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by

SAMUEL WALLACE MAC-DOWELL

CENTRO BRASILEIRO DE PESQUISAS FÍSICAS

Av. Wenceslau Braz, 71

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POLARIZATION OF SPIN ONE PARTICLES*

Samuel Wallace Mac-Dowell
Centro Brasileiro de Pesquisas Físicas
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The description of polarized states of particles with spin, involves two different problems. The first one concerns the characterization of the polarization state of the system and the discussion of its formal properties; this can be done by describ - ing the statistical properties of the system, by a density matrix, according to the von Neumann formalism. The second problem is that of obtaining the density matrix for some specific physical situation.

Many authors, namely W. Wolfenstein, (1952) have considered the problem of describing the polarization of spin one particles, such as deuterons, specially when produced by simple reactions in which both the initial and final states contain only two particles. The states of polarization have then a common property,

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that is, the spin vector is directed along a principal axis of the second rank tensor associated with the density matrix.

In this work, we discuss the general situation of polarized spin one system, determining the parameters which specify the pure mutually orthogonal states contained in the statistical mixture as well as the relative amounts of each state.

It is first shown how to express the percentages of polarization by means of two parameters. The incoherent states of polarization are next specified by six parameters. As the spin vectors of these states are in the same plane, we choose a reference system such that the x-axis is normal to this plane. Then for those states there is a constant relation $\frac{\lambda}{2}$, between the xz-component of the second rank tensor and the y-component of the spin, associated with them, whatever may be the choice of the axes y, z. The spin directions (five parameters), plus the number $\frac{\lambda}{2}$, proves to be a complete set to characterize the incoherent states of polarization. This doesn't hold if the average spin is directed along a principal axis of the second rank tensor; although many authors have already considered such cases, we also treat them by the method developed in this paper for the sake of completeness.

I. GENERAL FORMULATION

The state of a system of identical particles, may be described statistically, by a density matrix ρ, with unit trace, whose eigenvalues are the percentages of the corresponding eigenfunctions which are the pure orthogonal states contained in the statistical mixture. The average value of an observable represented by a matrix O is then given by the trace of the product ρO:

$$\langle O \rangle = \mathrm{Tr.}\,(\rho\, O)$$

If the system is in a well defined state in momentum space, the polarization state is described by a density matrix in spin space, which may be expressed as a linear combination of independent operators whose number equals the square of the dimensionality of that space.

Following Wolfenstein, we start by choosing the operators T_{JM} which transform under rotations like the spherical harmonics Y_{JM}. For spin one particles we may define them in the following way:

$$I = T_{00}$$

$$\vec{S}\cdot\vec{\mathcal{E}} = S_i\,\mathcal{E}^i = \frac{\sqrt{8\pi}}{3}\sum_M T_{1M}\,Y_{1M}^* \qquad (1)$$

$$(\vec{S}\cdot\vec{\mathcal{E}})^2 - \frac{2}{3}I = Q_{ij}\,\mathcal{E}^i\,\mathcal{E}^j = \frac{1}{3}\sqrt{\frac{8\pi}{5}}\sum_M T_{2M}\,Y_{2M}^*$$

In these relations $\vec{\mathcal{E}}$ is any unit vector with components $\mathcal{E}^i$, S_i and Q_{ij} are respectively the cartesian components of the spin vector and symmetric second rank tensor, which further on will be used instead of T_{JM}. The components of Q_{ij} are not all of them independent. Indeed, if $\vec{\mathcal{E}}_1$, $\vec{\mathcal{E}}_2$, $\vec{\mathcal{E}}_3$ are orthogonal unit vectors we have:

$$\sum_n \left[(\vec{S}\cdot\vec{\mathcal{E}}_n)^2 - \frac{2}{3}I \right] = 0$$

and consequently:

$$\sum_n Q_{ij} \; \varepsilon_n^i \; \varepsilon_n^j = Q_{ij} \; \delta^{ij} = 0 \tag{2}$$

The T_{JM} form a complete set of matrices satisfying the orthogonality and normalization conditions:

$$\frac{1}{3} \, T_r \left(T_{JM} \, T^+_{J'M'} \right) = \delta_{JJ'} \, \delta_{MM'}$$

