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Non-Abelian BF Theories With Sources and 2-d Gravity

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Abstract

We study the interaction of non-Abelian topological BF theories defined on two dimensional manifolds with point sources carrying non-Abelian charges. We identify the most general solution for the field equations on simply and multiply connected two-manifolds. Taking the particular choice of the so-called extended Poincaré group as the gauge group we discuss how recent discussions of two dimensional gravity models do fit in this formalism.

Key-words: BF theories; 2-d gravity.

1 Introduction

The adequate description of classical and first quantized relativistic objects, like particles and strings, is an essential point of discussion in any attempt to set a unified model of physical interactions. Our understanding of this issue for relativistic objects has been continually improved in recent years mainly by the application of BRST techniques. In particular, a satisfactory relativistic covariant treatment of isolated scalar [1] and spinning particles [2], and more recently, superparticles [3] which allows to discuss most of the kinematical aspects of such systems have been developed. The interest of studying interactions of these systems with classical fields is manifold. In a quantum field theoretical approach, along the lines of the standard model, new insights on the classical structure underlying our view of particles interacting with each other through the interchanging of gauge bosons and gravitons should be of great interest. They surely would be useful for the advance in the resolution of the formidable technical and conceptual challenges posed by non perturbative gauge theories and quantum gravity. In the alternative approach given by Superstring models, a more complete picture of the classical dynamics of Superparticles and Superstrings interacting with fields, it is desirable to elucidate the interaction mechanism embodied in the vertex operators in order to advance in the development of a second quantized theory.

The case of particles carrying non-Abelian charge interacting with gauge fields is of special interest. For such a system, internal degrees of freedom in the form of a variable $Q(\tau)$ taking values on the algebra are introduced on the world line to take into account the non-Abelian charge. When the gauge fields are Yang-Mills fields, the dynamics of the system is derived from the Wong equations [4]:

$$\ddot{x}^{\mu} + \Gamma^{\mu}_{\nu\rho}\dot{x}^{\nu}\dot{x}^{\rho} = -eF^{a\mu}_{\nu}Q_a(\tau)\dot{x}_{\nu}(\tau), \tag{1}$$

$$\frac{d}{d\tau}Q(\tau) + e\dot{x}^{\mu}(\tau)\Big[A_{\mu}(x(\tau)), Q(\tau)\Big] = 0. \tag{2}$$

The first of these equations is the non- Abelian generalization of the Lorentz force equation. The second one, as we discuss below in some detail, may be geometrically interpreted as the parallel transport of the charge $Q(\tau)$ along the world line. Due to the non-linearity in the gauge field, even for flat space, solutions of the Wong equations for the Yang-Mills system are not easy to find and discussion of important and manageable issues in the Abelian case such as radiation and self-interaction is very difficult to pursue.

More recently, systems with alternative dynamics for the gauge fields given by the Chern-Simons action in 3 dimensions or the BF [6] in various dimensions have gained attention. In both cases, the pure gauge theory is topological in the sense that the partition function is metric independent but the gauge fields can produce definite effects acting on sources. In the first case, particles interacting via Abelian and non-Abelian Chern-Simons fields provide realizations for the novel possibilities of particle statistics in 3 dimensional space, namely anyonic and more generally, Braid statistics [5]. In this paper we will deal with the second possibility.

BF models were introduced in Ref.[6] and since then have attracted wide interest [7]. In particular, their quantization and, the definition and properties of their observables were the object of much attention Ref.[16]. In higher dimensions, due to the fact that they are written in terms of antisymmetric fields or equivalently p-forms with $p \geq 2$, BF models couple naturally with extended sources [8]. In four dimensions the abelian BF fields induce spin transmutation on stringlike sources [9].

In two dimensions the BF fields couple to particles. The interaction of the two dimensional abelian BF model with point sources was discussed in Ref.[17], where the general solution for the field equation was presented. Non-Abelian BF models interacting with sources have appeared in the formulation of two dimensional gravity models as gauge theories [10] [11] [12][13] [14] [15]. These models written originally in terms of the geometrical field and an additional dilaton field have been under study for some time as toy models of Quantum Gravity, and also in connection with String and Superstring models [10] [11] [12][13] [14] [15]. Their formulation as gauge theories [10] [11] is a step forward in the study of their quantum properties since it makes available the well understood machinery of gauge theories and also open space for a better discussion of the inclusion of non-minimal interactions with matter, a point which from the physical point of view is of crucial importance when studying black hole effects in the theory [15].

Our concern in this article is with the general formulation and solution of the equations governing the interaction of BF non-Abelian gauge fields with point sources carrying non-Abelian charge in 2 dimensional manifolds and their application to 2 dimensional gravity. In the next section we discuss such equations starting from the action principle developed in Ref.[20]. Since for BF theories the gauge connections are flat, we are able to find the general solution of the equation in simply and multiply connected surfaces in terms of path ordered operators and step functions. In the last section we show how our discussion is applied to the formulation of 2 dimensional gravity models with sources as gauge theories with minimal and non-minimal interactions.

