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# The Closure of the Beltrami Flux Turbulence Wave Equation Without Pressure

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Abstract

We point out the closure of turbulence wave equation in the case of pressureless turbulent Beltrami fluid flux relevant to physical fluid turbulence at vanishing viscosity regime. We remark its usefulness to discover string dynamics in turbulence.

**Key-words:** Turbulence; Beltrami flux.

The main task in the statistical approach to fluid turbulence is to solve the set of infinite hierarchical equations for the random fluid velocity correlation functions. There are two schemes to solve these equations: the first scheme consists in applying ad hoc closure approximations without any control ([1]); the second pionered by Hopf, consists in writing the above infinite set of equations as a single functional differential equation satisfying some suitable initial-time condition and trying to solve it by several functional procedures ([2]).

Our aim in this Rappid Communication is to present a closure scheme for the Hopf functional equation; (called by us of the turbulent wave equation); by considering the turbulence as a stochastic regime motion of fluids ensemble dominated by Beltrami fluxs defined by the condition $rot\vec{V}(\vec{r},t) = \lambda\vec{V}(\vec{r},t)$ and subject to an external Random Stirring.

Let us start with the statistical Burger equation for Beltrami Flux turbulence in the one-dimensional case ([3]) with initial condition

$$\begin{aligned} &\frac{\partial v}{\partial t} + vv_x = -\nu\lambda^2 v) + fv(x,0) = g(x) \\ &v(x,0) = g(x) \end{aligned} \tag{1}$$

where we have used the Beltrami flux definition and the identity $r_0t(r_0t\vec{V})$ =grad div $\vec{V} - \Delta V$, to replace the fluid shear stress viscosity term $\nu d^2v(x,t)/dx^2$ by the damping term $-\nu\lambda^2 U(x,t)$ and leading, *thus to a Brownian fluid motion picture for turbulence.* One of the most important observable object in the fluid turbulence is the fixed velocities measurements at the grid points $(x_j)$ and at the common observation time $t$.

$$\langle\prod_{j=1}^{N}\delta(v(x_j,t) - v_j)\rangle \tag{2}$$

where the average $\langle\rangle$ is defined by the stochastic process of the fluid velocity induced by the gaussian random external forces with spatial correlation force.

$$\langle f(x,t)f(x',t')\rangle = K(x-x')\delta(t-t') \tag{3}$$

In momentum space, the observable eq. (2) is given by the following grid dependent characteristic functional (the turbulence wave function)

$$\psi((x_1,\cdots,x_N);(p_1,p_N);t) = \langle exp(i\sum_{k=1}^{N} p_k v(x_k,t))\rangle \tag{4}$$

The turbulente Hopf-Schrödinger wave equation in our case is given thus, in the following closed form as a straightforward application of the analysis of ref.[3].

$$\begin{aligned} &-i\frac{\partial}{\partial t}\psi((x_1,\cdots,x_N);(p_1,\cdots,p_N);t) \\ &\left\{\sum_{\ell=1}^{N}\left[p_\ell\frac{\partial}{\partial p_\ell}\left(\frac{1}{p_\ell}\frac{\partial}{\partial x_\ell}\right) - \nu\lambda^2 p_\ell\frac{\partial}{\partial p_\ell}\right]\right. \\ &\left. + \sum_{\ell=1;\ell'=1}^{N}(K(x_\ell - x_{\ell'})p_\ell p_{\ell'}\right\}\psi((x_1,\cdots,x_N);(p_1,\cdots,p_N);t) \end{aligned} \tag{5}$$

Added with the deterministic initial date condition

$$\psi(x_1,\cdots,x_N);(p_1,\cdots,p_N);t\to 0^+)=exp\left(i\sum_{\ell=1}^{N}p_\ell g(x_\ell)\right) \tag{6}$$