Then the density matrix may be written as:

$$\rho = \frac{1}{3} \sum_{J,M} \langle T_{JM} \rangle \, T^+_{JM} \tag{3}$$

In order to express ρ in terms of $\langle s_i \rangle$ and $\langle Q_{ij} \rangle$ we proceed in the following way: Taking mean values in equations (1), multiplying them by the primitive ones and integrating over the angular variables, we get:

$$\frac{1}{2} \langle s_i \rangle \; s_i = \frac{2}{3} \sum_M \langle T_{1M} \rangle \; T^+_{1M}$$

and

$$\langle Q_{ij} \rangle \; Q_{ij} = \frac{1}{3} \sum_M \langle T_{2M} \rangle \; T^+_{2M}$$

then we may write:

$$\rho = \frac{1}{3} + \frac{1}{2} \langle s_i \rangle \; s_i + \langle Q_{ij} \rangle \; Q_{ij} \tag{4}$$

II. PERCENTAGES OF THE PURE ORTHOGONAL STATES

1) General Case

The percentages of polarization i.e. the eigenvalues of ρ are the roots of the secular equation:

$$\left(\rho' - \frac{1}{3}\right)^3 - 3\left(\frac{P}{3}\right)^2 \left(\rho' - \frac{1}{3}\right) - 2\left(\frac{P}{3}\right)^3 \cos\omega = 0 \qquad (5)$$

whose coefficents are the following invariants:

$$3\left(\frac{P}{3}\right)^2 = \frac{1}{2}\,\mathrm{Tr.}\left(\rho - \frac{I}{3}\right)^2,\quad 2\left(\frac{P}{3}\right)^3 \cos\omega = \frac{1}{3}\,\mathrm{Tr.}\left(\rho - \frac{I}{3}\right)^3 \qquad (6)$$

The solutions of equation (5) are:

$$\rho'_n = \frac{1}{3}\left(1 + 2\,P \cos\frac{\omega + 2\,n\,\pi}{3}\right), \quad (n = 1,\ 2,\ 3) \qquad (7)$$

We see that P can be taken as a positive number and cannot be larger than one, because all the roots ρ'_n are positive: $P \leq 1$. Although in general $\cos\omega$ take any value in the interval $(-1, +1)$ there are restrictions upon its possible values when $P > \frac{1}{2}$. Indeed, from the positiveness condition for the eigenvalues ρ'_n the first member of equation (5) take a negative value for $\rho' = 0$; thus we have:

$$3P^2 - 2\,P^3 \cos\omega \leq 1 \qquad (8)$$

This condition implies in restriction for the value of $\cos\omega$ only if $P > \frac{1}{2}$ when the lower limit of its interval of variation is greater than -1:

$$\cos\omega \geq -\cos\left(3\cos^{-1}\frac{1}{2\,P}\right) \qquad (9)$$

The invariants which appear in the expression (7) for the eigenvalues ρ_n', and are given by (6), may be related to the observables $\langle s_i \rangle$ and $\langle Q_{ij} \rangle$. To do that we choose the following representation:

$$\langle \psi_i \mid s_k \mid \psi_j \rangle = -i\, \epsilon_{ijk}, \langle \psi_i \mid Q_{kl} \mid \psi_j \rangle = \tag{10}$$

$$= \frac{1}{3}\, \delta_{ij}\, \delta_{kl} - \frac{1}{2}\left(\delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk} \right)$$

This corresponds to represent the state vectors by their cartesian components. The matrix elements of $\left(\rho - \frac{I}{3} \right)$ found from (4) are then

$$\left\langle \psi_i \left| \rho - \frac{I}{3} \right| \psi_j \right\rangle = -\left(\langle Q_{ij} \rangle + \frac{i}{2}\, \epsilon_{ijk} \langle s_k \rangle \right) \tag{11}$$

Taking (11) into (6) we find by straightforward computation the following results:

$$\left.\begin{aligned} 3\left(\frac{p}{3}\right)^2 &= \frac{1}{2}\left(\frac{1}{2}\langle s_j \rangle \langle s_j \rangle + \langle Q_{ik} \rangle \langle Q_{ik} \rangle\right. \\ 2\left(\frac{p}{2}\right)^3 \cos\omega &= \frac{1}{3}\left(\frac{3}{4}\langle s_i \rangle \langle s_k \rangle - \langle Q_{ij} \rangle \langle Q_{jk} \rangle\right) \langle Q_{ik} \rangle \end{aligned}\right\} \tag{12}$$