2 Non-Abelian BF-Theories With Sources

Let us consider a simply connected 2-dimensional surface M described locally by coordinates ξ^μ, $(\mu = 0, 1)$, and the principal fibre bundle $(\mathcal{P}, M, \pi, G)$ with structure group a Lie group G. Let $\mathcal{G}$ be the Lie algebra of G, whose generators T_a satisfy the following relations

$$[T_a, T_b] = f^c_{ab} T_c. \tag{3}$$

We introduce an inner product

$$< T_a, T_b >= h_{ab} \tag{4}$$

by means of an appropriate non-degenerate, non-singular matrix h_{ab} invariant in the sense that

$$h_{ab} U^a_c U^b_c = h_{cd} \tag{5}$$

for group elements U. The indices of the internal space can be raised and lowered by means of the h_{ab}. We restrict to the case when

$$f_{abc} = f^d_{ab} h_{dc} \tag{6}$$

is totally antisymmetric. Whenever G is semisimple the inner product is simply the trace and h_{ab} is the Cartan-Killing metric $h_{ab} = -(1/2) f^d_{ac} f^c_{bd}$. In the case that G is not semisimple, the Cartan-Killing metric is singular and the trace is not a suitable inner product.

The action that describes the non-Abelian BF theory is [6][7]

$$S_{BF} = \int_{M^2} < (B \wedge F) >, \tag{7}$$

where the field F is the curvature 2-form corresponding to the connection 1-form $A = A^a_\mu(\xi) T_a d\xi^\mu$ over the principal fibre bundle $(\mathcal{P}, M, \pi, G)$, defined by

$$F = \mathcal{D}A = dA + e[A, A], \tag{8}$$

and the field $B = B^a(\xi) T_a$ is a $\mathcal{G}$-valued 0-form transforming under the adjoint representation of $\mathcal{G}$, and can be geometrically interpreted as a section of the vector bundle $(E, M, \pi_E, \mathcal{G}, G)$ with typical fibre $\mathcal{G}$ associated to the fibre bundle $(\mathcal{P}, M, \pi, G)$ by the adjoint representation.The operator $\mathcal{D} = d + e[A,\ ]$ is the gauge covariant derivative acting in the adjoint representation.

The derivation of Wong equations using a variational principle is discussed in Ref. [20]. As we mentioned previously the non-Abelian charge carried by the particle is described by a $\mathcal{G}$-valued variable $Q(\tau) = Q^a(\tau) T_a$ transforming under the adjoint representation. In order to couple the non-Abelian particle to the gauge field A in a gauge-invariant manner the following interaction term with support on the world-line W of the particle is introduced [20],

$$S_{int} = \int_W d\tau < K, g^{-1}(\tau) D_\tau g(\tau) > \tag{9}$$

with $K = K^a T_a$ being a real constant element of the algebra that, as we shall see later on, can be interpreted as an initial condition and $g(\tau)$ a group element related to the internal variable $Q(\tau)$ by

$$Q(\tau) = g(\tau) K g^{-1}(\tau). \tag{10}$$

The operator D_τ is the temporal covariant derivative given by

$$D_\tau = \Big[\frac{d}{d\tau} + e\dot{x}^\mu(\tau) A_\mu(x(\tau))\Big]. \tag{11}$$

We are interested in studying the interaction of a BF field theory formulated in a 2-dimensional manifold with a set of non-Abelian charges. We introduce coordinates x_i^μ and the additional quantities g_i,Q_i and K_i related through (10) and incorporate a coupling term like S_{int} for each particle. The complete action is then given by

$$S = -m \sum_i \int_{W_i} d\tau \sqrt{\dot{x}_i^2} + \sum_i \int_{W_i} d\tau < K_i, g_i^{-1}(\tau) D_\tau g_i(\tau) > + \int_{M^2} < (B \wedge F) > \tag{12}$$

where W_i are the world-lines of the particles. The equation of motion obtained by annihilating the variations of S with respect to B is

$$F = 0 \tag{13}$$

stating that the connection A on the principal bundle $(\mathcal{P}, M, \pi, G)$ is flat.

The group parameters in this formulation are the additional dynamical variables associated to the non-Abelian charge and have to be varied independently. By varying g_i in the interaction term, employing (4), (5), (6) and (10), integrating by parts and imposing $\delta g_i(\tau_{in}) = \delta g_i(\tau_{fin}) = 0$ we obtain

$$\mathcal{D}Q_i = \frac{dQ_i}{d\tau} + \dot{x}_i^\mu(\tau)[A_\mu(x_i(\tau)), Q_i(\tau)] = 0 \tag{14}$$

which is a covariant conservation equation for the non-Abelian charges $Q_i(\tau)$. It gives the classical configuration for the new dynamical variables introduced in this case, namely, the group elements.

The equation obtained varying the coordinates $x_i^\mu(\tau)$ of any of the trajectories of the particles is the equation for the geodesics on M

$$\ddot{x}_i^\mu + \Gamma^\mu_{\nu\rho}\dot{x}_i^\nu\dot{x}_i^\rho = 0 \tag{15}$$

since in this case, due to the fact that $F = 0$, there is no generalized Lorentz force. Nevertheless as will be show below the interaction with the field still forces a restriction on the allowed set of geodesics.

