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Aperiodic Interactions on Hierarchical Lattices: an Exact Criterion for the Potts Ferromagnet Criticality

by

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Abstract

We discuss the critical behavior of the q-state Potts model on any diamond-like hierarchical lattice with ferromagnetic interactions according to an arbitrary aperiodic two-letter substitutional sequence. We show that the geometric (deterministic) fluctuations become relevant for $\omega > 1 - D/(2-\alpha_u)$, where ω is the wandering exponent of the substitutional sequence, D is the fractal dimension of the lattice, and α_u is the critical exponent associated with the specific heat of the uniform model. Also, we point out that the criteria for analysing the relevance of deterministic and random fluctuations are generically *different*.

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The introduction of quenched disorder is known to change the critical behavior of ferromagnetic systems whenever (but not only) the corresponding uniform model is characterized by a positive exponent α_u associated with the divergence of the specific heat [1, 2]. A similar effect may be anticipated if the exchange interactions are chosen according to an aperiodic, although deterministic, type of rule. Recently, Luck[3] proposed a heuristic criterion which indicates indeed that the geometric fluctuations produced by the aperiodic rule may be responsible for changing the nature of the critical behavior.

The discovery of quasi-crystals [4] motivated the investigation of different types of spin models with aperiodic interactions. Recent calculations for the ground state of a quantum Ising chain do support the heuristic criterion of Luck [3, 5]. In previous papers [6], one of us has taken advantage of the simplicity of diamond-type hierarchical lattices (DHL) [7, 8] to analyze the critical behavior of the Ising model with a distribution of ferromagnetic exchange interactions according to a certain class of two-letter substitutional sequences. In the present paper, we extend these results to the q-state Potts model with aperiodic ferromagnetic interactions on a general diamond-type hierarchical lattice, and derive an exact criterion to show the relevance of the geometric fluctuations above a critical number of states q_d. We also establish some contacts with calculations for the disordered Potts model.

The q-state Potts ferromagnet is given by the Hamiltonian

$$\mathcal{H} = -q \sum_{(i,j)} J_{ij} \delta_{\sigma_i,\sigma_j}, \tag{1}$$

where $\sigma_i = 1, 2, \cdots, q$ for all sites of a lattice, $J_{ij} > 0$, and the sum (i, j) refers to nearest-neighbor sites. To give a simple example, let us consider the simple diamond lattice (that is, a DHL with $m = 2$ branches in parallel, each of them with $b = 2$ bonds in series), and choose the ferromagnetic interactions J_{ij} according to the two-letter generalized Fibonacci sequence given by the substitutions $(A, B) \to (AB, AA)$, as indicated in Fig. 1 (to mimic a layered structure, the interactions are *aperiodic along the branches* of the lattice). At each stage of this construction, the numbers N'_A and N'_B, of letters A and B, can be obtained from those of the preceding level, N_A and N_B, from the recursion relations

$$\begin{pmatrix} N'_A \\ N'_B \end{pmatrix} = \mathbf{M} \begin{pmatrix} N_A \\ N_B \end{pmatrix}, \tag{2}$$

with the substitution matrix

$$\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \tag{3}$$

whose eigenvalues are $\lambda_1 = b = 2$ and $\lambda_2 = -1$.

The total number of couplings (letters), $N^{(n)}$, at a large order n of this construction, fluctuates asymptotically as $\Delta N^{(n)} \sim (N^{(n)})^{\omega}$, where in general

$$\omega = \frac{\ln|\lambda_2|}{\ln \lambda_1} \tag{4}$$

is the *wandering exponent* [3] of the geometric fluctuations. It should be remarked that, in the above (Eq. (3)) ferromagnetic $(J_A, J_B > 0)$ Ising case $(q = 2)$, whose ω vanishes, the critical behavior remains unchanged [6] with respect to the uniform system.

