV equations which are integrable. The integrability for these special values of a was at first suggested (and proven for $a = -2, 4$) in [3] after checking the existence of higher order hamiltonians in involution among themselves and with respect to the original a-dependent $N = 2$-invariant one. Later the integrability of $a = 1$ was proven in the first reference of [5] with the explicit construction of the corresponding Lax operators.

Here we extend the analysis of [3] to the $N = 4$ KdV case. In particular we are able to fully determine the moduli space of inequivalent $N = 4$ KdVs. Our results extend and complete those originally appeared in [6].

The most general $N = 4$-invariant hamiltonian of right dimension depends on 5 parameters (apart the overall normalization factor) and is explicitly given by

$$
\begin{aligned}
H \;=\; & \int dx[-2T^2 - 2Q'Q - 2Q'_\alpha Q_\alpha + 2J''_\alpha J_\alpha + x_\alpha T {J_\alpha}^2 + 2x_\alpha Q Q_\alpha J_\alpha - \epsilon_{\alpha\beta\gamma} x_\gamma Q_\alpha Q_\beta J_\gamma + \\
& \frac{1}{3}\epsilon_{\alpha\beta\gamma}(x_\beta - x_\alpha) J_\alpha J_\beta {J_\gamma}' - 2z_\alpha \epsilon_{\alpha\beta\gamma} T J_\beta J_\gamma - \\
& 2z_1 Q(Q_2 J_3 + Q_3 J_2) - 2z_2 Q(Q_3 J_1 + Q_1 J_3) - 2z_3 Q(Q_1 J_2 + Q_2 J_1) + \\
& 2z_1 Q_1(Q_2 J_2 - Q_3 J_3) + 2z_3 Q_3(Q_1 J_1 - Q_2 J_2) + 2z_2 Q_2(Q_3 J_3 - Q_1 J_1) - \\
& z_1 {J_1}'({J_2}^2 - {J_3}^2) - z_3 {J_3}'({J_1}^2 - {J_2}^2) - z_2 {J_2}'({J_3}^2 - {J_1}^2)],
\end{aligned}
\tag{7}
$$

where the convention over repeated indices is understood and α, β, γ are restricted to $1, 2, 3$, while $\epsilon_{123} = 1$.

In order to guarantee the $N = 4$ invariance the three parameters x_α must satisfy the condition

$$
x_1 + x_2 + x_3 \;=\; 0, \tag{8}
$$

so that only two of them are truly independent (together with the three z_α's they provide the five parameters mentioned above). However, the further requirement for the hamiltonian to be invariant not only under global $N=4$ supersymmetry, but also under the three involutions of the $N=4$ Superconformal Algebra (obtained by flipping the sign of the four fields J_α, Q_α, for $\alpha=1,2$, $\alpha=1,3$ and $\alpha=2,3$ respectively, while leaving unchanged the remaining four fields) kills the three z_α's parameters, which must be set equal to zero.

The most general hamiltonian of such a kind is therefore given by

$$\begin{aligned} H &= \int dx[-2T^2-2Q'Q-2Q'_\alpha Q_\alpha+2J''_\alpha J_\alpha+x_\alpha T{J_\alpha}^2+2x_\alpha QQ_\alpha J_\alpha-\epsilon_{\alpha\beta\gamma}x_\gamma Q_\alpha Q_\beta J_\gamma+ \\ &\quad \frac{1}{3}\epsilon_{\alpha\beta\gamma}(x_\beta-x_\alpha)J_\alpha J_\beta {J_\gamma}']. \end{aligned} \tag{9}$$

where of course (8) continues to hold.

Since any given ordered pair of the three parameters x_α can be chosen to be plotted along the x and y axis describing a real $x-y$ plane, it can be easily proven that the fundamental domain of the moduli space of inequivalent $N=4$ KdV equations can be chosen to be the region of the plane comprised between the real axis $y=0$ and the $y=x$ line (boundaries included). Five other regions of the plane (all such regions are related via an S_3-group transformation) could as well be chosen as the fundamental domain.

In the region of our choice, the $y=x$ line corresponds to an extra global $U(1)$-invariance, since the hamiltonian whose parameters live in this line is in involution with the global charge $\int dxJ_3$ (namely $\{H,\int dx\cdot J_3\}=0$). The origin, that is $x_1=x_2=x_3=0$, is the most symmetric point, corresponding to a global $SU(2)$ invariance, the given hamiltonian being in involution with respect to the three $\int dx\cdot J_\alpha$ charges.

