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Detectability of cosmic topology in flat universes

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Abstract

A combination of recent astrophysical and cosmological observations seems to indicate that we live in an accelerating Friedmann-Lemaître-Robertson-Walker (FLRW) universe whose spatial sections are nearly or exactly flat ($\Omega_0 \simeq 1$). Motivated by this, and in order to complete our previous investigations on the detectability of *nearly* flat hyperbolic and spherical universes, we study here the problem of observational detection of the topology of FLRW universes with (exactly) flat spatial sections. To this end, we first give a complete description of the diffeomorphic classification of compact flat 3-manifolds, and determine the expressions for the injectivity radii (r_{inj}), and for the volume of each class of Euclidean 3-manifolds. There emerges from our calculations the undetectability conditions for each (topological) class of flat universes. We also study how the bounds provided by recent cosmological observations can be used to identify flat models having undetectable topologies. To materialize and quantify the study of the detectability of flat topologies we use the undetectability conditions and an assumption by Bernshteĭn and Shvartsman which permits to establish a relation between topological typical lengths to the dynamics of flat models. As a particular result we show that none of the models of two specific classes of flat universes which satisfy the Bernshteĭn and Shvartsman condition has an undetectable topology, even if current catalogues of clusters of galaxies are used. A modified version of Bernshteĭn-Shvartsman assumption is also suggested and used to construct a great number of flat universes with undetectable topology, even if cosmic microwave background radiation is used.

Key-words: Cosmic topology; Nearly flat universe; Compact universe; Shape of the Universe; BOOMERANG and MAXIMA Experiments; Detectibility of Cosmic Topology; Topology of Friedmann-Robertson-Walker Universes.

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1 Introduction

A great deal of work has recently gone into studying the possibility that the universe may possess compact spatial sections with a nontrivial topology, including the construction of different topological indicators (see, for example, refs. [1] – [3]). A fair number of these studies have concentrated on cases where the densities corresponding to matter and vacuum energy are substantially smaller than the critical density. This was motivated by the fact that until very recently observations favoured a low density universe. However, recent measurements of the position of the first acoustic peak in the angular power spectrum of cosmic microwave background radiation (CMBR) anisotropies, by BOOMERANG-98 and MAXIMA-I experiments, seem to provide strong evidence that the corresponding ratio for the total density to the critical density, Ω_0, is close to one [4, 5].

In addition an important aim of most of previous works in Cosmic Topology has often been to produce examples where the topology of the universe has strong observational signals, and can therefore be detected and even determined. Until very recently it was never considered in detail the possibility that the topology of the universe may not be detectable from the current astro-cosmological observations due to its almost *flatness*.

In two previous articles [6, 7] we have studied the question of detectability of a possible nontrivial compact topology in locally homogeneous and isotropic universes with total density parameter close to (but different from) one, i.e. the so-called *nearly flat hyperbolic or spherical* universes. We have employed an indicator T_{inj} which is defined by the ratio of the *injectivity radius*, r_{inj} , to the depth of a given catalogue, d_{obs}. In a recent article Gausmann *et al.* [8] have also studied which spherical topologies are likely to be detectable by using crystallographic methods. An important outcome from these studies [6] – [8] is that by using any method of detection of topology which rely on observations of repeated patterns the topology of an increasing number of nearly flat (hyperbolic and spherical) becomes undetectable as $\Omega_0 \to 1$. Thus, it would appear at first sight that in the limiting case $\Omega_0 = 1$ the topology of FLRW universes would definitely be undetectable. It turns out, however, as we discuss in the present work, that when Ω_0 is exactly one the topological possibilities for the universe are completely different, and the detectability of cosmic topology may again become possible.

In this article we complete our previous works [6, 7] by extending the analysis of detectability of cosmic topology to the flat cases. We also study how the bounds provided by recent cosmological observations can be used to identify flat models having undetectable topologies. The underlying cosmological setting and new results of this paper are stated and structured as follows.

In Section 2 we give an account of the cosmological framework employed throughout this work. Section 3 gives a complete description of the diffeomorphic classification of compact flat 3-manifolds. In this section we also determine the expressions for the injectivity radii r_{inj} and present the formulae (derived in Appendix B) for the volume for each class of Euclidean compact 3-manifolds. In Section 4 we present a brief discussion of the question of detectability of topology, recasting (and refining upon) some of the detectability aspects discussed in our previous articles [6, 7]. In Section 5 we use the results of Sections 3 and 4 in connection with an assumption first suggested by Bernshteĭn and Shvartsman [9] to materialize and quantify the problem of the detectability of flat topologies. We note that with Bernshteĭn-Shvartsman assumption one can establish a relation between topological typical lengths to the dynamics of flat models.[1] As a particular result we show that all models of two specific classes of flat universes which satisfy Bernshteĭn-Shvartsman hypothesis have a *detectable* topology (at least in principle for some observers) if the existing catalogues of clusters of galaxies are used. However, we also show that an alternative version of Bernshteĭn-Shvartsman assumption leads to flat universes with *undetectable* topologies, according to the most recent observations, and even if CMBR is used. Section 6 contains a summary of our main results and further remarks. The details of the isometric classification of compact flat 3-dimensional manifolds is treated in Appendix A. Finally in Appendix B we present the relevant piece of calculations which lead to the expressions of the volume of all compact Euclidean 3-manifolds.

2 Cosmological preliminaries

We shall assume that the universe is modelled by a 4-manifold $\mathcal{M}$ which allows a $(1+3)$ splitting, $\mathcal{M} = R \times M$, with a locally isotropic and homogeneous Robertson-Walker metric

$$ds^2 = -c^2dt^2 + R^2(t) \left[d\chi^2 + f^2(\chi)(d\theta^2 + \sin^2\theta \, d\phi^2)\right] \,, \tag{2.1}$$

where t is a cosmic time, c is the speed of light, $f(\chi) = \chi$, $\sin\chi$, or $\sinh\chi$, depending on the sign of the constant spatial curvature ($k = 0, \pm 1$), and $R(t)$ is the scale factor. Furthermore, although we will deal only with models with Euclidean spatial sections, for the sake of comparison we shall initially consider that the 3-space M is a multiply connected compact manifold, i.e. that $M = \widetilde{M}/\Gamma$, where Γ is a discrete group of isometries of the covering space $\widetilde{M}$ acting freely on $\widetilde{M}$. The covering space $\widetilde{M}$ takes one of the forms E^3, S^3 or H^3 [corresponding, respectively, to flat ($k=0$), spherical ($k>0$) and hyperbolic ($k<0$) spaces]. The group Γ is called the covering group of M, and is isomorphic to the fundamental group $\pi_1(M)$.