Finally, by taking variations with respect to the gauge field A^a_μ we obtain the following equation :

$$\epsilon^{\mu\nu}[\partial_\nu B^a + f^a_{bc}A^b_\nu B^c] + \sum_i \int_{W_i} d\tau Q^a_i \delta^2(\xi - x_i(\tau))\dot{x}^\mu_i = 0. \tag{16}$$

This can be written also in the form

$$\mathcal{D}B(\xi) = (\partial_\mu B + e[A_\mu, B])d\xi^\mu =^* J \tag{17}$$

where * is the Hodge duality operator and J is the current 1-form associated to the particle

$$J = J^a_\mu T_a d\xi^\mu, \qquad {}^*J = \epsilon_{\mu\rho}J^a_\mu T_a d\xi^\rho,$$

$$J^a_\mu(\xi) = \sum_i \int_{W_i} d\tau Q^a_i(\tau)\delta^2(\xi - x(\tau))\dot{x}_{i\mu}(\tau). \tag{18}$$

We now turn to the problem of solving equations (14), (16) and (17). Since the field B depends via the particle current J on the non abelian charges Q_i, one has to give solutions $Q_i(\tau)$ in order to describe the field B. The equations of motion (14) for the non-Abelian charges $Q_i(\tau)$ are parallel-transport equations on the vector bundle associated to the principal bundle by the adjoint representation. The covariantly conserved non-Abelian charges $Q_i(\tau)$ are then given by the parallel transport of the initial values $Q_i(0)$ lying on the fibre over $x_i(0)$, along the particle trajectories W_i. Therefore in terms of the path-ordered exponentials we have

$$Q_i(\tau) = \left(exp : -\int_{x_i(0)}^{x_i(\tau)} A :\right) Q_i(0) \left(exp : -\int_{x_i(\tau)}^{x_i(0)} A :\right) \tag{19}$$

from which one can explicitly determine the form of the group elements $g_i(\tau)$ appearing in (10). Let us stress again that the connection A appearing in (19) is flat (cf. eq. 13). For this reason provided that the base manifold is simply connected, the ordered exponential along any closed path is the identity element. Thus, the ordered exponentials appearing in (19) do not depend on the particle trajectory itself but only on its extreme points.

When the base manifold M is multiply connected, there are inequivalent solutions. Let $(\mathcal{P}, M, \pi, G)$ be the principal fibre bundle associated to G, with a base manifold M which now is a Riemann surface of genus $g \geq 1$ and π denoting the projection map. Let C be a closed curve on M based on a point $x \in M$ and $p \in \mathcal{P}$ such that $x = \pi(p)$. Parallel transport of p along C results in an element

$$exp : -\int_C A : p$$

belonging to the fibre on x called the holonomy of p over the curve C with respect to the connection A. Varying the closed paths C based on $x = \pi(p)$, the corresponding path-ordered integrals are the elements of a subgroup of G, the holonomy group of the connection A in p. In the particular case that A is a flat connection, the holonomy group is isomorphic to the fundamental homotopy group of M $\pi_1(M)$. Using the non-triviality of the homotopy group let us define $Q_i^{mn}(\tau)$ as

$$Q_i^{mn}(\tau) = \left(exp : -\int_{C_m} A :\right) Q_i^0(\tau) \left(exp : -\int_{C_n} A :\right), \tag{20}$$

where the ordered exponentials are computed on closed loops of the homotopy classes m and n based on $x_i(\tau)$. Calculating the dot derivative of $Q_i^{mn}(\tau)$, it is straightforward to verify that $Q_i^{mn}(\tau)$ is also a solution to (14)

$$D_\tau Q_i^{mn}(\tau) = 0. \tag{21}$$

Therefore, on a multiply connected manifold M, the covariant conservation equation $D_\tau Q(\tau) = 0$ has a family of solutions $\{\ Q_i^{mn}(\tau)\}$ each one depending on the homotopy classes m, n of closed loops C_m and C_n based on $x_i(\tau)$ on which one calculates the path ordered exponentials.

We can now construct a solution to the equation (17). The field B is a $\mathcal{G}$-valued 0-form transforming under the adjoint representation and can thus be interpreted as a section of the vector bundle $(E, M, \pi_E, G, \mathcal{G})$ associated to the principal bundle $(\mathcal{P}M, \pi, G)$ by the adjoint representation. The equation (17) is non- homogeneous and linear in the field B. The general solution is then the sum of the solution to the homogeneous equation $DB = 0$, whose solution is similar in form to the one already found for $Q_i(\tau)$, plus a particular solution to the non-homogeneous equation. We write at an arbitrary point $P \in M$

$$B(P) = \left(exp : -\int_{P_0}^{P} A :\right) f_W \left(exp : -\int_{P}^{P_0} A :\right) \tag{22}$$

where f_W is a 0-form depending on the world-lines of the particles W_i that has to be determined, $P_0 \in M$ is a fixed but arbitrary point and the path ordered integrals are computed over any path connecting P_0 and P. Taking the exterior derivative of B at the point P, using the definition of the ordered exponential, and the Leibniz rule one obtains

$$dB = -AB + BA + \left(exp : -\int_{P_0}^{P} A :\right) df_W \left(exp : -\int_{P}^{P_0} A :\right) \tag{23}$$