The equations of motion for the whole class of inequivalent $N=4$ KdV's are given by

$$\begin{aligned} \dot{T} &= -T'''-12T'T-6Q''Q-6{Q_\alpha}''Q_\alpha+(4+\frac{x_\alpha}{2})J'''_\alpha J_\alpha+\frac{3}{2}x_\alpha {J_\alpha}''{J_\alpha}'+3x_\alpha(T{J_\alpha}^2)'+ \\ &\quad 6x_\alpha(QQ_\alpha J_\alpha)'-3x_\gamma\epsilon_{\alpha\beta\gamma}(Q_\alpha Q_\beta J_\gamma)'+\epsilon_{\alpha\beta\gamma}(x_\gamma-x_\beta)({J_\alpha}''J_\beta J_\gamma-J_\alpha {J_\beta}'J'_\gamma), \\ \dot{Q} &= -Q'''-6(TQ)'-(4+\frac{x_\alpha}{2})({Q_\alpha}'J_\alpha)'+(2-\frac{x_\alpha}{2})(Q_\alpha {J_\alpha}')'+3x_\alpha(Q{J_\alpha}^2)'- \\ &\quad \epsilon_{\alpha\beta\gamma}(x_\gamma-x_\beta)(Q_\alpha J_\beta J_\gamma)', \\ \dot{Q}_\alpha &= -{Q_\alpha}'''-6(TQ_\alpha)'+(4+\frac{x_\alpha}{2})(Q'J_\alpha)'-(2-\frac{x_\alpha}{2})(Q{J_\alpha}')'+3x_\beta(Q_\alpha {J_\beta}^2)'+ \\ &\quad \epsilon_{\alpha\beta\gamma}(x_\gamma-x_\beta)(QJ_\beta J_\gamma)'+\epsilon_{\alpha\beta\gamma}(4+\frac{x_\gamma}{2})({Q_\beta}'J_\gamma)'-\epsilon_{\alpha\beta\gamma}(2-\frac{x_\gamma}{2})(Q_\beta {J_\gamma}')'+ \\ &\quad 2(x_\beta-x_\alpha)(1-\delta_{\alpha\beta})(J_\alpha Q_\beta J_\beta)', \\ \dot{J}_\alpha &= -{J_\alpha}'''-(4+\frac{x_\alpha}{2})(TJ_\alpha)'+(2-\frac{x_\alpha}{2})(QQ_\alpha)'-2(x_\alpha+x_\beta)Q_\alpha Q_\beta J_\beta- \\ &\quad \epsilon_{\alpha\beta\gamma}(1-\frac{x_\alpha}{4})(Q_\beta Q_\gamma)'-2\epsilon_{\alpha\beta\gamma}(x_\gamma-x_\beta)QQ_\beta J_\gamma+\epsilon_{\alpha\beta\gamma}(4+\frac{x_\gamma}{2})({J_\beta}'J_\gamma)'+ \\ &\quad 3x_\beta {J_\alpha}'{J_\beta}^2+2(1-\delta_{\alpha\beta})(x_\beta-x_\alpha)J_\alpha {J_\beta}'J_\beta. \end{aligned} \tag{10}$$

where the constraint $x_1+x_2+x_3=0$ is satisfied and (x_1,x_2) take value either in the region $I\equiv\{x_1,x_2|x_2\geq x_1\geq 0\}$ or in $II\equiv\{x_1,x_2|x_2\leq x_1\leq 0\}$. Each given pair $(x_1,x_2)\in I\cup II$ labels an inequivalent $N=4$ KdV equation.

The three involutions (each one associated to any given imaginary quaternion) allows to perform three consistent reduction of the $N=4$ KdV equation to an $N=2$ KdV, by setting simultaneously equal to 0 all the fields associated with the τ's which flip the sign (confront the discussion in the previous section). Therefore the first involution allows to consistently set equal to zero the fields $J_2 = J_3 = Q_2 = Q_3 = 0$, leaving the $N=2$ KdV equation for the surviving fields T, Q, Q_1, J_1. Similarly, the second and the third involution allows to set equal to zero the four fields labeled by $1,3$ and $1,2$ respectively. It turns out that to each such reduction only one free parameter survives, namely x_1, x_2 or respectively x_3.

This remaining free parameter coincides up to a normalization factor with the free parameter a of reference [3]. More specifically

$$a = \frac{1}{4} x_\alpha \qquad (11)$$

with $\alpha = 1,2,3$ according to the reduction.