For non-flat models ($k \neq 0$), the scale factor $R(t)$ is identified with the curvature radius of the spatial section of the universe at time t, and thus χ can be interpreted as the distance of any point with coordinates (χ, θ, ϕ) to the origin of coordinates (in $\widetilde{M}$), in units of curvature radius, which is a natural unit of length and suitable for measuring areas and volumes. For flat models, χ can still be interpreted as the distance of any

[1] The Bernshteĭn and Shvartsman hypothesis is an attempt to establish a connection between local and global properties of the universe, and may be looked upon as a physical ground for the existence of a fundamental length in flat cosmological models.

point with coordinates (χ, θ, ϕ) to the origin of coordinates in units of $R(t)$, but since the curvature radius of Euclidean 3-space is infinite, in this case there is no natural unit of length, so we will measure lengths in megaparsecs (Mpc).

In the light of current observations, we assume the current matter content of the universe to be well approximated by pressureless dust plus a cosmological constant [10]. The redshift-distance relation in FLRW models with Euclidean spatial sections can be written in the form

$$d(z) = \frac{c}{H_0} \int_1^{1+z} \frac{dx}{\sqrt{\Omega_{\Lambda 0} + \Omega_{m0} x^3}} \ , \tag{2.2}$$

where Ω_{m0} and $\Omega_{\Lambda 0}$ are, respectively, the matter and the cosmological density parameters. For flat spatial sections we obviously have the constraint $\Omega_0 \equiv \Omega_{m0} + \Omega_{\Lambda 0} = 1$. Finally we note that the horizon radius d_{hor} is defined as the limit of (2.2) when $z \to \infty$.

To close this section we remark that it is unlikely that astro-cosmological observations can fix the density parameter Ω_0 to be exactly equal to one. However, since the algebraic structure of the fundamental group of a constant curvature manifold is different for each one of the three constant curvature 3-geometries, the identification of the topology of space would unambiguously fix the sign of the 3-curvature. Moreover, if it turns out that Ω_0 is exactly equal to one, topology seems to be the only way to precisely determine such a sharp value for Ω_0 [9].

3 Compact flat space forms

In this section we describe the classification of all flat compact 3-manifolds and calculate the explicit expressions for the corresponding injectivity radii. Further, we also present the formulae for the volumes of these 3-manifolds, which are obtained with some details in the Appendix B. The expressions for the injectivity radii are needed to build the topological indicator T_{inj} (Section 4), which together with the expressions for the volumes are employed in Section 5 to concretely discuss the detectability of the topology of classes of compact flat cosmological models.

The diffeomorphic classification of Euclidean 3-dimensional space forms is well known [11]. There are ten classes of compact Euclidean 3-manifolds, six of which are orientable. Tables 3 and 3 give the diffeomorphism classes of orientable and non-orientable compact Euclidean 3-dimensional space forms, respectively. In these tables, the triple $\{a, b, c\}$ is a set of three linearly independent vectors in Euclidean 3-space (basis). An isometry in Euclidean 3-space is denoted by (A, a), where a is a vector and A is an orthogonal transformation, and the action of (A, a) is given by

$$(A, a) : x \mapsto Ax + a \ , \tag{3.3}$$

for any vector x. An isometry of the form (I, a), where I is the identity transformation, is written simply as a.

Class	Generators of Γ	r_{inj}	Volume
$\mathcal{G}_1$	$a,\ b,\ c$	$\frac{1}{2}\lvert a\rvert$	$\lvert a\times b\cdot c\rvert$
$\mathcal{G}_2$	$(A_1,a),\ b,\ c$	$\frac{1}{2}\min\{\lvert a\rvert,\lvert b\rvert\}$	$\lvert a\rvert\,\lvert b\times c\rvert$
$\mathcal{G}_3$	$(B,a),\ b,\ c$	$\frac{1}{2}\min\{\lvert a\rvert,\lvert b\rvert\}$	$\frac{\sqrt{3}}{2}\lvert a\rvert\,\lvert b\rvert^2$
$\mathcal{G}_4$	$(C,a),\ b,\ c$	$\frac{1}{2}\min\{\lvert a\rvert,\lvert b\rvert\}$	$\lvert a\rvert\,\lvert b\rvert^2$
$\mathcal{G}_5$	$(D,a),\ b,\ c$	$\frac{1}{2}\min\{\lvert a\rvert,\lvert b\rvert\}$	$\frac{\sqrt{3}}{2}\lvert a\rvert\,\lvert b\rvert^2$
$\mathcal{G}_6$	(A_1,a), $(A_2,b+c)$, $(A_3,a+b+c)$	$\frac{1}{2}\min\{\lvert a\rvert,\lvert b\rvert,\lvert c\rvert\}$	$2\lvert a\rvert\,\lvert b\rvert\,\lvert c\rvert$

Table 1: Compact orientable 3-dimensional Euclidean space forms. Diffeomorphism classes of compact orientable 3-dimensional Euclidean space forms. The first column contains Wolf's notation for each class. The second gives the generators of the corresponding covering groups. The third column gives the injectivity radius, and the fourth the volume. An isometry in Euclidean 3-space is written as (A,a), where A is an orthogonal transformation and a is a vector. The action of the isometry (A,a) on Euclidean 3-space is given by (3.3), while the orthogonal transformations in the second column are given by (3.4). From the isometric classification of flat 3-manifolds (Appendix A) one has that for class $\mathcal{G}_1$ the vectors a,b,c can always be ordered such that $|a|\leq|b|\leq|c|$. Further, for the class $\mathcal{G}_2$ the vectors b and c can be always be ordered such that $|b|\leq|c|$. Finally, for the classes $\mathcal{G}_3$ – $\mathcal{G}_5$ one always has $|b|=|c|$. This makes apparent why the parameters $|b|$ and $|c|$ do not appear in the expression of r_{inj} for the class $\mathcal{G}_1$, and also why $|c|$ does not appear in the expressions for r_{inj} for the classes $\mathcal{G}_2-\mathcal{G}_5$.