In the physical grid on $R^3=\{x_k^{(a)};a=1,2,3;k=1,\cdots,N\}$, eq. (5) naturally reads

$$\begin{aligned}&-i\frac{\partial}{\partial t}((x_1^{(a)},\cdots,x_N^{(a)});(p_1^{(a)},\cdots,p_N^{(a)});t)\\&\sum_{\ell=1}^{N}\sum_{a=1}^{3}\left[p_\ell^{(a)}\frac{\partial}{\partial p_\ell^{(a)}}\left(\frac{1}{p_\ell^{(a)}}\frac{\partial}{\partial x_\ell^{(a)}}\right)-\nu\lambda^2P_\ell^{(a)}\frac{\partial}{\partial p_\ell^{(a)}}\right]+\psi((x_1,\cdots,x_N^{(a)});(p_1^{(a)},\cdots,p_N^{(a)});t)+\\&\left[\sum_{\ell=1,\ell'=1}^{N}K_{ab}(x_\ell^{(a)}-x_{\ell'}^{(a)})p_\ell^{(a)}p_\ell^{(b)}\right]\psi((x_1^{(a)},\cdots,x_N^{(a)});(p_1^{(a)},\cdots p_N^{(a)});t)\end{aligned} \tag{7}$$

Hereafter we will present our study of eq. [7] one dimensional case (eq. (5)-eq. (6)). By introducing the mixed coordinates defined by the transformation law.

$$p_j+x_j=U_j \quad ; \quad p_j-x_j=v_j. \tag{8}$$

The turbulent wave equation eqs. (5)-(6) takes the more invariance form similar to a many-particle Schrödinger equation in quantum Mechanics.

$$\begin{aligned}&-i\frac{\partial}{\partial t}((U_1,\cdots,U_N);(v_1,\cdots v_N);t)\\&=\sum_{\ell=1}^{N}\frac{1}{4}\left[\frac{\partial^2}{\partial^2U_\ell}-\frac{\partial^2}{\partial^2v_\ell}-\left(\frac{2}{U_\ell+v_\ell}\right)\left(\frac{\partial}{\partial U_\ell}-\frac{\partial}{\partial v_\ell}\right)-\frac{\nu\lambda^2}{2}(U_\ell+v_\ell)\left(\frac{\partial}{\partial U_\ell}\right)+\frac{\partial}{\partial v_\ell}\right]\\&\psi((U_1,\cdots,U_N);(v_1,\cdots,v_N);t)+\frac{1}{4}\left[\sum_{\ell=1;\ell^1=1}^{N}(U_\ell+v_\ell)(U_{\ell'}+v_{\ell'})K\left(\frac{U_\ell-U_{\ell'}+(v_{\ell'}-v_\ell)}{2}\right)\right]\\&\psi((U_1,\cdots,U_N);(p_1,\cdots,p_N);t)\end{aligned} \tag{9}$$

and

$$\psi((U_1,\cdots,U_N);(v_1,\cdots v_N);0)=exp\left(i\sum_{\ell=1}^{N}\left(\frac{U_\ell+v_\ell}{2}\right)g\left(\frac{U_\ell-v_\ell}{2}\right)\right) \tag{10}$$

The above written closed partial differential equation is the main result of this paper. At this point we can implementar perturbative calculations for our turbulent wave equation by considering a physical a physical slowly varying (even function) correlation function the form([4]).

$$\begin{aligned}&K(x)\approx K(0)-\frac{\kappa_0}{2}x^2 \quad ; \quad |x|<<\left(\frac{x(0)}{x_0}\right)^{1/2}\equiv L\\&0 \quad ; \quad |x|>>L\end{aligned} \tag{11}$$

Which by its turn leads to the Harmonic and quartic anharmonic potential of the form.

$$\sum_{\ell=1;\ell'=1}^{N}\{K(0)(U_\ell U_{\ell'}+v_\ell v_{\ell'}+U_\ell v_{\ell'}+v_\ell U_\ell)$$
$$-\frac{1}{8}\kappa_0(U_\ell+v_\ell)(U_{\ell'}+v_{\ell'}[U_\ell^2+U_{\ell'}^2+v_{\ell'}^2+v_\ell^2-2U_\ell U_{\ell'}-2v_\ell v_{\ell'}]\Big\} \tag{12}$$

In the important case of the single fluid velocity average, our turbulent wave equation takes the following form, after making an analitic continuation $v \to iv$; namely

$$i\frac{\partial}{\partial t}\psi(U,v;t)=(\mathcal{L}_0+\mathcal{L}_1)\psi(U,v;t) \tag{13}$$

With the initial condition

$$\psi(U,v;t\to 0^+)=exp\left[i\left(\frac{U+iv}{2}\right)g\left(\frac{U-iv}{2}\right)\right] \tag{14}$$