As a consequence, a necessary condition for the integrability of the $N=4$ KdV requires that for a given pair $(x_1, x_2) \in I \cup II$ each one of the three reductions produce for a one of the known integrable values of a, namely $-2, 1, 4$. It is then easily checked that there are only two points in $I \cup II$, both in the $U(1)$-invariant $x_1 = x_2$ line, implying integrability for the three reduced $N=2$ KdV's. The solutions are

i) $x_1 = x_2 = -8$, $(x_3 = 16)$ and

ii) $x_1 = x_2 = 4$, $(x_3 = -8)$.

The first point, which produces the $a=-2$ and the $a=4$ integrable $N=2$ KdV's after reduction, is the integrable point discussed in [6]. For what concerns the second point, despite the fact that it allows the reduction to the $a=1$ and the $a=-2$ integrable $N=2$ KdV's, it does not seem to correspond to an $N=4$ integrable hierarchy. We explicitly constructed the most general global $N=4$ and $U(1)$ invariant hamiltonian whose hamiltonian density has total dimension dimension 6. This would correspond to the third hamiltonian in the KdV hierarchy ((9) would be the second hamiltonian). This hamiltonian however fails to be compatible with the third hamiltonian of the corresponding integrable $N=2$ KdV's. More precisely, the three (two independent) reductions to $N=2$ produce hamiltonians which should coincide with the third hamiltonian of the $N=2$ KdV for the corresponding value of a. While this is true for the first solution ($x_1 = x_2 = -8$, $x_3 = 16$), this is no longer true for the second choice of values ($x_1 = x_2 = 4$, $x_3 = -8$), as we explicitly verified.

This computation does not yet rule out the possibility that *ii)* would be a point of integrability for the $N=4$ KdV. It would still be possible that it corresponds to an integrable hierarchy with a "missed" hamiltonian for the hamiltonian density of dimension 6.

The origin ($x_1 = x_2 = x_3 = 0$) corresponds to the most symmetric point, being associated to a global $SU(2)$ invariance, as already remarked. In any case it does not correspond to an integrable point of the $N=4$ KdV since its reductions to $N=2$ KdV do not lead to one of the three integrable values of a.

4 The $N = 8$ SuperKdV.

In this section we construct the first example of an $N = 8$ supersymmetric extension of the KdV equation. In order to be able to realize an $N = 8$ KdV we extend the method discussed in the previous section to the case of the "Non-Associative $N = 8$ Superconformal Algebra" (6). The reason why we are forced to make use of a non-associative algebra has been discussed in the Introduction.

More specifically, we started with the most general hamiltonian of right dimension (its hamiltonian density having dimension equal to 4) constructed with the 16 (8 bosonic and 8 fermionic) fields entering (6). Later we imposed some constraints on it. At first we restricted the free coefficients in order to make the resulting hamiltonian invariant under the whole set of seven involutions of the $N = 8$ superconformal algebra. This is the $N = 8$ extension of a requirement already encountered in the $N = 4$ case. The seven involutions are so defined. The fields T, Q are unchanged, as well as the 6 fields Q_α, J_α, for the α's taking value in one of the seven triples entering (2). The 8 remaining fields Q_β, J_β, with β labeling the four complementary values (for any given choice of the original triple), have the sign flipped ($Q_\beta \mapsto -Q_\beta$, $J_\beta \mapsto -J_\beta$). After having constructed the most general hamiltonian H invariant under the whole set of seven involutions, we started imposing the invariance under the $N = 8$ global supersymmetric transformations, that is we required

$$\{\int dx \cdot Q_a(x), H\} \quad = \quad 0, \tag{12}$$

for $a = 0, 1, 2, ..., 7$ (here $Q_0 \equiv Q$), while $\{\star, \star\}$ denotes the generalized Poisson brackets given by the Non-associative $N = 8$ SCA (6).

It is worth to point out that for this generalized hamiltonian system, the Poisson brackets are assumed to be classical. In particular they satisfy the Leibniz property (or, better, its graded version due to the supersymmetry of (6)). The only feature of the non-associativity of the octonions lie in the non-vanishing of the Jacobi identities for the structure constants of the (6) algebra. The fields entering (6) are assumed to be ordinary (bosonic and fermionic) real fields.

Needless to say, the get the final answer we heavily relied on Mathematica's computations for classical OPE's, based both on the Thielemans' package [13] and on our own package to deal with octonionic structure constants.