Using (14) one then has that the 1-form df_W satisfies

$$df_W = \left(exp : -\int_P^{P_0} A :\right) {}^*J \left(exp : -\int_{P_0}^{P} A :\right) \tag{24}$$

and thus, due to the delta function in the current J has support in $\bigcup W_i$. According to this, f_W is of constant value at each side of W_i and is recognized as a step function. Examining the action of df_W on test 1-forms φ of compact support one obtains

$$\int_{M^2} df_W \wedge \varphi = \sum_i q_i \int_{W_i} d\tau \varphi_\mu \left(x(\tau)\right) \dot{x}_i^\mu(\tau) \tag{25}$$

where q_i are the quantities

$$q_i := \left[\left(exp : -\int_{x_i(\tau)}^{P_0} A :\right) Q_i(\tau) \left(exp : -\int_{P_0}^{x_i(\tau)} A :\right)\right] \tag{26}$$

which, due to the conditions $F = 0$ and $\mathcal{D}_\tau Q = 0$ are constants. We note that if we choose P_0 to be $x_i(0)$ then $K_i = Q_i(0)$. In general K_i and $Q_i(0)$ are related by fixed factors expressed as path ordered exponentials. In obtaining (26) we take (19) for simplicity instead of the more general (20).

The solution to equation (25) for any test 1-form φ do exists when $\bigcup_i W_i$ may be organized in set of paths any of which divide the manifold and furthermore constitute the boundary of a 2-chain in M. When this condition is fulfilled the solution is obtained by taking f_W to be an adequate superposition of Heaviside functions with discontinuty q_i when we cross W_i from right to left.

In the simplest case of a simply connected unbounded manifold the physical geodesics are lines coming from the region of the manifold with minus infinite time coordinate and going to the region with plus infinite time coordinate. In this case each individual trajectory W_i divide the space in two sub-manifolds M_i^+ and M_i^- and is the boundary of any of them. The solution for B is then

$$B_{part} = \left(exp : -\int_{P_0}^{P} A :\right) \sum_i q_i \Theta(M_i^+) \left(exp : -\int_{P}^{P_0} A :\right). \tag{27}$$

As explained before a different choose for P_0 amounts to a change in the initial conditions $Q(0)$. The general solution is simply the sum of the particular solution (27) plus a solution to the homogeneous equation constructed in a similar fashion to the one for $Q(\tau)$.

In the case in which we have only one particle, the quantity $q\Theta(W)$ in (27) can be expressed alternatively in an integral form related to the intersection index of curves on M. Let C_s be an auxiliary curve transverse to W described locally by coordinates z with $z(0) = P_0$ and $z(s) = P_s$. Let us consider now the expression

$$I(C_s, W) = q^{-1} \int_{C_s} \left[\left(exp : -\int_{P_s}^{P_0} A :\right) {}^*J(P_s) \left(exp : -\int_{P_0}^{P_s)} A :\right)\right] \tag{28}$$

Writing explicitly ${}^*J(P_s)$ and using (25) we have that

$$I(C_s, W) = \int_{C_s} ds \int_W \epsilon^{\mu\nu} \dot{x}^\mu(\tau) z'^\nu(s) \delta(x(\tau) - z(s)) d\tau \tag{29}$$

and recognize $I(Cs, W)$ as the intersection index between C_s and W. It is evident that if C_s is closed then $I(Cs, W) = 0$. Thus, $I(Cs, W)$ is non-zero only if C_s intersects W an odd number of times. Choosing P_0 at the left of W it is clear that

$$I(C_s, W) = \Theta(W) \tag{30}$$

thus obtaining an integral form for the step function in terms of the intersection index. This equation can be generalized for the case of a system of particles by weighting the index with the charges q_i each time we cross W_i from right to left and with $-q_i$ when we cross from left to right.

When the 2-dimensional base manifold M is multiply connected the trajectories of particles interacting with non-Abelian BF fields are topologically selected to be homologous to zero as were in the abelian case [17]. To illustrate this point let us consider M to be the manifold $\mathbf{R} \times S^1$ with S^1 identified as the space manifold. The physical trajectories of a particle are spiral lines coming from $t = -\infty$ and going to $t = \infty$. The propagation of a single particle in such manifold is inconsistent with (16), (17). The spiral line on the cylinder does not divide the manifold and it is not possible to find f_W to satisfy (26). The only geodesics on the cylinder that divide the manifold are the circles $t = constant$ but this are not physical trajectories. On the other hand if one consider two particles with opposite non-Abelian charges q moving in parallel trajectories the manifold is divided in two strips and we may construct the solution (27) with the support of the Heaviside function on one of the strips.