The orthogonal transformations that appear in the classification of the Euclidean space forms take, in the basis $\{a,b,c\}$, the following matrix forms [11]:

$$A_1=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}\;,\quad A_2=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\;,\quad A_3=\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\;, \tag{3.4}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix},$$

for the rotations, and

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}. \tag{3.5}$$

for the reflections.

Class	Generators of Γ	r_{inj}	Volume
$\mathcal{B}_1$	(E,a), b, c	$\frac{1}{2}\min\{\lvert a\rvert, \lvert b\rvert, \lvert c\rvert\}$	$\lvert a \times b\rvert\, \lvert c\rvert$
$\mathcal{B}_2$	(F,a), b, c	$\frac{1}{2}\min\{\lvert a\rvert, \lvert b\rvert, \lvert c\rvert\}$	$\lvert a \times b\rvert\, \lvert c - (a + \frac{1}{2}b)\rvert$
$\mathcal{B}_3$	(A_1,a), (E,b), c	$\frac{1}{2}\min\{\lvert a\rvert, \lvert b\rvert, \lvert c\rvert\}$	$\lvert a\rvert\, \lvert b\rvert\, \lvert c\rvert$
$\mathcal{B}_4$	(A_1,a), $(E,b+c)$, $2c$	$\frac{1}{2}\min\{\lvert a\rvert, \lvert b\rvert, 2\lvert c\rvert\}$	$2\,\lvert a\rvert\, \lvert b\rvert\, \lvert c\rvert$

Table 2: Compact non-orientable 3-dimensional Euclidean space forms. Diffeomorphism classes of compact non-orientable 3-dimensional Euclidean space forms. The contents of the columns are of the same type as in Table 3. The orthogonal transformations in the second column are given by (3.4) and (3.5).

Euclidean manifolds are not *rigid*, in the sense that they can be deformed while still conserving its zero curvature at every point. As a consequence, manifolds in each class are topologically equivalent but can have different sizes, and even shapes, i.e. although diffeomorphic they may not be isometric. For example, a 3-torus can be constructed by taking any parallelepiped and identifying opposite faces by translations. Usually, it is considered a parallelepiped with mutually orthogonal faces (a brick), the simplest example being that of a cube. However, note that parallelepipeds with different volumes give rise to non-isometric torii. Stretching the parallelepiped in one or more directions while leaving the volume constant also gives rise to non-isometric torii.

More exactly, the generators of the covering group of a torus form a basis in Euclidean 3-space. But the only condition for three vectors in Euclidean 3-space to form a basis is to be linearly independent. No restriction is imposed to the lengths of these vectors, nor to their mutual orientations, i.e. the angles between them. These six parameters (three lengths and three angles) uniquely characterize "locally" a torus in 6-dimensional parameter space formed by the lengths of the basis vectors and the angles between them. So, two torii with these six parameters approximately equal are non-isometric, but very similar in shape and size. However, this characterization is not unique ("global"?). This can be seen by considering any basis $\{a, b, c\}$, and using it to construct another basis, say $\{a, a+b, c\}$, that generates the same fundamental group, and hence the same torus. Clearly the lengths of the vectors and the angles between them are very different for these two bases. Thus we have two different sets of these six parameters characterizing the same torus.

It is therefore clear that, to uniquely characterize isometrically a torus, one has to perform some identifications in the 6-dimensional parameter space formed by the lengths of the basis vectors and the angles between them. This resulting quotient space is a kind of modular space for the torus, and uniquely ("globally") gives the isometric classification for it. In general the modular spaces are not manifolds, as it is in the special case of the torus. The isometric classification, i.e. the modular spaces, of Euclidean 3-dimensional compact space forms is given with some details in Appendix A.

A natural way to characterize the shape of compact manifolds is through the size of their closed geodesics. A suitable indicator is constructed using the injectivity radius defined by (see [6, 7])

$$r_{inj} = \frac{1}{2} \min_{(g,x)\in\widetilde{\Gamma}\times P} \{\delta_g(x)\} \;, \tag{3.6}$$

where $\widetilde{\Gamma}$ denotes the covering group without the identity map, i.e. $\widetilde{\Gamma} = \Gamma \setminus \{id\}$, and P is any fundamental polyhedron for M. The distance function $\delta_g(x)$ for a given isometry $g \in \Gamma$ is defined by

$$\delta_g(x) = d(x, gx) \;, \tag{3.7}$$

for all $x \in P$. Clearly here d is the Euclidean metric. The distance function gives the length of the closed geodesic associated with the isometry $g \in \Gamma$ that passes through the projection of x onto M. So, from (3.6) and (3.7) one has that the injectivity radius is half of the length of the smallest closed geodesic in M, or equivalently, the radius of the smallest sphere inscribable in M.[2]

In a globally homogeneous manifold, the distance function for any covering isometry g is constant, and so is the length of the closed geodesic associated to g and that passes through any $x \in M$. However, this is not the case in a locally, but not globally,

[2]Incidentally, as far as we are aware, the injectivity radius was introduced in Cosmic Topology by Sokolov and Shvartsman [12] who called it the "minimum gluing parameter l". It was defined by saying that "at distances $r < l$ there isn't a single ghost". In modern terms it means that any survey with depth smaller than the injectivity radius does not present multiple images nor pattern repetitions (see Section 4).

homogeneous manifold, so the calculation of r_{inj} for these cases requires some careful work.