2) Degenerate Cases

A density matrix whose eigenvalues are not all distinct represents a degenerate state of the system. When two roots are equal, the system is composed of a certain pure state, the eigenstate corresponding to the single eigenvalue and of a random mixture of all states normal to that one; the percentage of such state is equal to its eigenvalue and the total amount of the remaining normal states is equal to twice the double root. The condition for this to happen is that $\cos\omega = \pm 1$ (if $P > \frac{1}{2}$ only the positive value is to be considered). The eigenvalues of ρ are then:

$$\rho_1' = \rho_2' = \frac{1}{3}(1 \mp P) \ , \ \rho_3' = \frac{1}{3}(1 \pm 2P)$$

Therefore the irreducible equation satisfied by ρ is:

$$\left(\rho - \frac{I}{3}\right)^2 \mp \frac{P}{3}\left(\rho - \frac{I}{3}\right) - 2\left(\frac{P}{3}\right)^2 = 0$$

Substituting in this equations ρ as is given by the expression (4) we obtain, after some developements:

$$\langle Q_{ij}\rangle\langle Q_{jk}\rangle \pm \frac{P}{3}\langle Q_{ik}\rangle + \frac{1}{4}\left(\langle s_j\rangle\langle s_j\rangle\delta_{ik} - \langle s_i\rangle\langle s_k\rangle\right) = 2\left(\frac{P}{3}\right)^2\delta_{ik} \tag{13}$$

$$\langle Q_{ij}\rangle\langle s_j\rangle = \pm\frac{P}{3}\langle s_i\rangle$$

From these equations it results that if the spin vector $\langle\vec{s}\rangle$ is different from zero, the spin direction is a principal axis of the second rank tensor with corresponding principal value $\pm\frac{P}{3}$. On the other hand if the spin is zero, the first equation is an eigen value equation for the tensor $\langle Q_{ij}\rangle$ which is also degenerate, i.e. has cylindrical symmetry about the principal axis corresponding to the principal value $\mp 2\frac{P}{3}$. In the normal directions to that one, the tensor has a constant value $\pm\frac{P}{3}$.

Particularly important cases of degeneracy are the totally polarized states and the completely unpolarized one. In the first case, two roots vanish and P = 1; the principal values of $\langle Q_{ij}\rangle$ are found to be:

$$\frac{1}{3},\ \frac{1}{2}\left(-\frac{1}{3}+\sqrt{1-\langle\vec{s}\rangle^{2}}\right),\ \frac{1}{2}\left(-\frac{1}{3}-\sqrt{1-\langle\vec{s}\rangle^{2}}\right) \quad (14)$$

In the second case P = 0, as the density matrix reduces to $\rho=\frac{I}{3}$.

III. DETERMINATION OF THE PURE ORTHOGONAL STATES

Now we shall find the eigenstates $|\chi_n\rangle$ of ρ

If the expectation values s_n^i and Q_n^{ij} of the operators s_i and Q_{ij} for the eigenfunctions $|\chi_n\rangle$, were known, it was possible to construct these eigenfunctions; indeed, they would be obtained as a result of applying some unitary transformations to the eigenfunction $|\chi_+\rangle$ belonging to the eigenvalue + 1 of s_z ($s_z|\chi_+\rangle = |\chi_+\rangle$). The first one $e^{-i\alpha_n Q_{xy}}$ would reduce the modulus of the average spin from one to $s_n = \cos\alpha_n$ leaving unchanged its direction if $\sin\alpha_n > 0$. The sucessive transformations would be rotations bringing the coordinate axes to coincide with the principal directions of Q_n^{ij} Therefore:

$$|\chi_n\rangle = e^{-i\psi_n s_z}\, e^{-i\theta_n s_x}\, e^{-i\varphi_n s_z}\, e^{-i\alpha_n Q_{xy}}|\chi_+\rangle \quad (15)$$

where ψ_n, θ_n, φ_n are the Euler angles for the principal axes of Q_n^{ij}, so oriented that the z-axis is along the spin direction and* $Q_n^x \geq Q_n^y$.