Introducing the transmissivity variable [9],

$$t = \frac{1 - \exp\left(-q\beta J\right)}{1 + (q-1)\exp\left(-q\beta J\right)}, \tag{5}$$

and using the break-collapse techniques [10], it is straightforward to write the exact recursion relations

$$t'_A = \frac{2t_A t_B + (q-2)\, t_A^2 t_B^2}{1 + (q-1)\, t_A^2 t_B^2}, \tag{6}$$

and

$$t'_B = \frac{2t_A^2 + (q-2)\, t_A^4}{1 + (q-1)\, t_A^4}, \tag{7}$$

where t_A and t_B are associated with $J_A > 0$ and $J_B > 0$, respectively. Now it is easy to show that the only physical fixed points are along the diagonal $t_A = t_B$ of the parameter space. There are two trivial stable fixed points ($t_A = t_B = 0$, and $t_A = t_B = 1$), and the non-trivial uniform fixed point, $0 < t_A = t_B = t_u^*(q) < 1$, where $t_u^*(q)$ comes from the equation $(q-1)\,(t_u^*)^3 + (t_u^*)^2 + t_u^* - 1 = 0$ (the function $t_u^*(q)$ decreases monotonically from 1 to 0 as q varies from 0 to ∞). The linearization in the neighborhood of this uniform fixed point yields the matrix relation

$$\begin{pmatrix} \Delta t'_A \\ \Delta t'_B \end{pmatrix} = C(q)\mathbf{M}^T \begin{pmatrix} \Delta t_A \\ \Delta t_B \end{pmatrix}, \tag{8}$$

where the prefactor $C(q)$ depends on q but does not depend on the particular two-letter sequence, and $\mathbf{M}^T$ is the transpose of the substitution matrix. The eigenvalues of this transformation are $\Lambda_1(q) = C(q)\lambda_1 = 2C(q)$ and $\Lambda_2(q) = C(q)\lambda_2 = -C(q)$. The expression for the largest eigenvalue, $\Lambda_1(q)$, also corresponds to the thermal eigenvalue of the linearization about the non-trivial fixed point of the corresponding uniform model (that is, with $J_A = J_B > 0$). Therefore, it is straightforward to write an expression for $C(q)$. For $\Lambda_1 = 2C(q) > 1$ (as in the uniform model), and $|\Lambda_2(q)| = C(q) < 1$, the fixed point in the (t_A, t_B) parameter space is of a hyperbolic character as illustrated in Fig. 2a (which indicates the existence of a critical line in the phase diagram in terms of the temperature and the ratio $r = J_B/J_A$). In this case, the critical behavior is characterized by the same critical exponents of the uniform model. For $C(q) > 1$, however, the uniform fixed point is totally unstable (as illustrated in Fig. 2b), which indicates a change in the character of the transition. From the condition $C(q) = 1$, we obtain the critical value $q = q_d = 4 + 2\sqrt{2}$ (where the subscript d stands for deterministic). For $q > q_d$, that corresponds to $C(q) > 1$, the uniform fixed point is fully unstable. The geometric fluctuations are irrelevant for $q < q_d$, as in the case of the Ising model ($q = 2$), but become relevant for $q > q_d$. It should be remarked that, as shown by Derrida and Gardner [11], the same value $q_r = 4 + 2\sqrt{2}$ (where r stands for random) corresponds to the crossover between uniform and disordered fixed points in the case of the disordered ferromagnetic Potts model on the simple diamond hierarchical lattice we are discussing (see Eq. (3)).

Now we consider a Potts model on a *general* DHL, with m branches in parallel, each one of them with b bonds in series (and hence a chemical length b), and with ferromagnetic interactions according to the two-letter substitution $(A, B) \to (A^{n_1} B^{b-n_1}, A^{n_2} B^{b-n_2})$, with

$0 \leq n_1 < b$, $0 < n_2 \leq b$, and where the order of the letters A and B does not matter. This family of hierarchical structures includes the lattices that represent the Migdal-Kadanoff renormalization-group approximations for this model on a d-dimensional hypercubic Bravais lattice (d coincides with their fractal dimension). The substitution matrix is given by

$$\mathbf{M} = \begin{pmatrix} n_1 & n_2 \\ b - n_1 & b - n_2 \end{pmatrix}, \tag{9}$$

with eigenvalues $\lambda_1 = b$ and $\lambda_2 = n_1 - n_2$. Hence, from Eq. (4):

$$\omega = \frac{\ln |n_1 - n_2|}{\ln b} \; . \tag{10}$$

Using techniques of graph theory, as in the work of Essam and Tsallis [12], it is not difficult to write the recursion relations