To compute r_{inj} it is convenient to choose a faithful representation of M, which is the Dirichlet domain P of Γ, to be centered at the origin of Euclidean 3-space. Consider the subset $\Delta \subset \widetilde{\Gamma}$ consisting on the isometries that transform P to a neighbouring cell in the correspondent tessellation. Thus for all $g \in \Delta$, $gP \cap P$ is either a face, an edge, or a vertex of P. Now, for any $g = (A, a_g) \in \Delta$, the set of points that A leaves unchanged (the axis of rotation if $\det(A) = 1$, or the plane of reflection if $\det(A) = -1$) passes through the origin and thus intersects P, so

$$\min_{x\in P}\{\delta_g(x)\} = |a_g^{||}| \,, \tag{3.8}$$

where $a_g^{||}$ is the projection of a_g onto the set of fixed points of A. From eqs. (3.6) and (3.8) and taking into account the above reasoning we have

$$\begin{aligned} r_{inj} &= \frac{1}{2} \min_{(g,x)\in\Delta\times P}\{\delta_g(x)\} && (3.9)\\ &= \frac{1}{2}\min_{g\in\Delta}\left\{\min_{x\in P}\{\delta_g(x)\}\right\} && (3.10)\\ &= \frac{1}{2}\min_{g\in\Delta}\{|a_g^{||}|\}\,. && (3.11) \end{aligned}$$

Analyzing separately each case from Tables 3 and 3 one can compute the injectivity radius for any compact Euclidean 3-dimensional space form. The results are exhibited in the third column of each table. Let us illustrate this procedure for the case of manifolds of class $\mathcal{G}_6$. Firstly note that the matrices A_1, A_2 and A_3 satisfy the products $A_iA_j = A_k$, where the indices i, j, k run in a cyclic order. Secondly note that the vectors a, b, and c, are parallel to the axes of rotation of A_1, A_2 and A_3 respectively. As a consequence, any isometry of the covering group of a manifold of class $\mathcal{G}_6$ is of the form $\lambda = (A_i, u)$, where $i = 1, 2, 3$ and u is a linear combination of the vectors a, b, and c. Since λ has no fixed points, then the coefficient of a is a non-zero integer if $i = 1$, and similarly for other values of i. It is now clear that $u^{||}$ is a multiple of a, b, or c for $i = 1, 2$ or 3 respectively. Thus the minimum length of a closed geodesic in a $\mathcal{G}_6$ manifold is $\min\{|a|, |b|, |c|\}$.

To conclude this section note that the volumes of all closed Euclidean 3-manifolds are listed in the fourth column of Tables 3 and 3. The relevant calculations are given in Appendix B.

4 Detectability problem in cosmic topology

One may conjecture whether there are fundamental laws that can restrict or even predict the topology of the universe. Nevertheless, its detection and determination is ultimately an observational problem. At present it is becoming clear that the detection of a possible nontrivial topology of the universe may be a difficult problem to accomplish in view of the bounds on the cosmological parameters set by recent observations [6] – [8]. Indeed, it was shown in [6, 7] (see also [8] for detectability of spherical spaces) that, if one uses pattern repetitions, increasing number of nearly flat spherical and hyperbolic possible topologies for the universe become undetectable as $\Omega_0 \to 1$. It would appear at first sight that in the limiting case $\Omega_0 = 1$ the topology of such universes would definitely be undetectable. It turns out, however, that when Ω_0 is exactly one the topological possibilities for the universe are completely different, and the detectability of cosmic topology may again become possible. In this section we shall briefly review some indicators of our approach to the detectability of cosmic topology in order to make explicit in Section 5 that the detectability of the topology for the flat cases becomes indeed a concrete possibility.

For cosmological models with compact spatial sections M which have nontrivial topology it is clear that any attempt at the discovery of such a topology through observations must start with the comparison between the horizon radius and suitable characteristic sizes of the manifold M. We use the injectivity radius as a characteristic size of M. The ratio of the injectivity radius to the horizon radius,

$$T_{inj}^{hor} = \frac{r_{inj}}{d_{hor}} \, , \tag{4.12}$$

is very useful to identify cosmological models whose topology is undetectable through methods that rely on the existence of multiple images or pattern repetitions in any survey. In fact, for the case in which $T_{inj}^{hor} \geq 1$ the whole observable universe lies inside a fundamental polyhedron of M, no matter what is the location of the observer in the manifold (universe). In such cases no multiple images (or pattern repetitions) will arise from any survey. Thus, any method for the search of cosmic topology based on the existence of repeated patterns (multiple images) will not be capable of detecting the cosmic topology — the topology of the universe is definitely undetectable in such cases.

We shall now discuss the detectability problem when we restrict the search to specific catalogues. There are basically three types of catalogues which can possibly be used in order to search for repeated patterns in the universe: namely, clusters of galaxies, containing clusters with redshifts of up to $z \approx 0.3$; active galactic nuclei (mainly QSO's and quasars), with a redshift cut-off of $z_{max} \approx 4$; and maps of the CMBR with a redshift of $z \approx 10^3$. In this way, instead of d_{hor} it is observationally more suitable to consider the largest distance $d_{obs} = d(z_{max})$ covered by a given survey, and define the indicator

$$T_{inj} = \frac{r_{inj}}{d_{obs}} \, . \tag{4.13}$$

In this context, we shall refer to the region covered by a given survey by "observed universe".

Now, in the cases when $T_{inj} > 1$, every source in the survey is inside a fundamental polyhedron of M, no matter the location of the observer within the manifold. Actually, the whole "observed universe" lies inside a fundamental polyhedron of M. So, there are no multiple images in that survey and every method for the search for topology based on the existence of multiple images is not capable of detecting the cosmic topology in this cases — the topology of the universe is undetectable with this specific survey. Thus, there emerges trivially from eq. (4.13) and the expressions for the injectivity radii for all classes of flat manifods given in Section 3 the undetectability conditions for each (topological) class of flat universes. Equations (5.17) of Section 5 constitute an example of such conditions for the flat classes $\mathcal{G}_3$ and $\mathcal{G}_5$ (see also [13] for another explicit example).