The final result is the following. There exists a unique hamiltonian which is invariant under the whole set of global $N = 8$ supersymmetries. It admits no free parameter (apart the trivial normalization factor) and is quadratic on the fields. It is explicitly given by

$$H \quad = \quad \int dx[-2T^2 - 2Q'Q - 2Q'_\alpha Q_\alpha + 2J''_\alpha J_\alpha], \tag{13}$$

(here $\alpha = 1, 2, ..., 7$ and the summation over repeated indices is understood). This result implies that there is only one $N = 4$ KdV system which can be consistently extended to $N = 8$ KdV, namely the one which corresponds to the origin of the coordinates ($x_1 = x_2 = x_3 = 0$), that is the most symmetric point. While the corresponding hamiltonian for the $N = 4$ case admits a global $SU(2)$-invariance, the $N = 8$ hamiltonian (13) is invariant with respect to each one of the seven global charges $\int dx \cdot J_\alpha(x)$, that is

$$\{\int dx \cdot J_\alpha(x), H\} \;=\; 0. \tag{14}$$

The seven charges $\int dx \cdot J_\alpha(x)$ generates a symmetry which extends $SU(2)$; it does not correspond to a group due to the non-associative character of the octonions.

Despite the apparent simplicity and the fact that it is quadratic in the fields, the hamiltonian (13) generates an $N = 8$ supersymmetric extension of KdV which is not integrable. Better stated, even its $N = 4$ KdV reduction does not correspond to an integrable point of the $N = 4$ KdV.

The equations of motion of the $N = 8$ KdV are obtained through

$$\dot{\Phi}_i \;=\; \{\Phi_i, H\}, \tag{15}$$

where Φ_i collectively denote the fields entering (6).

We explicitly obtain

$$\begin{aligned}
\dot{T} &= -T''' - 12T'T - 6Q_a''Q_a + 4J_\alpha''J_\alpha,\\
\dot{Q} &= -Q''' - 6T'Q - 6TQ' - 4Q_\alpha''J_\alpha + 2Q_\alpha J_\alpha'' - 2Q_\alpha'J_\alpha',\\
\dot{Q}_\alpha &= -{Q_\alpha}''' - 2QJ_\alpha'' - 6TQ_\alpha' - 6T'Q_\alpha + 2Q'J_\alpha' + 4Q''J_\alpha -\\
&\quad 2C_{\alpha\beta\gamma}(Q_\beta J_\gamma'' - Q_\beta'J_\gamma' - 2Q_\beta''J_\gamma),\\
\dot{J}_\alpha &= -{J_\alpha}''' - 4T'J_\alpha - 4TJ_\alpha' + 2QQ_\alpha' + 2Q'Q_\alpha - C_{\alpha\beta\gamma}(4J_\beta J_\gamma'' + 2Q_\beta Q_\gamma').
\end{aligned} \tag{16}$$

It is a simple exercise to prove that the equations of motion (16) are compatible with the $N = 8$ global supersymmetries generated by $\int dx \cdot Q_a(x)$ ($a = 0, 1, 2, ..., 7$) which provide the transformations

$$\delta_a\Phi_i(y) \;=\; \{\int dx \cdot Q_a(x), \Phi_i(y)\}. \tag{17}$$

The above system of equations corresponds to the first known example of an $N = 8$ supersymmetric extension of KdV.

5 On Global $N = 3$ and $N = 4$ Extended SuperKdVs Based On the $N = 8$ SCA.

The authors of [10] proved the existence of integrable systems, obtained in terms of the $N = 4$ Superconformal algebra, which admit only an $N = 2$ global supersymmetry.

It is worth considering in our context, which involves a larger number of supersymmetries, which kind of extended supersymmetric systems are supported by the Non-associative $N = 8$ SCA. We present the complete analysis of the $N = 3$ and the $N = 4$ solutions. We construct the most general $N = 3$ and $N = 4$ superextensions of KdV admitting the Non-associative $N = 8$ SCA as generalized Poisson brackets. Both such cases turn out to be parametric-dependent.

Apart the unique $N = 8$ solution, $N = 4$ is the largest number of supersymmetries which can be consistently imposed (by assuming an $N > 4$ invariance we automatically recover the full $N = 8$ invariance).

Both in the $N = 3$ and the $N = 4$ cases, without loss of generality, one of the invariant supersymmetric charges can always be assumed to be $\int dxQ(x)$, with $Q(x)$ entering (6). In the $N = 3$ case the two remaining invariant supersymmetric charges (associated with imaginary octonions) can be chosen at will, since all pairs of imaginary octonions are equivalent. In the formula below, without loss of generality, we chose the invariant supersymmetric charges being given by $\int dxQ_1(x)$ and $\int dxQ_2(x)$.