In the case that the base manifold M is simply connected, given the particle world line W and initial conditions $Q_i(0)$ and $B(0)$ the solutions for the covariantly conserved non abelian charge $Q_i(\tau)$ and the field B are uniquely determined and given respectively by (19) and (27). Whenever M is multiply connected there will be an additional dependence on the homotopy classes of the curves along which the ordered exponentials are calculated. In particular, there exists a family of solutions Q_i^{mn} to the equation $D_\tau Q(\tau) = 0$ given by (20). From the Q_i^{mn} one constructs the currents J^{mn} associated to the field equations $DB^{mn} =^* J^{mn}$, with particular solutions of the form (27). Let us note, that the solution to the homogeneous equation $DB_{hom} = 0$ has an additional topological information. Just as in the case with $Q(\tau)$, one can construct, with the aid of Wilson operators, topologically inequivalent solutions

$$B_{hom,pq}(P) = \left(exp : -\int_{P_0}^{P} A :\right)_p B(0) \left(exp : -\int_{P}^{P_0} A :\right)_q \tag{31}$$

for arbitrary homotopy classes p and q. Therefore, for solving the homogeneous equation $DB_{hom} = 0$ one has to specify the initial condition $B(0)$ as well as the homotopy classes of the curves associated to the Wilson operators. The difference between two solutions B_{mn} and B'_{mn} to the equation $DB_{mn} =^* J_{mn}$ is a solution to the homogeneous equation of the form (31) for some p, q. In this fashion, given a sector associated to homotopy classes m, n via the current J_{mn}, there will be subsectors associated to the classes p, q of the solutions of the homogeneous equation. Let us note, however, that by varying p, q it is not possible to jump to another sector associated to a current $J_{m'n'}$; this can only be done by modifying $Q_{mn} \longrightarrow Q_{m'n'}$. Given a particular solution to $DB_{mn} =^* J_{mn}$ one is confined to a sector (m,n) even though there are several subsectors within associated to the solutions (31) to the homogeneous equation. Other distinct sectors (m',n') are associated to different field equations via different currents $J_{m'n'}$.

3 Applications to 2-d Gravity

The Einstein theory of gravity in 2 dimensions is trivial since the Einstein tensor is in this case identically zero. Various alternative models which introduce an additional dilaton field are currently under study [18] [19]. The string inspired linear gravity [15], is specially relevant since it has a classical black-hole type solution and a gauge theoretical formulation of the model is available [11]. In its original form [15], the dilaton gravity model is described by the action

$$S = \int d^2x \sqrt{\overline{g}} e^{-2\phi} \left(R + 4\overline{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \Lambda \right) \tag{32}$$

where Λ is the cosmological constant and ϕ is the dilaton field. Taking new variables variables $g_{\mu\nu} = e^{-2\phi}\overline{g}_{\mu\nu}$ and $\eta = e^{-2\phi}$ [12] this can be written in the form

$$S = \int dx^2 \sqrt{g} \left(\eta R - \Lambda \right). \tag{33}$$

The equation of motion of this action allow classical solutions of the black-hole type [15].

In what follows we will work with the *Zweibein* e^a_μ, related to the metric $g_{\mu\nu}$ by

$$g_{\mu\nu} = e^a_\mu e^b_\nu h_{ab}. \tag{34}$$

As usual, lower case greek indices refer to the space-time (base) manifold, take the values 0 and 1, and are raised and lowered by the metric $g_{\mu\nu}$. Lower case indices take the values 0 and 1, refer to the tangent space and are raised and lowered with the Minkowski metric $h_{ab} = diag(1,-1)$. The antisymmetric tensor is normalized by $\epsilon^{01} = 1 = -\epsilon_{01}$. The determinants of the metric and the *Zweibein* are given respectively by $g = detg_{\mu\nu}$ and $\sqrt{-g} = dete^a_\mu = -\frac{1}{2}e^a_\mu e^b_\nu \epsilon_{ab}$ We shall also need the spin connection ω_μ, and the Christoffel symbol $\Gamma^\alpha_{\mu\nu}$ related by

$$\partial_\mu e^a_\nu + \epsilon^a_b \omega_\mu e^b_\nu = \Gamma^\alpha_{\mu\nu} e^a_\alpha. \tag{35}$$

The null-torsion condition

$$\epsilon^{\mu\nu} \left(\partial_\mu e^a_\nu + \epsilon^a_b \omega_\mu e^b_\nu \right) = 0 \tag{36}$$

determines the spin connection in terms of the *Zweibein* and leads to

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu} \right). \tag{37}$$

Finally, the scalar curvature R is obtained by

$$\partial_\mu \omega_\nu - \partial_\nu \omega_\mu = -\frac{1}{2}\sqrt{-g}\,\epsilon_{\mu\nu} R. \tag{38}$$