The bounds provided by recent cosmological observations [4, 5] can be used to identify flat models having undetectable topologies, since an absolute lower bound of r_{inj} for undetectability of flat universes can be obtained by calculating the horizon radius corresponding to the limiting values of the density parameters. Indeed, for $\Omega_{m0} = 0.4$ and $\Omega_{\Lambda 0} = 0.6$, one obtains $d_{hor} = 12600\, Mpc$ from equation (2.2). Thus, flat universes for which $r_{inj} \geq 12600$ have undetectable topologies.[3]

Some remarks are in order here. The indicator T_{inj} is useful for the identification of cosmological models whose topology is undetectable by search methods based on the presumed existence of multiple images, for when $T_{inj} \geq 1$, the whole "observed universe" (region covered by a specific survey) lies inside a fundamental polyhedron of M. However, it should be noted that without further considerations, nothing can be said when $T_{inj} < 1$. In fact in this case, despite the radius of the depth of a given survey ("observed universe") be larger than r_{inj}, it may be that, due to the location of the observer, the "observed universe" would still be inside a fundamental polyhedron of M making the topology undetectable. This is the case when the smallest closed geodesic that passes through the observer is larger than $2d_{obs}$.

There is the case when d_{obs} is larger than the length of the smallest closed geodesic that passes through the position of the observer, but not too much larger, so that only a small fraction of the "observed universe" contains multiple images. Current methods that look for multiple images are not sensitive enough to detect this small quantity of copies, so even in this case the topology of the universe would be in practice undetectable until the suitable refined new methods are developed (and used). These and other points will be dealt with in future works.

The topology of a given cosmological model being undetectable for a given survey up to a depth z_{max}, clearly may be detectable by using other deeper survey. However, the deepest catalogue (survey) ever constructed will have $z_{max} < z_{SLS}$ ($z_{SLS} \approx 10^3$ being the redshift of the surface of last scattering). Thus the quotient (4.13) computed with z_{SLS} is a lower bound for the indicator T_{inj}. It turns out that, in practice, there is almost no difference if we push z_{SLS} to infinite, and take T_{inj}^{hor} as the lower bound for T_{inj} (see [6, 7]).

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[3]Note that, e.g., for current catalogues of clusters ($z_{max} = 0.3$), the undetectable flat universes are now those for which $r_{inj} \geq 1170\, Mpc$, making clear that relative lower bounds arise when different catalogues are used.

5 Detectability of flat topologies: a case study

Since there is no natural unit of length in Euclidean geometry, there is no natural way to relate the typical lengths of the spatial sections of a flat cosmological model with the density parameters within the General Relativity framework, as in the case of non-flat cosmological models (see e.g. [9] and [6]). So we have to rely on other kinds of grounds to establish such a connection.

Following Bernshteĭn and Shvartsman [9] we will suppose that the total number of baryons in our universe equals the reciprocal of the square of the gravitational fine structure constant,

$$N = \alpha_{gr}^{-2} = \left(\frac{G m_p^2}{\hbar c} \right)^{-2} \approx \, 2.87 \cdot 10^{76} \; , \tag{5.14}$$

where G is the gravitational constant, $h = 2\pi\,\hbar$ is Planck's constant and m_p is the proton mass. This hypothesis enables one to construct cosmological models in which the volume of the universe is related to cosmological parameters and fundamental physical constants.

A word of clarification is in order here. We remark that we are not claiming that (5.14) is a realistic assumption for constructing theoretical models for our universe. Instead, our intention here is, in the one hand, to illustrate how a hypothesis like (5.14), that may arrive from, e.g., a fundamental theory unifying elementary particles with gravity, can be used to construct models that can in principle be confronted with cosmological observations. On the other hand, we will use Bernshteĭn-Shvartsman assumption (5.14) to materialize and quantify the study of the detectability of flat topologies, since it permits to establish a relation between topological typical lengths (volume) and the dynamics of flat models.

Since the total baryonic mass is $M_b = N m_p$, the volume of our universe is

$$V = \frac{8\pi}{3} \frac{G N m_p}{\Omega_b H_0^2} \; , \tag{5.15}$$

where we have used $\Omega_b = \frac{8\pi G \rho_b}{3 H_0^2}$, for the baryon density parameter. Taking the current value for $\Omega_b h^2 = 0.03$ and $H_0 = 100\, h \, km \, Mpc^{-1}/s$, with $h = 0.7$ (see, e.g. [5]), one obtains

$$V \approx 2.9 \cdot 10^9 Mpc^3 \tag{5.16}$$

for the volume of the universe.

In order to illustrate the method of analysis of detectability of topology in these models, let us restrict to those ones whose spatial sections have a $\mathcal{G}_3$ or $\mathcal{G}_5$ topology. Note that the expressions for both r_{inj} and the volumes of these two classes are identical, and the other classes can be treated in a similar way. Let us also consider that we are looking for multiple images with a catalogue of clusters of galaxies with redshift cut-off $z_{max} = 0.3$. Using the values $\Omega_{m0} = 0.3$ and $\Omega_{\Lambda 0} = 0.7$ (see [4, 5]) one obtains $d_{obs} = 1200\, Mpc$. From Table 3 we have that the region of undetectability is given by

$$\begin{aligned} |b| > 2\, d_{obs} \qquad & \text{if} \qquad |a| \geq |b| \,, \\ |a| > 2\, d_{obs} \qquad & \text{if} \qquad |a| < |b| \,. \end{aligned} \tag{5.17}$$

This region is shown in Figure 1, together with a curve of constant volume given by (5.16). This figure shows that the region of undetectability does not intersect the curve of constant volume, making clear that it may be possible to detect the topology in such universes. However, the possibility of detecting and even deciding the topology in these models depends on other factors such as the location of the observer in M, and the quality and reliability of the catalogues and methods used for the search of repeated patterns. We will deal with these problems in forthcoming articles.