The situation is different in the $N = 4$ case. Now we have three extra invariant supersymmetric charges to be associated with imaginary octonions. However, two inequivalent ways in choosing a triple of imaginary octonions exist, depending on whether the chosen triple corresponds to one of the seven values in (2) (i.e. the triples associated to an $su(2)$ subalgebra), or not. Two inequivalent classes of solutions, labelled by $N = 4$ (I) and $N = 4$ (II) are respectively obtained. The first (I) class can be individuated by choosing, without loss of generality, the three extra supersymmetric charges to be given by $\int dxQ_1(x)$, $\int dxQ_2(x)$ and $\int dxQ_3(x)$. The second class (II), without loss of generality, can be produced by assuming invariance under $\int dxQ_1(x)$, $\int dxQ_2(x)$ and $\int dxQ_4(x)$.

Let us present now the complete solutions.

The most general $N = 3$ invariant hamiltonian depends (up to the normalization factor) on 6 free parameters entering x and x_τ $(\tau = 1, 2, ..., 7)$.

The seven x_τ's satisfy two constraints

$$\begin{aligned} x_1 + x_2 + x_3 &= 0, \\ x_4 + x_5 + x_6 + x_7 &= 0. \end{aligned} \tag{18}$$

The most general hamiltonian is given by

$$\begin{aligned} H = \int dx[&-2T^2 - 2Q'Q - 2Q'_\alpha Q_\alpha + xQ_\mu{}'Q_\mu + 2J''_\alpha J_\alpha - xJ_\mu{}''J_\mu + x_\alpha TJ_\alpha{}^2 + x_\mu TJ_\mu{}^2 + \\ &2x_\alpha QQ_\alpha J_\alpha + 2x_\mu QQ_\mu J_\mu - x_\gamma C_{\alpha\beta\gamma}Q_\alpha Q_\beta J_\gamma - x_\nu C_{\alpha\mu\nu}Q_\alpha Q_\mu J_\nu + \\ &(x_\mu + x_\nu)C_{\mu\nu\alpha}Q_\mu Q_\nu J_\alpha + \frac{1}{3}C_{\alpha\beta\gamma}(x_\beta - x_\alpha)J_\alpha J_\beta J_\gamma{}' + 2x_\mu C_{\alpha\mu\nu}J_\alpha J_\mu J_\nu{}']. \end{aligned} \tag{19}$$

where α, β, γ are restricted to take the values $1, 2, 3$, while μ, ν are restricted to the complementary values $4, 5, 6, 7$.

The equations of motion for this $N = 3$ generalization of KdV are directly computed from (19) by applying the Poisson brackets, like in (15).

The complete set of equations is written down in 37 pages of LaTex. For that reason they are not being reported here. The corresponding LaTex file however is available upon request.

For what concerns the $N = 4$ cases, the (I) class of solutions involve three free parameters (up to the normalization factor) entering x and x_α $(\alpha = 1, 2, 3)$, where the x_α's are constrained to satisfy $x_1 + x_2 + x_3 = 0$.

The most general $N = 4$ -invariant hamiltonian of type (I) is given by

$$H = \int dx[-2T^2 - 2Q'Q - 2Q'_\alpha Q_\alpha + xQ_\mu{}'Q_\mu + 2J''_\alpha J_\alpha - xJ_\mu{}''J_\mu + x_\alpha TJ_\alpha{}^2 + x_\mu TJ_\mu{}^2 +$$

$$
\begin{aligned}
&2x_\alpha QQ_\alpha J_\alpha + 2x_\mu QQ_\mu J_\mu - x_\gamma C_{\alpha\beta\gamma} Q_\alpha Q_\beta J_\gamma - x_\nu C_{\alpha\mu\nu} Q_\alpha Q_\mu J_\nu + \\
&(x_\mu + x_\nu) C_{\mu\nu\alpha} Q_\mu Q_\nu J_\alpha + \frac{1}{3} C_{\alpha\beta\gamma}(x_\beta - x_\alpha) J_\alpha J_\beta {J_\gamma}' + 2x_\mu C_{\alpha\mu\nu} J_\alpha J_\mu {J_\nu}'].
\end{aligned} \tag{20}
$$

As before $\alpha, \beta, \gamma = 1, 2, 3$, while μ, ν take the values $4, 5, 6, 7$.