$$t'_A = \frac{N\left(t_A, t_B; n_1\right)}{D\left(t_A, t_B; n_1\right)}, \quad \text{and} \quad t'_B = \frac{N\left(t_A, t_B; n_2\right)}{D\left(t_A, t_B; n_2\right)}, \tag{11}$$

where

$$N\left(t_A, t_B; n\right) = \sum_{l=1}^{m} \frac{F\left(q, G_{l+1}\right)}{(q-1)} \, t_A^{nl} t_B^{(b-n)l} C_l^m, \tag{12}$$

and

$$D\left(t_A, t_B; n\right) = 1 + \sum_{l=2}^{m} F\left(q, G_l\right) t_A^{nl} t_B^{(b-n)l} C_l^m \; , \tag{13}$$

where G_l is the graph formed by l parallel edges, C_l^m is a combinatorial number and $F(q, G_l)$ is the flow polynomial [12] associated with G_l. For example, $F(q, G_2) = (q-1)$, $F(q, G_3) = (q-1)(q-2)$, $F(q, G_4) = (q-1)(q^2 - 3q + 3)$, and we can use the deletion-contraction rule to write the recursion relation

$$F(q, G_l) = (q-1)^{l-1} - F(q, G_{l-1}). \tag{14}$$

From these equations, we can easily derive Eqs. (6) and (7) for the simple diamond hierarchical lattice. For a general DHL, the fixed points in the two-parameter space include those of the uniform case (for which $t_A = t_B$). Again, besides the trivial fixed points, there is a non-trivial uniform fixed point, $0 < t_A = t_B = t_u^*(q) < 1$. As in the previous example, the linearization of the recursion relations in the neighborhood of this uniform fixed point, $t_u^*(q)$, still leads to the same form of matrix relation given by eq. (8), with $C(q) = \Lambda_1(q)/b$, where $\Lambda_1(q)$ is the thermal eigenvalue of the uniform model ($J_A = J_B > 0$). In fact, the prefactor $C(q)$ can be calculated from the renormalized transmissivity $t'(t_1, t_2, \cdots t_{mb})$ of the DHL under consideration,

$$C(q) = m \left. \frac{\partial t'}{\partial t_i} \right|_{t_u^*(q)}, \tag{15}$$

where the ith bond ($i = 1, 2, \cdots, mb$) has a transmissivity t_i, and where $t_1 = t_2 = \cdots = t_{mb} = t_u^*(q)$. Due to the invariance of $t'(t_1, \cdots, t_{mb})$ under any permutation of the t_i's, all the mb derivatives $\partial t'/\partial t_i|_{t_u^*}$ are equal among themselves. Derrida et al. [2] have shown that, if this symmetry condition holds for the quenched disordered Potts model on a hierarchical lattice, then we can use the Harris criterion, that is, disorder is relevant (irrelevant) when the critical exponent α_u of the uniform case is positive (negative). In the absence of this symmetry condition, the disorder is relevant for α_u above a negative critical value. In the symmetric case, disorder starts to become relevant at a critical number q_r of states, corresponding to the vanishing of α_u, such that

$$\left.\frac{\partial t'}{\partial t_i}\right|_{t_u^*(q_r)} = \frac{1}{\sqrt{bm}}. \tag{16}$$

For the aperiodic Potts model of this paper, the eigenvalues of the linearization of the recursion relations in the neighborhood of $t_u^*(q)$ are $\Lambda_1(q) = \lambda_1 C(q) = bC(q)$ and $\Lambda_2(q) = \lambda_2 C(q) = (n_1 - n_2)C(q)$. Therefore, as $\Lambda_1 > 1$, the uniform fixed point becomes fully unstable for

$$|\Lambda_2(q)| = |n_1 - n_2| C(q) > 1. \tag{17}$$

From eq. (15), the number of states q_d associated with the onset of relevance of the geometrical fluctuations is given by

$$\left.\frac{\partial t'}{\partial t_i}\right|_{t_u^*(q_d)} = \frac{1}{m|n_1 - n_2|}. \tag{18}$$