Before closing this section we mention that if one introduces a modified version of Bernshteĭn-Shvartsman assumption in which N equals the reciprocal of the square of a new gravitational fine structure constant now given by

$$\alpha_{gr}^{-2} = \left(\frac{G m_p m_e}{\hbar c} \right)^{-2} \approx 9.68 \cdot 10^{82} \,, \tag{5.18}$$

where m_e is the electron mass, following the above reasoning one obtains a figure in which the curve of constant volume lies in the undetectability region, making clear that the topology of such flat universes of class $\mathcal{G}_3$ is undetectable even CMBR is used. We shall not go into details of such a simple (and rather similar) calculations here for the sake of brevity.

To close this section we note that the above results are insensitive to substantial variations of the values of the cosmological density parameters Ω_{m0} and $\Omega_{\lambda 0}$. In fact, even for the extreme cases of Einstein-de Sitter ($\Omega_{m0} = 1$) and pure cosmological constant ($\Omega_{m0} = 0$), the constant volume curve does not cross the undetectability region of Figure 1 if Bernshteĭn-Shvartsman assumption is used, whereas if the above mentioned modified version of this assumption is employed one arrives at flat universes with undetectable topologies.

6 Conclusion and further remarks

It is becoming generally known that the detection of a possible nontrivial topology of the universe may be a difficult problem to accomplish in view of the bounds on the cosmological parameters set by recent observations which seem to indicate that we live in FLRW accelerating universe with nearly flat spatial sections [6] – [8]. Indeed, we have shown in [6, 7] that if one uses pattern repetitions, an increasing number of nearly flat spherical and hyperbolic possible topologies for the universe become undetectable as $\Omega_0 \to 1$. It would appear at first sight that in the limiting case $\Omega_0 = 1$ the topology of such universes would definitely be undetectable.

However, when Ω_0 is exactly one the topological set of possibilities for the universe is completely different from the nearly (but not exactly) flat cases, and we have shown that the detectability of cosmic topology may again becomes possible according to the most recent astrophysical and cosmological observations.

We have given a first prime of the diffeomorphic classification of compact flat 3-manifolds, and determined the expressions for both the injectivity radius r_{inj} and the volume for each class of Euclidean compact 3-manifolds.

We have used the expression of the injectivity radius for the classes $\mathcal{G}_3$ and $\mathcal{G}_5$, and the topological indicator T_{inj} in connection with Bernshteĭn-Shvartsman [9] assumption to materialize and quantify the problem of the detectability of flat topologies. As a particular result we have shown that all models of these two classes of flat universes that satisfy Bernshteĭn-Shvartsman hypothesis have a *detectable* topology (at least in principle for some observers) if the existing catalogues of clusters of galaxies are used. We have also shown that a modified version of Bernshteĭn-Shvartsman assumption leads to flat universes of classes $\mathcal{G}_3$ and $\mathcal{G}_5$ with *undetectable* topologies.

We have also presented a brief discussion of the question of detectability of topology and refined upon some of the detectability aspects discussed in our previous articles [6, 7].

Finally, we note that it is believed that a theory of quantum gravity or quantum cosmology would be able to set close relations between the dynamics of the universe, its material content, shape and size. Such a theory should, in principle, predict the detectability (or undetectability) of the topology of our universe given the current bound for cosmological density parameters. Thus the considerations and the results of the present paper as well as those in refs. [6, 7] may become a framework of the observational tests of this kind of fundamental theories.

Acknowledgments

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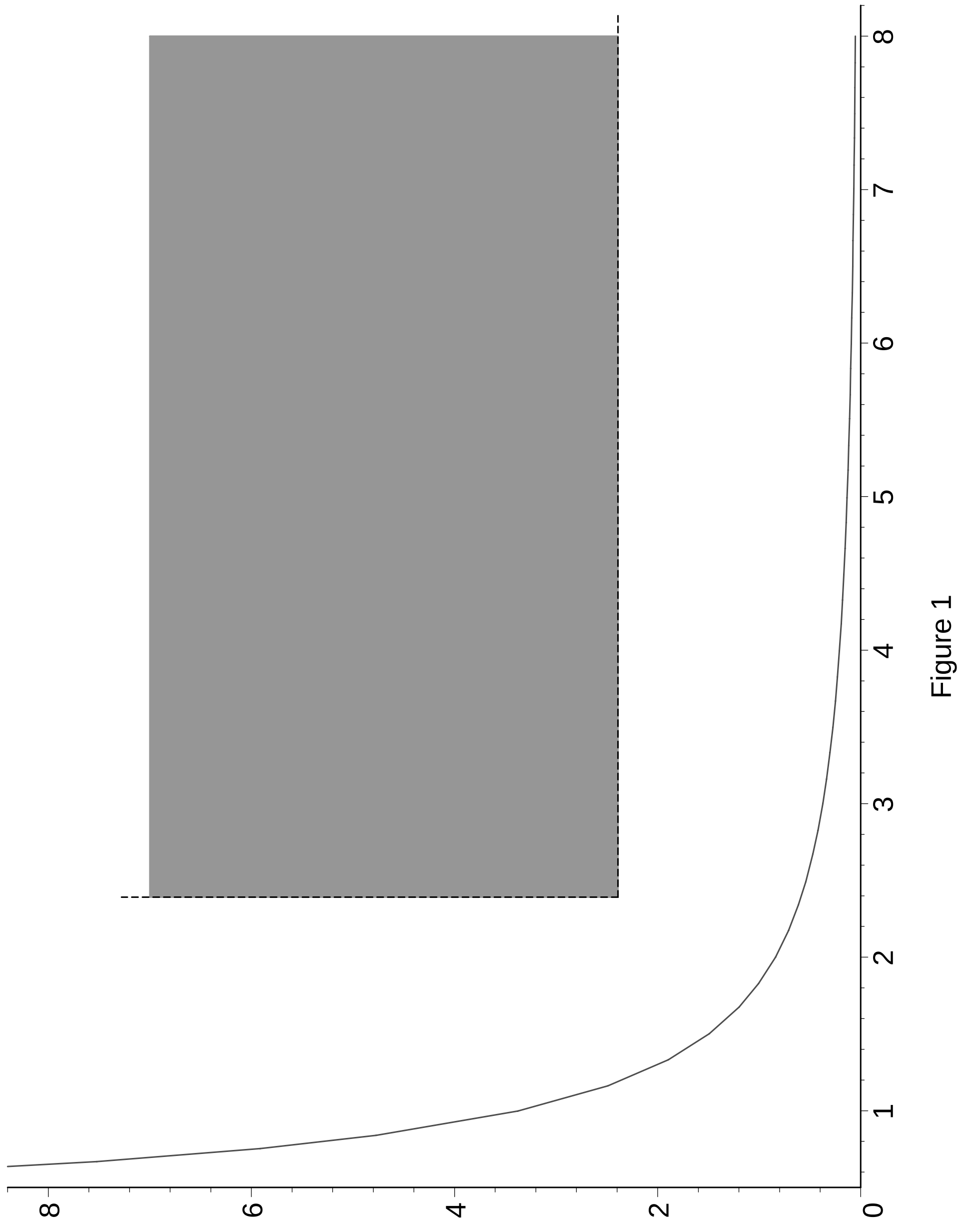