The $N = 4$ (I) equations of motion are explicitly given by

$$
\begin{aligned}
\dot{T} &= -T''' - 12T'T - 6Q''Q - 6{Q_\alpha}''Q_\alpha + (4 + \frac{x_\alpha}{2}) J'''_\alpha J_\alpha + \frac{3}{2} x_\alpha {J_\alpha}'' {J_\alpha}' + 3x {Q_\mu}'' Q_\mu - 2x {J_\mu}''' J_\mu + \\
&\quad 3x_\alpha (T {J_\alpha}^2)' + 6x_\alpha (QQ_\alpha J_\alpha)' - 3x_\gamma C_{\alpha\beta\gamma} (Q_\alpha Q_\beta J_\gamma)' + C_{\alpha\beta\gamma}(x_\gamma - x_\beta)({J_\alpha}'' J_\beta J_\gamma - J_\alpha {J_\beta}' J'_\gamma), \\
\dot{Q} &= -Q''' - 6(TQ)' - (4 + \frac{x_\alpha}{2})({Q_\alpha}' J_\alpha)' + (2 - \frac{x_\alpha}{2})(Q_\alpha {J_\alpha}')' + 2x({Q_\mu}' J_\mu)' - x(Q_\mu {J_\mu}')' + \\
&\quad 3x_\alpha (Q {J_\alpha}^2)' - C_{\alpha\beta\gamma}(x_\gamma - x_\beta)(Q_\alpha J_\beta J_\gamma)', \\
\dot{Q}_\alpha &= -{Q_\alpha}''' - 6(TQ_\alpha)' + (4 + \frac{x_\alpha}{2})(Q' J_\alpha)' - (2 - \frac{x_\alpha}{2})(Q {J_\alpha}')' + 3x_\beta (Q_\alpha {J_\beta}^2)' - \\
&\quad 2x C_{\alpha\mu\nu} ({Q_\mu}' J_\nu)' + x C_{\alpha\mu\nu} (Q_\mu {J_\nu}')' + C_{\alpha\beta\gamma}(x_\gamma - x_\beta)(Q J_\beta J_\gamma)' + \\
&\quad C_{\alpha\beta\gamma}(4 + \frac{x_\gamma}{2})({Q_\beta}' J_\gamma)' - C_{\alpha\beta\gamma}(2 - \frac{x_\gamma}{2})(Q_\beta {J_\gamma}')' + 2(x_\beta - x_\alpha)(1 - \delta_{\alpha\beta})(J_\alpha Q_\beta J_\beta)', \\
\dot{Q}_\mu &= \frac{x}{2} {Q_\mu}''' + (x - 4) T {Q_\mu}' - 6T' Q_\mu + 4Q'' J_\mu + 2Q' {J_\mu}' + xQ {J_\mu}'' + 4C_{\mu\alpha\nu} {Q_\alpha}'' J_\nu + 2C_{\mu\alpha\nu} {Q_\alpha}' {J_\nu}' + \\
&\quad x C_{\mu\alpha\nu} Q_\alpha {J_\nu}'' - 2x C_{\mu\nu\alpha} {Q_\nu}'' J_\alpha - x C_{\mu\nu\alpha} {Q_\nu}' {J_\alpha}' - 2C_{\mu\nu\alpha} Q_\nu {J_\alpha}'' + x_\alpha C_{\mu\alpha\nu} QQ_\alpha Q_\nu - \\
&\quad x_\alpha C_{\mu\alpha\nu} Q J_\alpha {J_\nu}' - 2x_\alpha C_{\mu\alpha\nu} Q {J_\alpha}' J_\nu - 2x_\alpha C_{\mu\alpha\nu} Q' J_\alpha J_\nu + 2x_\alpha {Q_\alpha}' J_\alpha J_\mu + x_\alpha {Q_\mu}' {J_\alpha}^2 + \\
&\quad 3x_\alpha Q_\mu {J_\alpha}' J_\alpha - x_\alpha C_{\mu\nu\alpha} T Q_\nu J_\alpha + x_\alpha Q_\alpha J_\alpha {J_\mu}' + 2x_\alpha Q_\alpha {J_\alpha}' J_\mu + \frac{1}{2} C_{\mu\alpha\beta\nu}(x_\alpha + x_\beta) Q_\alpha Q_\beta Q_\nu - \\
&\quad 2x_\beta C_{\mu\alpha\beta\nu} {Q_\alpha}' J_\beta J_\nu + x_\alpha C_{\mu\nu\alpha\beta} Q_\nu J_\alpha {J_\beta}' - 2x_\beta C_{\mu\alpha\beta\nu} Q_\alpha {J_\beta}' J_\nu - x_\beta C_{\mu\alpha\beta\nu} Q_\alpha J_\beta {J_\nu}', \\
\dot{J}_\alpha &= -{J_\alpha}''' - (4 + \frac{x_\alpha}{2})(T J_\alpha)' + (2 - \frac{x_\alpha}{2})(QQ_\alpha)' - 2(x_\alpha + x_\beta) Q_\alpha Q_\beta J_\beta - \\
&\quad C_{\alpha\beta\gamma}(1 - \frac{x_\alpha}{4})(Q_\beta Q_\gamma)' + x C_{\alpha\mu\nu} {Q_\mu}' Q_\nu - 2C_{\alpha\beta\gamma}(x_\gamma - x_\beta) QQ_\beta J_\gamma + C_{\alpha\beta\gamma}(4 + \frac{x_\gamma}{2})({J_\beta}' J_\gamma)' - \\
&\quad 2x C_{\alpha\mu\nu} {J_\mu}'' J_\nu + 3x_\beta {J_\alpha}' {J_\beta}^2 + 2(1 - \delta_{\alpha\beta})(x_\beta - x_\alpha) J_\alpha {J_\beta}' J_\beta, \\
\dot{J}_\mu &= \frac{1}{2} x {J_\mu}''' - 4(T J_\mu)' + 2Q' Q_\mu - xQ {Q_\mu}' - 2C_{\mu\alpha\nu} {Q_\alpha}' Q_\nu + x C_{\mu\alpha\nu} Q_\alpha {Q_\nu}' - 4C_{\mu\nu\alpha} {J_\nu}'' J_\alpha + \\
&\quad 2x C_{\mu\alpha\nu} J_\alpha {J_\nu}'' + 2x_\alpha C_{\mu\alpha\nu} T J_\alpha J_\nu - x_\alpha C_{\mu\nu\alpha} QQ_\nu J_\alpha + 2x_\alpha C_{\mu\alpha\nu} QQ_\alpha J_\nu + x_\alpha Q_\alpha J_\alpha Q_\mu + \\
&\quad x_\alpha {J_\mu}' {J_\alpha}^2 + 2x {J_\alpha}' J_\alpha J_\mu + 2x_\alpha C_{\mu\alpha\beta\nu} J_\alpha {J_\beta}' J_\nu + x_\beta C_{\mu\alpha\beta\nu} Q_\alpha J_\beta Q_\nu + (x_\alpha + x_\beta) C_{\mu\alpha\beta\nu} Q_\alpha Q_\beta J_\nu.
\end{aligned} \tag{21}
$$