Comparing eqs. (16) and (18), we see that q_r coincides with q_d if

$$b = m|n_1 - n_2|^2 \ . \tag{19}$$

Now we investigate the implications of the condition (Eq. (17)) under which the non-trivial uniform fixed point becomes fully unstable. Let us consider the recursion relation associated with the uniform model ($J_A = J_B > 0$). From the linearization about the non-trivial fixed point, we have

$$\Lambda_1 = bC(q) = b^{y_t}, \tag{20}$$

with the thermal exponent [8, 11, 13]

$$y_t = \frac{D}{2 - \alpha_u}, \tag{21}$$

where $D = \ln(bm) / \ln b$ is the fractal dimension of the DHL. Therefore,

$$C(q) = b^{\frac{D}{2-\alpha_u} - 1}. \tag{22}$$

From eq.(10) we also have

$$|n_1 - n_2| = b^{\omega}. \tag{23}$$

Inserting the expressions for $C(q)$ and $|n_1 - n_2|$ into eq. (17), we show that the geometric fluctuations become relevant for

$$\omega > 1 - \frac{D}{2 - \alpha_u} \tag{24}$$

and irrelevant for $\omega < 1 - D/(2-\alpha_u)$. Now, it should be remarked that condition (24) reduces to the inequality $\alpha_u > 0$ only if $\omega = 1 - D/2$, which occurs for the substitution sequences that fulfil the equality (19).

As an example, let us consider again the q-state Potts model on the simple diamond lattice ($b = 2$, $m = 2$) with aperiodic interactions according to the two-letter substitution $(A, B) \to (AB, AA)$ (that is, with $n_1 = 1$ and $n_2 = 0$). As $\omega = 0$ and $D = 2$, the geometric fluctuations become relevant for $\alpha_u > 0$, which is identical to the criterion of Derrida and Gardner [11] for the relevance of disorder in the ferromagnetic Potts model on the simple diamond lattice. Also, $\alpha_u > 0$ is associated with $q > q_d = q_r = 4 + 2\sqrt{2}$.

To give another example, consider the q-state Potts model on a DHL with $b = 3$ bonds per branch and $m = 3$ branches (fractal dimension $D = 2$), and with ferromagnetic aperiodic interactions according to the two-letter substitution $(A, B) \to (ABB, AAA)$ (that is, $n_1 = 1$ and $n_2 = 3$, and hence $b \neq m|n_1 - n_2|^2$). As $\omega = \ln 2/\ln 3$, the geometric fluctuations become relevant for $\alpha_u > -2\,(\ln 2)\,/\ln(3/2)$, that corresponds to $q > q_d = 0.226414....$ Therefore, the critical behavior of the Ising version of this model ($q = 2$) is drastically affected by the geometric fluctuations. However, quenched disorder is still irrelevant up to much bigger values of q (in this example, the crossover to a disordered fixed point only occurs for $q > q_r = 7.722361...$).

In conclusion, deterministic geometric fluctuations and random disorder are both capable of introducing drastic changes in the critical behavior of a statistical model. We have established an *exact* criterium to check the relevance of geometric fluctuations in the critical behavior of ferromagnetic Potts models on diamond-type hierarchical lattices. Geometrical and random fluctuations, however, are distinct phenomena. For example, in the case of the q-state Potts ferromagnet, the threshold for the onset of changes in the critical behavior may occur at different values, $q_d \neq q_r$, in the deterministic and the random cases. More precisely, all situations are possible ($q_d > q_r$, $q_d = q_r$ or $q_d < q_r$).

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Figure Captions

Figure 1- Some stages of the construction of a DHL with chemical length $b = 2$ and $m = 2$ branches (the simple diamond lattice) for the period-doubling sequence $(AB) \rightarrow (AB, AA)$ (letters A and B indicate the exchange interactions, $J_A > 0$ and $J_B > 0$).

Figure 2- Schematic representations of the flow diagrams for the ferromagnetic Potts model in the (t_A, t_B) parameter space: (a) for $q < q_d$, and (b) for $q > q_d$. The arrows indicate the sense of the flow for consecutive (alternating) iterations when the smallest eigenvalue $\Lambda_2(q)$ of the map is positive (negative) i.e., when $n_1 > n_2$ ($n_1 < n_2$). Squares, full dots, and empty dots, represent fully stable, semi-stable and unstable fixed points, respectively. The diagonal $t_A = t_B$ is an invariant subspace under the renormalization-group transformation.