The determination of s_n^i and Q_n^{ij} is undertaken in

Appendix A; here we present only the results:

$$\left.\begin{aligned} s_n^i &= \frac{4\sin\frac{\omega+2n\pi}{3}}{3\sin\omega}\left\{Q_o^{ij}s_o^j + s_o^i\cos\frac{\omega+2n\pi}{3}\right\} \\ Q_n^{ij} &= -\frac{4\sin\frac{\omega+2n\pi}{3}}{3\sin\omega}\left\{Q_o^{ik}Q_o^{kj} + \frac{1}{4}(s_o^k s_o^k\delta^{ij} - s_o^i s_o^j) - \right. \\ &\qquad \left. - Q_o^{ij}\cos\frac{\omega+2n\pi}{3} - \frac{1}{2}\delta^{ij}\right\} \end{aligned}\right\} \quad (16)$$

where: $s_o^i = \frac{3}{2P}\langle s_i\rangle$ and $Q_o^{ij} = \frac{3}{2P}\langle Q_{ij}\rangle$

and

$$\sum_n s_n^i = 0\ ,\ \sum_n Q_n^{ij} = 0 \qquad (17)$$

IV. GEOMETRIC INTERPRETATION AND SPECIFIC PARAMETERS

Geometric interpretation of these states may be settled in analogy with light polarization. In general, each state $|\chi_n\rangle$ may be represented by an ellipse on a plane normal to the spin direction and oriented according to its orientation. Such ellipse is associated with the projection of the tensor

$\left(Q_n^{ij} + \frac{2}{3}\right)$ on that plane and the semiaxes values computed from (14) are given by:

$$\frac{1}{a_n^2} = \frac{1}{2}\left(1 \mp \sqrt{1 - s_n^2}\right) \tag{18}$$

The system is then composed of three elliptically polarized states on coaxial planes. Limiting cases are the circularly polarized states, when $s_n = 1$ and the linearly polarized states, those of spin zero.

While we have already the necessary elements for the determination of the polarization ellipses, that is, the characteristic elements ψ_n, θ_n, φ_m, α_n of the pure states, they do not form an irreducible set of parameters, because of the orthogonality conditions holding between the eigenfunctions $|\chi_n\rangle$. It is easy to see that we need only a set of six independent parameters just sufficient to fix the elements of the polarization ellipses.

We shall distinguish two cases:

1) General Case: the spin direction is not a principal axis of the second rank tensor.

The vectors s_o^i and $q_o^i = Q_o^{ij} s_o^j$ determine a plane A and all the spin vectors $\vec{S}_n$ are parallel to this plane. However these vectors don't determine uniquely the state of the system, since there are two states with the same parameters (P, $\cos\omega$) and vectors ($\vec{s}_0$, $\vec{q}_0$) but symmetric tensors in recpect to the axis normal to the plane A.

We shall prove that the unit vectors $\frac{\vec{s}_n}{s_n}$ and the number $\frac{\lambda}{2} = \frac{Q_o^{ij} q_o^i u^j}{|\vec{s}_o \wedge \vec{q}_o|}$, with $\vec{u} = \frac{\vec{s}_o \wedge \vec{q}_o}{|\vec{s}_o \wedge \vec{q}_o|}$ form a complete set to describe the component states of polarization.

The geometrical meaning of the parameter $\frac{\lambda}{2}$ can be found out with the help of equations (16); they give:

$$Q_n^{ij} s_o^i u^j = - \frac{4 \sin \frac{\omega + 2 n \pi}{3}}{3 \sin \omega} \; Q_o^{ij} q_o^i u^j =$$

$$= (\vec{s}_n \wedge \vec{s}_o \cdot \vec{u}) \frac{Q_o^{ij} q_o^i u^j}{|\vec{s}_o \wedge \vec{q}_o|}$$

$$Q_n^{ij} s_o^i u^j = \frac{\lambda}{2} (\vec{s}_n \cdot \vec{s}_o \wedge \vec{u})$$

It is easy to understand that this result still holds, for any vector normal to $\vec{u}$ replacing $\vec{s}_o$. Then in a reference system with $\vec{u}$ as x-axis and axes y, z, arbitrarily chosen, there is a constant relation between the xz-component of the second rank tensor and the y-component of the spin, the same one for the three pure states and just equal to $\frac{\lambda}{2}$.