The second (II) class of $N = 4$ solutions is two-parametric. The free parameters can be chosen to be x_1 and x_2, while the remaining x_τ parameters entering the hamiltonian below are restricted to be

$$
\begin{aligned}
x_3 = x_4 &= -(x_1 + x_2), \\
x_5 &= 0, \\
x_6 &= x_1, \\
x_7 &= x_2.
\end{aligned} \tag{22}
$$

The most general $N = 4$ (II) hamiltonian is given by

$$
H = \int dx[-2T^2 - 2Q'Q - 2Q'_\alpha Q_\alpha + 2J''_\alpha J_\alpha + x_\alpha T {J_\alpha}^2 + 2x_\alpha QQ_\alpha J_\alpha + C_{\rho\sigma\lambda}(x_\rho + x_\sigma) Q_\rho Q_\sigma J_\lambda +
$$

$$C_{\rho\lambda\sigma}(x_\rho+x_\lambda)Q_\rho Q_\lambda J_\sigma - C_{\lambda\mu\nu}(x_\lambda+x_\mu)Q_\lambda Q_\mu J_\nu + C_{\lambda\mu\rho}(x_\lambda+x_\mu)Q_\lambda Q_\mu J_\rho +$$
$$2x_\mu C_{\lambda\mu\rho}Q_\lambda J_\mu Q_\rho - 2x_\rho C_{\rho\lambda\sigma}J_\rho J_\lambda J_\sigma{}' + \frac{1}{3}C_{\mu\nu\lambda}(x_\mu - x_\nu)J_\mu J_\nu J_\lambda{}' - 2x_\mu C_{\mu\rho\nu}J_\mu J_\rho J_\nu{}']. \quad (23)$$

where now $\alpha = 1, 2, ..., 7$, while $\rho, \sigma = 1, 2, 4$ and $\lambda, \mu, \nu = 3, 5, 6, 7$.