The gauge formulation equivalent to this model discussed in Ref.[11] is based on the extended Poincaré group,[11] [12], whose Lie algebra reads

$$[P_a, J] = \epsilon_a^b P_b \qquad [P_a, P_b] = \epsilon_{ab} I \qquad [J, I] = [P_a, I] = 0 \tag{39}$$

which differs from the ordinary Poincaré algebra in that the translation generators do not commute due to the presence of the central element I. Introducing the notation $T_A = (P_a, J, I)$, $(A = a, 2, 3)$, the algebra $[T_A, T_B] = f_{AB}^C T_C$ is 4-dimensional. The Cartan-Killing metric $f_{AD}^C f_{BC}^D$ is singular since the group is semisimple; but an invariant, non-singular, bilinear form is available with the tensor

$$h_{AB} = \begin{bmatrix} h_{ab} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \tag{40}$$

which satisfies (5). It is employed to raise and lower the algebra indices. One verifies that F_{ABC} is totally antisymmetric, the only non-zero components being the permutations of $f_{ab2} = -\epsilon_{ab}$. The invariant product formed from (40) is thus

$$\langle W, V\rangle = W^A h_{AB} V^B = W^A V_A = W^b V_b - W_2 V_3 - W_3 V_2 = W^b V_b - W^3 V^2 - W^2 V^3 \tag{41}$$

In the dilaton gravity model (33) the dynamical variables are the multiplier η and the metric $g_{\mu\nu}$ (or equivalently, the dilaton ϕ and the metric). In the equivalent extended Poincaré gauge formulation one introduces a connection 1-form A on a principal bundle with the extended Poincaré group as structure group, whose expression on the base manifold is

$$A = A_\mu^A d\xi^\mu T_A = e_\mu^a d\xi_\mu P_a + \omega_\mu d\xi^\mu J + a_\mu d\xi^\mu I. \tag{42}$$

The components of this gauge potential are the dynamical variables of the theory: the *Zweibein* e_μ^a, the spin connection ω_μ, and the potential a_μ. Parameterizing the group element g in the form

$$g = e^{\theta_a P^a} e^{\alpha J} e^{\beta I} \tag{43}$$

the transformation laws for the components are obtained for the usual rule for the connection. We have

$$\begin{gathered} \omega_\mu \to \omega_\mu + \partial_\mu \alpha \\ e_\mu^a \to \left(L^{-1}\right)_b^a \left(e_a^b + \epsilon_c^b \theta^c \omega_\mu + \partial_\mu \theta^b\right) \\ a_\mu \to a_\mu - \theta^a \epsilon_{ab} e_\mu^b - \frac{1}{2}\theta^a \theta_a \omega_\mu + \partial_\mu \beta + \frac{1}{2}\partial_m u \theta^a \epsilon_{ab} \theta^b \end{gathered} \tag{44}$$

where L_b^a is the Lorentz transformation of rapidity α

$$L_b^a = \delta_b^a cosh\alpha + \epsilon_b^a sinh\alpha \tag{45}$$

From A one constructs the curvature 2-form F

$$F = dA + A\wedge A = F^A T_A = (de^a + \epsilon_b^a \omega e^b) + d\omega J + (da + \frac{1}{2} e^a \epsilon_{ab} e^b) I = f^a P_a + d\omega + \nu I. \tag{46}$$

The components along the translations and rotations are the torsion and the scalar curvature respectively. The two terms in the component along I are the field strength associated to the potential A and the volume element expressed in terms of the *Zweibein*. By incorporating a Lie algebra-valued Lagrange multiplier $\eta^A = (\eta^a, \eta^2, \eta^3)$ and using the inner product (41) one constructs the BF-type action

$$S_{\eta F} = \int < \eta, F > . \tag{47}$$

This action has been shown to be classically equivalent to (33).

A consistent interaction of this system with particles may be obtained directly from (12) and our discussion of the previous section (identifying the B field with the multiplier η). For a single particle we have

$$S = -m \int_W d\tau \sqrt{\dot{x}^2} + \int_W d\tau < K, g^{-1}(\tau) D_\tau g(\tau) > + \int_{M^2} < (\eta, F) > . \tag{48}$$

We have to take care of the fact that the kinetic term of the the action for particle, explicitly includes the metric $g_{\mu\nu}$ evaluated in $x(\tau)$ and therefore, since the *Zweibein* is a dynamical variable as a component of the connection A, it is necessary to consider the extra contribution of this term while performing variations with respect to A. This is done by observing that the corresponding field equation may be written in the form:

$$D_\mu \eta = \epsilon_{\mu\nu} \int_W d\tau \widetilde{Q} \delta^2(\xi - x(\tau)) \dot{x}^\nu \tag{49}$$

where

$$\widetilde{Q} = Q - \frac{m}{\sqrt{\dot{x}^2}} \dot{x}^\mu e^a_\mu P_a. \tag{50}$$

The other field equations are the zero curvature condition (13) and the covariant conservation of $Q(\tau)$. The important point is that then $\widetilde{Q}$ also satisfies

$$D_\tau \widetilde{Q} = 0 \tag{51}$$

since the extra term in (50) is also conserved as can be verified by direct computation. So the solution to (49) may be found directly from (27) and the expression for (50) is given by (19)