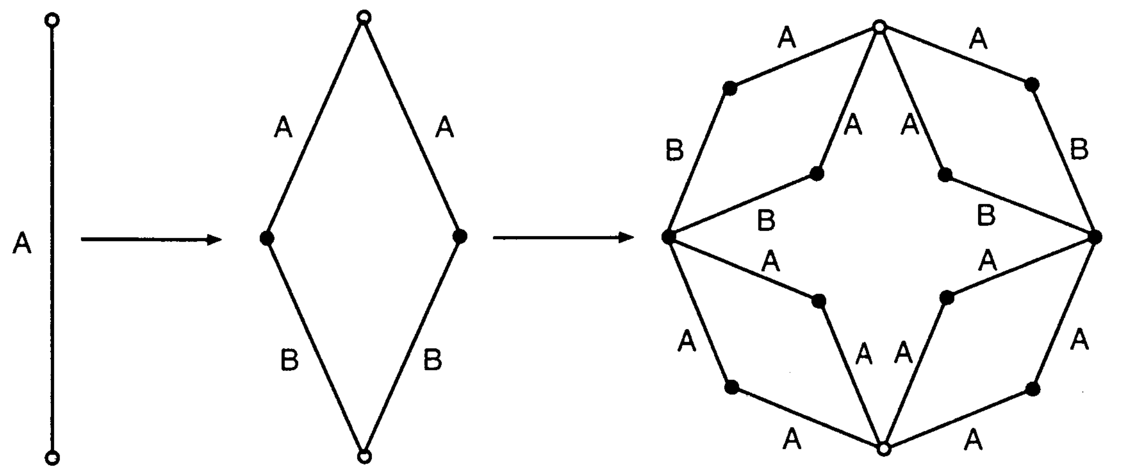

Figure 1

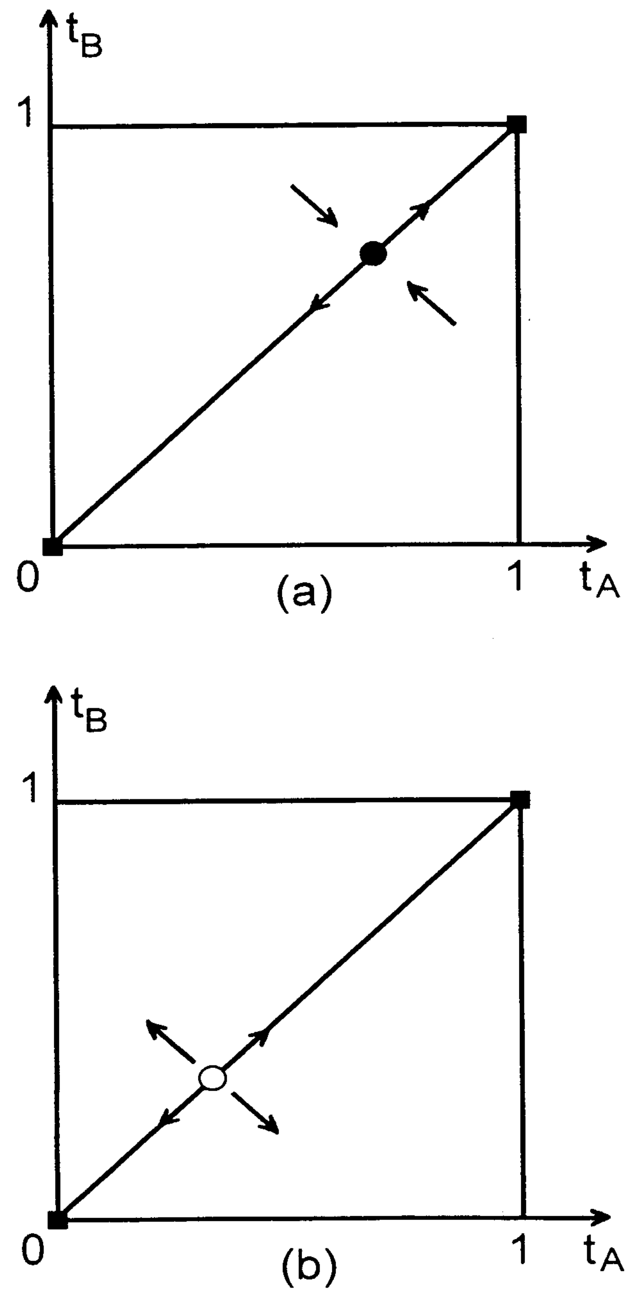

Figure 2

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