The complete set of equations of motion for the $N = 4$ (II) case occupies 13 pages in LaTex. The given file is available upon request. Just like the $N = 3$ case and contrary to the $N = 4$ (I) case, these equations of motion cannot be easily compactified since the field labels $1 \leftrightarrow 2$, 3, 4, 5 and $6 \leftrightarrow 7$ all play a different role.

Let us conclude this section with a final comment. The two classes (I) and (II) of $N = 4$ solutions are obviously inequivalent. For what concerns the first class we can notice that by suitably choosing the parameters x_α's being given by $x_1 = x_2 = -8$ $(x_3 = 16)$, the resulting generalized KdV system extends the integrable $N = 4$ KdV based on the "minimal $N = 4$ SCA". This leaves the possibility that the $N = 4$ (I) KdV, for the given values of the x_α's parameters and for some $x \neq 0$, could be an integrable system. We plan to address this issue in the future.

6 Conclusions.

In this paper we investigated the issue of large N supersymmetric extensions of the KdV equation. The construction of extended supersymmetrizations is important in connection with integrable hierarchies since extended supersymmetric theories provide the unification of otherwise unrelated bosonic or lower supersymmetric hierarchies. The case mentioned throughout the paper of the integrable $N = 4$ KdV based on the $N = 4$ SCA, which encompasses both the inequivalent $a = -2$ and the $a = 4$ $N = 2$ KdVs, is a nice example of that.

From what concerns the applications of supersymmetry many good reasons are found to investigate extended supersymmetries. We refer to [15] for a detailed discussion of various aspects. In this cited paper the matrix-representations of the N extended supersymmetries have been classified (see also [16]). Besides matrix representations however, just in the case of the $N = 8$ supersymmetry, a specific realization for it can be obtained via the non-associative division algebra of the octonions (a detailed discussion of this topic can be found in [17]). From the point of view of superextensions of KdV, it would be then quite natural to expect the octonionic realization of the $N = 8$ supersymmetry being related with the "Non-associative $N = 8$ Superconformal algebra" introduced in [8]. The "non-associativity" is here referred to the fact that this algebra does not satisfy the (super) Jacobi identities. This apparent drawback turns out to be an advantage since it allows to overcome a no-go theorem which prevented so far to construct N-supersymmetric KdVs for $N > 4$, due to the fact that no central extension is allowed for superconformal algebras (of standard type) for $N > 4$ (see [9]).

In the present paper we used the "Non-associative $N = 8$ SCA" as a tool to produce the first example of an $N = 8$ supersymmetric extension of the KdV equation. The system under consideration involves the 8 bosonic and the 8 fermionic fields entering the $N = 8$ SCA. We constructed the $N = 8$ superKdV equations by deriving them from a generalized hamiltonian system admitting the "Non-associative $N = 8$ SCA" as generalized Poisson brackets. To our knowledge this is also the first example of a (generalized) dynamical system associated to the given $N = 8$ SCA.

The main results of this paper can be summarized as follows. We reviewed at first the $N = 4$ KdV based on the "minimal $N = 4$ SCA" and constructed the fundamental domain for its inequivalent supersymmetrizations. Later we investigated the possibility for an $N = 8$ superKdV based on the $N = 8$ SCA. We arrived at a uniquely specified system of equations given by formula (16). This system corresponds to the $N = 8$ superextension of the most symmetric (the $SU(2)$-invariant) point in the fundamental domain of the $N = 4$ KdV. Despite its enlarged symmetry this point is however not an integrable point of the $N = 4$ KdV.

In the following we investigated which N supersymmetric extensions (for $N > 2$) of KdV are supported by the "Non-associative $N = 8$ SCA" generalized Poisson brackets. The complete results are stated as follows. Besides the unique $N = 8$ case, such extensions are found for $N = 3$ and $N = 4$.

The class of solutions of the $N = 3$ case depends on 6 free parameters and is reported in formula (19). For what concerns the $N = 4$ cases two inequivalent classes of solutions, named "(I)" and "(II)", are found. The first class depends on three free parameters, while the second one depends on just two free parameters. They are given in formulas (20) and (23) respectively. For a convenient choice of the parameters of the class (I) solution, the resulting system of equations generalizes the integrable point of the "minimal" $N = 4$ KdV, leaving room to the possibility that a global $N = 4$ system involving the whole set of $N = 8$ SCA fields could correspond to an integrable hierarchy. This is an issue that we are planning to address in a future work.

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