Let us consider the solution to the equation (51) with initial condition $Q(0) = Q^a_0 P_a + Q^3_0 I$

$$\widetilde{Q}(\tau) = \left(exp : - \int_{x(0)}^{x(t)} A : \right) (Q^a_0 P_a + Q^3_0 I) \left(exp : - \int_{x(t)}^{x(0)} A : \right) \tag{52}$$

and let us calculate explicitly the components $\widetilde{Q}^A$ (This is done for later use when we will to show the equivalence with the formulation of Cangemi and Jackiw). Since the central element I commutes with all the generators our labour is simplified. We are then interested in computing

$$(1 - A\Delta)_n \cdots (1 - A\Delta)_1 P_a (1 - A\Delta)_1 \cdots (1 - A\Delta)_n \tag{53}$$

using the definition of the path-ordered exponential as the limit $\Delta \to 0$ of the product of factors $(1 - A\Delta)_i$, $\Delta = \Delta p_i$ with p_i being points on the path W and the subindex

i meaning that the expression between parentheses is evaluated in p_i. We fix $x(0) = p_1$ and $x(\tau) = p_n$. Noting that

$$(1 - A\Delta)_1 P_a (1 - A\Delta)_1 = \left(\delta^a_b + \epsilon^b_a \omega\Delta\right)_1 P_b + \epsilon_{ab}(e^b\Delta)_1 I + O(\Delta^2) \tag{54}$$

we have

$$\prod_{i=1}^{n-1} \left(\delta^{a^{i+1}}_{a_i} + \epsilon^{a_{i+1}}_{a_i}\omega\Delta\right)_i Q_0^{a_1} P_{a_n} + \sum_{i=1}^{n} \Delta\epsilon_{a_i b}(e^b)_i \left[\prod_{j=1}^{i} \left(\delta^{a_j}_{a_{j-1}} + \epsilon^{a_j}_{a_{j-1}}\omega\Delta\right)_{j-1} Q_0^{a_1}\right] I \tag{55}$$

Taking $\Delta \to 0$ it becomes

$$Q^a(\tau)P_a = exp : -\int_{x(0)}^{x(\tau)} \epsilon^a_b \omega : Q_0^b P_a \tag{56}$$

$$Q^3(\tau)I = \int_0^\tau e^a(s) p_a(w(s)) I + Q_0^3 I \tag{57}$$

An alternative method for switching on the interaction was originally proposed by Cangemi and Jackiw [12][13] introducing the degrees of freedom associated to the non-Abelian charge in a different way. They construct their formulation in terms of a set of tangent space coordinates and momenta q^a and p_a. That such formulation may be equivalent to construction presented in the previous section is suggested by the fact discussed in [20] that the $\mathcal{G}$-valued variables Q are related by a canonical transformation to the conjugate momenta of the group elements g used in the lagrangian formulation. Actually we will show presently that (56) and (57) are solutions to the equations of the system discussed by Cangemi and Jackiw [12],[13] but we stress that they have the advantage that can be easily generalized to global solutions in multiply connected manifolds.

In its most compact [24] form, the particle action of Cangemi and Jackiw is written in the form

$$S_p = \int d\tau \left[p_a (D_\tau q)^a - \frac{1}{2} N(p^2 + m^2)\right] \tag{58}$$

where $(D_\tau q)^a = \dot{q}^a + \epsilon^a_b (q^b \omega_\mu - e^b_\mu)\dot{x}^\mu$, $x^\mu(\tau)$ are the trajectory coordinates on the manifold and q^a are the Poincaré parameters identified as mentioned with the coordinates of tangent space. This action is invariant under the gauge transformation (44) if q^a and p_a transform like

$$q^a \to (\lambda^{-1})^a_b q^b + \rho^a, \quad p_a \to \lambda^b_a p_b \tag{59}$$

with λ and ρ evaluated on the trajectory.

The interacting system defined by (47) and (58)

$$S_{CJ} = S_{\eta F} + S_p \tag{60}$$

leads to the flat curvature condition $F = 0$ and the equations of motion

$$\dot{p}_a + \epsilon^b_a p_b \omega_\mu \dot{x} I \mu = 0 \tag{61}$$

$$(D_\tau q)^a = N p^a \tag{62}$$

and

$$\partial_\mu \eta + [A_\mu, \eta] = \epsilon_{\mu\nu} J^\nu. \tag{63}$$

The matter current J^μ may be written in the form

$$J^\mu(\xi) = \int d\tau j \delta^2(\xi - x(\tau)\dot{x}^\mu(\tau) \tag{64}$$

with the components in $j = j^a P_a + j^2 J + j^3 I$ such that

$$j = -\epsilon^{ab} p_b P_a - \epsilon_a^b q^a p_b I. \tag{65}$$

Then then using (62) we have

$$\frac{dj^a}{d\tau} + \epsilon_b^a \omega_\mu \dot{x}^\mu j^b = 0 \tag{66}$$

$$\frac{dj^3}{d\tau} - e_\mu^a \dot{x}^\mu p_a = 0. \tag{67}$$

The solution of this equations are given by (56), (57) if we make the following identifications