Figure 1: The undetectability region in the $(|a|, |b|)$ plane [defined by (5.17)] for universes with $\mathcal{G}_3$ and $\mathcal{G}_5$ topologies. Also a constant volume curve, whose expression is given in Table 1, for the value of the volume given by (5.16). The vertical axis represents $|a|$, while the horizontal axis gives $|b|$. Since the curve does not intersect the undetectability region it is potentially possible to detect the shape of such universes, with $\mathcal{G}_3$ and $\mathcal{G}_5$ topologies, using catalogues of clusters of galaxies ($z_{max} = 0.3$).

Appendices

A Isometric classification of flat 3-manifolds

In this appendix we shall present in full details the isometric classification of compact Euclidean 3-dimensional space forms following the reasoning scratched by Wolf (see Lemma 3.5.11 and comments below on pages 123 – 124 of [11]). The isometric classification can be collected together in the following theorem:

Theorem A.1 *The isometric classification of Euclidean* 3*-dimensional compact space forms is as follows:*

1. *The class $\mathcal{G}_1$ is parametrized by the equivalence classes $SL(3,Z)\cdot A(T^3)\cdot O(3)$, where $A(T^3)$ is the matrix whose rows are formed by the vectors of a base of Euclidean* 3*-space that generates the covering group of the torus.*
2. *The class $\mathcal{G}_2$ is parametrized by the ordered triples $(|a|, r, w)$, where the pair (r, w) parametrizes the bidimensional torus generated by the vectors b and c.*
3. *The classes $\mathcal{G}_3$, $\mathcal{G}_4$ and $\mathcal{G}_5$ are parametrized by the ordered pairs $(|a|,|b|)$.*
4. *The classes $\mathcal{G}_6$, $\mathcal{B}_3$ and $\mathcal{B}_4$ are parametrized by the ordered triples $(|a|,|b|,|c|)$.*
5. *The classes $\mathcal{B}_1$ and $\mathcal{B}_2$ are parametrized by the triples $(r, w, |c|)$, where the pair (r, w) parametrizes the double covering of the modular space of the bidimensional torus generated by the vectors a and b, which interchanges the generators of the lattice. Additionally, for the class $\mathcal{B}_2$, we have the constraint $|c| > |a + \frac{1}{2}b|$.*

Proof. We begin by constructing the modular space for the simplest classes, namely $\mathcal{G}_3$, $\mathcal{G}_4$ and $\mathcal{G}_5$. Consider first the class $\mathcal{G}_3$ and note that $Ba = a$, $Bb = c$, and $Bc = -(b + c)$. Since B is an orthogonal transformation, it follows that $|b| = |c| = |b + c|$, and from

$$\begin{aligned}\langle b, c\rangle &= \langle Bb, Bc\rangle \\ &= -\langle c, b+c\rangle \\ &= -\langle b, c\rangle - |c|^2 ,\end{aligned}$$

one finds that the angle between b and c is $\pi/3$. Furthermore, a is orthogonal to b and c. We have then that all the angles between the basis vectors are fixed, and two lengths (those of b and c) are equal, leaving just two free parameters, namely $|a|$ and $|b|$. It is now clear that the ordered pair $(|a|, |b|)$ parametrizes isometrically manifolds of class $\mathcal{G}_3$.

Manifolds of classes $\mathcal{G}_4$ and $\mathcal{G}_5$ are analyzed in a similar way, by just looking at the action of the transformations C and D on the basis vectors, respectively. One obtains the following:

$\mathcal{G}_4$. The three basis vectors are mutually orthogonal, and b and c are of the same length. The ordered pair $(|a|, |b|)$ parametrizes isometrically this class.

$\mathcal{G}_5$. The vectors b and c are of the same length and the angle between them is $\pi/6$, furthermore a is orthogonal to both b and c. Again, the ordered pair $(|a|, |b|)$ parametrizes isometrically this class.

Let us now turn our attention to manifolds with 3-dimensional modular spaces, i.e. manifolds of classes $\mathcal{G}_6$, $\mathcal{B}_3$ and $\mathcal{B}_4$. In all these cases the basis vectors are mutually orthogonal, as can be seen by analyzing the action of the transformations A_1, A_2, A_3 and E on the vectors a, b and c. Moreover, there is no restriction on the lengths of these vectors, so the parameter space is 3-dimensional. In order to see that these classes are parametrized by the ordered triple $(|a|, |b|, |c|)$ note that the vector a is the only one that enters as a translation along the axis of rotation of a screw motion. Also, for the classes $\mathcal{B}_3$ and $\mathcal{B}_4$, the vector b is in the invariant subspace of the reflection E, while c is not. On the other hand, for the class $\mathcal{G}_6$, the vector b is in the invariant subspace of the rotation in the screw motion $(A_2, b+c)$, while c does not have an analog property. Clearly, the vectors a, b and c play distinguished roles as translations in the generators of the corresponding fundamental groups.