Here the Kinetic and perturbation terms are:

$$\mathcal{L}_0=-\frac{1}{4}\left(\frac{\partial^2}{\partial U^2}+\frac{\partial^2}{\partial v^2}\right)+\frac{k(0)}{4}(U^2-v^2)$$
$$\mathcal{L}_1=\frac{2}{U+iv}\left(\frac{\partial}{\partial U}-\frac{1}{i}\frac{\partial}{\partial v}\right)-\nu\lambda(U+iv)\left(\frac{\partial}{\partial U}+\frac{1}{i}\frac{\partial}{\partial v}\right) \tag{15}$$

The Harmonic oscilator propagator of the kinetic term eq. (15) is determined in a straightforward way and a Feynman diagramatic analysis may be easily implemented for $\nu << 1$ the same perturbative procedure used in quantum mechanical problems ([5]). Similar remarks hold true in the general case eq. (9).

It is worth point out that analogous results are easily obtained in the physical case of turbulent Beltrami flux in the three dimensional case.

Let us comment the case of general turbulent flux. In this case, although being impossible to write a closed partial differential equation as we did in this paper, we can develop approximate schemes to solve the full functional Hopf equation (3) by approximating the fluid shear stress tensor by finite differences, namelly:

$$\nu\frac{d^2v(x_j;t)}{dx_j^2}\approx\frac{\nu}{\Delta_j}(-2v(x_j,)t+v(x_{j+1};t)v(x_{j-1},t)) \tag{16}$$

With the grid spacing $\Delta_j=|x_{j+1}-x_j|$.

Finally, we point out the usefulness of stochastic Beltrami flux for its reformulation as dynamics of Random surfaces ([6], [7]) by using the usual Wilson Loop defined by a spatial loop C as a complex parameter to describe collective fluid motions as usually done in quantum chromodynamics.

$$w[c]=\langle exp\{+i\oint_c\vec{V}(\vec{r},t)d\vec{r}\}\rangle_f \tag{17}$$

Its a straightforward consequence of the definition of Beltrani flux to re-write eq. (7) as a sum of Random Sheets S bounded by the closed loop C

$$w[c] = \sum_{\{S\}} \langle\ exp\{+i\lambda \int_S \vec{V}(\vec{r},t).d\vec{S}\}\rangle$$
$$\approx \sum_{\{S\}} \langle exp\{\sum_k (+i\lambda)(\vec{j}_k(S)\cdot\vec{V}(\vec{r}_k;t)\}\rangle \tag{18}$$

where we have defined the "occupation time" of the fluid flux over the Random Sheets $S = \{\vec{x}(\bar{u},\bar{v});\ 0 \le \bar{u}1;\ 0 \le \bar{v} \le 1\}$

$$\vec{j}_k(S) = \int_0^1 d\bar{v} \int_0^1 d\bar{v} \sqrt{g(\bar{v},\bar{v})} \left(\frac{\vec{x}_{\bar{u}} \times \vec{x}_{\bar{v}}}{|\vec{x}_{\bar{u}} \times \vec{x}_{\bar{v}}|}\right) (\bar{u},\bar{v}) \delta^{(3)}(\vec{x}(\bar{u},\bar{v}) - \vec{r}_k) \tag{19}$$

and the sum over $S$ is defined by the closed string propagator defined by $C$.

$$\sum_S = \int_{\partial S = C} \prod_{(\bar{u},\bar{v})} d[\vec{x}(u,v)\sqrt{det(g_{ab}(\bar{u},\bar{v})}]exp\left(-\frac{1}{2}\int_0^1 d\bar{u} \int_0^1 d\bar{v}\sqrt{det(\bar{u},\bar{v})}\right) \tag{20}$$

with $g_{ab}(\bar{u},\bar{v}) = \partial_a\vec{x}(\bar{u},\bar{v})\cdot\partial_b\bar{x}(\bar{u},\bar{v})$ denoting the Riemman metric on $S$ by the parametrization $\{\vec{x}(\bar{u},\bar{v})\}$.

Note that since the loop $C$ can be the boundary of a set of infinite surfaces, we have to sum all such surfaces in eq. (19) in order to obtain a pure loop dependent functional ([6][7]). A exact evaluation of eq. (18) (without the sum over the sheets) can be done, in principle, by solving the turbulent wave equation eq. (7) with the identification $\vec{p}_k = \vec{j}_k(S)$ at $R^3$. It is expected, thus, the area law behavior at large loops $C$ for the collective field eq. (17).

Work in the precise connection of Random Surfaces and turbulence are in progress and will appear in a more extended paper.

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