$$Q^a(\tau) = j^a(\tau) = -\epsilon^{ab} p_b, \tag{68}$$

$$Q^3(\tau) = j^3(\tau) = -\epsilon_a^b q^a p_b. \tag{69}$$

The treatment described above can also be applied to the more general situation studied in [13] where non-minimal gravitational interactions are considered. Let us consider the gauge invariant action [13]

$$S_t = \int \eta_A F^A + \int d\tau \left[p_a (D_\tau q)^a - \frac{1}{2} N(p^2 + m^2) + q_a A_\mu^A \dot{x}^\mu - \frac{1}{2} q^a \epsilon_{ab} q^b \right] \tag{70}$$

where now $q^3 = (1/2) q^a q_a + \mathcal{A}$ ($\mathcal{A}$ is a constant). The extra terms in (70) introduce a forcing term in the geodesic equation arising solely from gravitational variables. After choosing the "physical gauge" $q^a = 0$, performing variations with respect to x^μ leads to the geodesic equation, modified by a gravitational force [13]

$$\frac{d}{d\tau} \frac{1}{N} \dot{x}^\mu + \frac{1}{N} \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta + (\frac{1}{2} \mathcal{A} R + 1) g^{\mu\alpha} \sqrt{g} \epsilon_{\alpha\beta} \dot{x}^\beta = 0. \tag{71}$$

The term including $\mathcal{A}R$ is non-minimal and vanishes in flat-spacetime. The second term resembles the interaction with an external electromagnetic field in two dimensions. In our framework, this additional interaction can be incorporated by adding to the action (12) the gauge-invariant term

$$- \int d^2\xi \int d\tau \delta^2(\xi - x(\tau)) < J + \mathcal{A} I, A_\nu > . \tag{72}$$

Using (38) one can verify that the interaction term above originates, upon variation with respect to x^μ, the forced geodesic equation (70). The equation of motion for the multiplier η is again (49) but now

$$\tilde{Q} = Q - \frac{\dot{x}^\mu}{N} e_\mu^a P_a - J - \mathcal{A} I \tag{73}$$

and once more, after some algebra, one gets $D_\tau Q = 0$. In order to obtain $Q(\tau)$ proceed as above with the initial condition $Q(0) = Q^a P_a + Q^2 J + Q^3 I$. The components of $Q(\tau)$ proportional to P_a and J are respectively

$$Q^a(\tau) = \left(exp: -\int_0^t \epsilon\omega :\right)^a_b Q_0^b P_a - \left(exp: -\int_0^t \epsilon\omega : \int_0^t ds exp : \int_0^s \epsilon\omega : \epsilon e(s)\right)^a Q_0^2 P_a,$$

$$Q^2(\tau) = Q_0^2 J. \tag{74}$$

Let us compare again with the formulation of Cangemi and Jackiw. The equations of motion obtained from the action (70) besides $F = 0$ are

$$D_\tau q = N(p + < p, q > I),$$

$$D_\tau p + < D_\tau p, q > I = [D_\tau q, q]$$

$$D_\mu \eta = \epsilon_{\mu\nu} J^\nu \tag{75}$$

where the current J^ν is given by

$$J^\nu(\xi) = \int d\tau([p(\tau), q(\tau)] - q(\tau))\dot{x}(\tau)\delta^2(\xi - x(\tau)) \tag{76}$$

From the equations of motion (75) above, we note that $D_\tau([p,q] - q) = 0$, so that it is possible to identify $[p,q] - q$ with $Q(\tau)$. In fact, defining $[p,q] - q = h^A T_A$, one gets that the components h^a and h^2 satisfy

$$\dot{h}^2 = 0$$

$$\dot{h}^a = -(\dot{q}^a + \epsilon^{ab}\dot{p}_b) = (-\epsilon^a_b \omega h^b + \epsilon^a_b e^b) \tag{77}$$

The solutions to these equations are precisely the expressions (74) for the components Q^a and Q^2 obtained when solving the covariant conservation equation for $Q(\tau)$ with the aid of the path-ordered exponentials. In this fashion, we are allowed to identify $Q^a(\tau) = ([p,q] - q)^a = h^a$ and $Q^2 = ([p,q] - q)^2 = h^2 = -1$. The expression for $Q^3(\tau)$ is calculated by taking into account the terms proportional to I, and the identification with h^3 is obtained after following the procedure shown above.

4 Discussion

The method proposed in Ref.[20] in the context of Yang-Mills theory provides a general rule for the coupling of non-Abelian point sources with gauge fields. Following this approach we constructed te action describing the interaction of non-Abelian point sources with a set of BF fields. For this system the condition $F = 0$ determines that the particles suffer no Lorentz like forces and this allows the equations of motion to be integrated explicitly. The solutions for the B fields may then be written in terms of steps functions depending only on the trajectories of the particles which are geodesics. When, using the equivalence of the two dimensional gravity models with corresponding BF models, this results lead to a new formulation of the interaction of two dimensional gravity with sources in terms of the geometrical variables which is equivalent to the results of Ref.[11] [13] .

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