We now briefly review the isometric classification of the bidimensional torus in order to continue with the proof of theorem A.1. The generators of the fundamental group of a 2-torus are two linearly independent vectors in the Euclidean plane, say $\xi = \{a, b\}$. These two vectors generate a lattice in the plane, namely

$$\Lambda_\xi = \{na + mb \ : \ n, m \text{ are integers}\} \ .$$

Actually, the lattice Λ_ξ is the orbit of the origin under the action of the group generated by the basis ξ.

It is clear that two different bases give rise to equivalent lattices if they are related by an orthogonal transformation of the plane, so one can always order the generators so that (i) $|a| \leq |b|$, (ii) a lies along the positive direction of the $x-$axis, and (iii) b is in the upper half plane with non-negative first component. Moreover, the bases $\{a, b\}$ and $\{a, na + b\}$, with n being a positive integer, give rise to the same lattice, so b can be taken to be such that its projection to the $x-$axis lies between 0 and $|a|/2$. Thus a 2-torus is characterized isometrically by the pair (r, w), where $r = |a|$, and w is the complex number representing the same point in the plane as the vector $\frac{1}{r}\, b$. The parameter w satisfies the following conditions:

1. $|w| \geq 1$, and

2. $0 \leq \mathrm{Re}\, w \leq 1/2$.

Next we study classes with 4-dimensional parameter spaces. Class $\mathcal{G}_2$ is the simplest, the vector a is orthogonal to b and c, and there is no restriction to the angle between these two vectors, nor to the lengths of the three basis vectors. Thus ordering the vectors b and c such that $|b| \leq |c|$ one concludes that the modular space of the class $\mathcal{G}_2$ is $(|a|, r, w)$, where (r, w) is the modular space for the bidimensional torus generated by b and c.

For the class $\mathcal{B}_1$ one notes that the vectors a and b are in the invariant subspace of the reflection E, and c is orthogonal to it. Since there is no restriction to the angle between a and b, these two vectors form a bidimensional torus. However, one cannot simply set the condition $|a| \leq |b|$ without loss of generality, since the vectors a and b enter in a very different way in generating the fundamental group of the manifold, i.e. b is a pure translation while a enters as part of a glide reflection. Hence we use a double covering of the modular space of the 2-torus generated by a and b to parametrize the class $\mathcal{B}_1$, one sheet parametrizing the case $|a| \leq |b|$, the other parametrizing the case $|b| \leq |a|$. The class $\mathcal{B}_2$ is similar to the class $\mathcal{B}_1$, except that c is not orthogonal to the plane formed by a and b, but has a projection $(a + \frac{1}{2}b)$ on this plane. This is the origin of the constraint $|c| > |a + \frac{1}{2}b|$.

Finally let us consider the class $\mathcal{G}_1$, or the 3-dimensional torus. The isometric classification of the torus has been partially discussed in Section 3. There, we have seen that bases related by orthogonal transformations give rise to isometric torii, and furthermore, one has to perform additional identifications on the set of bases of 3-dimensional Euclidean space to give a unique characterization of isometric torii. To put this in mathematical (and useful) language, let us first choose as a reference basis the canonical one ($\hat{\imath} = (1,0,0)$, $\hat{\jmath} = (0,1,0)$, and $\hat{k} = (0,0,1)$) to write the components of vectors and transformations. Now, let us write a basis in Euclidean 3-space as a square matrix whose rows are the components of the vectors of the basis. If the basis $\xi = \{e_1, e_2, e_3\}$ is related to the basis $\eta = \{f_1, f_2, f_3\}$ by an orthogonal transformation, one can write

$$\eta = \xi B \ ,$$

where $B \in O(3)$. In this case, the bases ξ and η generate two isometric torii.

A lattice in Euclidean 3-space is the orbit of the origin by the action of a group generated by three linearly independent vectors. Thus if $\xi = \{e_1, e_2, e_3\}$, the lattice generated by ξ is the set

$$\Lambda_\xi = \{n_1 e_1 + n_2 e_2 + n_3 e_3 \ : \ n_1, n_2, n_3 \text{ are integers}\} \ .$$

Now, two bases ξ and η may produce the same lattice, and hence generate the same torus. In fact, consider the group $SL(3, Z)$ formed by the square matrices of order three with integer entries and determinant unity. Any two bases related by a matrix $A \in SL(3, Z)$ generates the same lattice, for if $\eta = A\xi$, each f_i is an integer linear combination of ξ, and thus $\Lambda_\eta \subseteq \Lambda_\xi$. On the other side, $\xi = A^{-1}\eta$, thus $\Lambda_\xi \subseteq \Lambda_\eta$, and so $\Lambda_\xi = \Lambda_\eta$. We have that two bases ξ and η generate the same torus if there exist $A \in SL(3, Z)$ and $B \in O(3)$ such that $\eta = A\xi B$.

B Volumes of flat 3-manifolds

In this appendix we show how to calculate the volumes of compact flat 3-manifolds in terms of their modular spaces. The calculations are based on the following simple observations:

1. The volume of a compact manifold of constant curvature equals the volume of any of its fundamental domains.
2. If Γ is the covering group of a compact flat 3-dimensional manifold in the form given in Tables 3 and 3, the orbit of the origin by Γ is a lattice.
3. A fundamental domain of the lattice generated by the base $\xi = \{a, b, c\}$ is the parallelepiped naturally constructed with these vectors, and thus its volume is $|a \cdot b \times c|$.

Now, in terms of the correspondent modular spaces given in Theorem A.1, the orbit of the origin by the covering group of a manifold of any of the classes $\mathcal{G}_1$–$\mathcal{G}_5$ or $\mathcal{B}_1$–$\mathcal{B}_3$, is the lattice generated by the vectors a, b and c. While the orbit of the origin by the covering group of a manifold of class $\mathcal{G}_6$ is a lattice generated by the vectors a, $b + c$ and $b - c$. Finally, note that a manifold of class $\mathcal{B}_4$ is a double covering of the manifold of class $\mathcal{B}_3$ with the same parameters, hence its volume doubles that of the corresponding manifold of class $\mathcal{B}_3$.

The results of this appendix are listed in the fourth column of Tables 3 and 3.

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