On the other hand, observing Figure 1, we have:

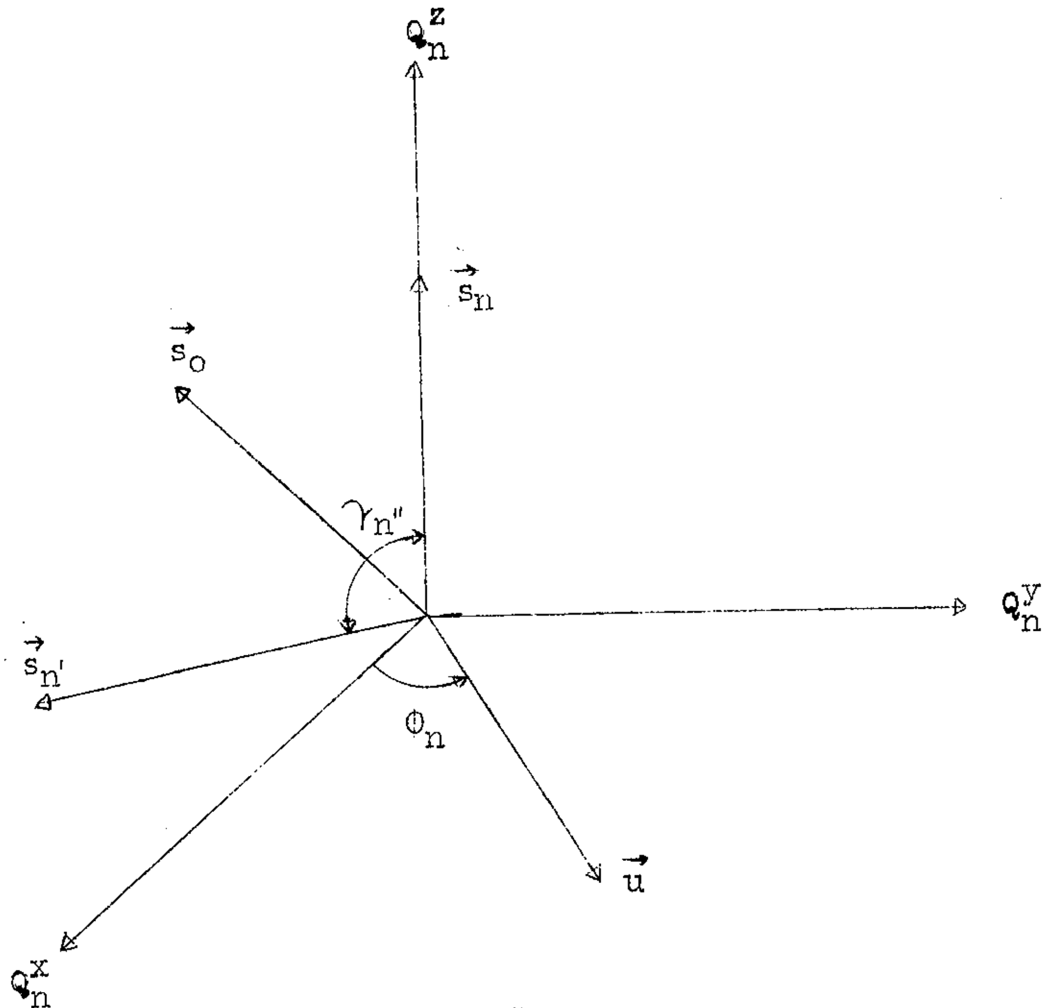

Figure 1

$$Q_n^{ij} s_o^i u^j = -\frac{1}{2}\left(Q_n^x - Q_n^y\right) \frac{\vec{s}_n \wedge \vec{s}_o \cdot \vec{u}}{s_n} \sin 2\phi_n =$$

$$= -\frac{1}{2}\sqrt{1 - s_n^2}\, \frac{\vec{s}_n \cdot \vec{s}_o \wedge \vec{u}}{s_n} \sin 2\phi_n$$

The second step is performed, taking in mind the result (14). Therefore comparing with the preceding computation we get:

$$\sqrt{1 - s_n^2}\, \sin 2\phi_n = -\lambda s_n \tag{19}$$

The positions of the axes corresponding to the principal values Q_n^x are determined without ambiguity by means of (19) and (1,B) of Appendix B,

Since $\sum_n \vec{s}_n = 0$ the spin vectors $\vec{s}_n$ can be disposed as to form a triangle. Let us introduce the auxiliary parameter r, which is the radius of the circumscribed circle about this triangle. Then we have:

$$s_n = 2\, r \sin \gamma_n \tag{20}$$

where γ_n is the angle between $\vec{s}_{n'}$ and $\vec{s}_{n''}$. In Appendix B is deduced an equation (3,B) which may be written in a suitable form to calculate r, as follows:

$$(1 + \lambda^2)\, r^2 + \prod_{n'} \frac{\cos \gamma_{n'}}{\sin^2 \gamma_{n'}} = 0 \tag{21}$$

All the elements of the polarization ellipses, may be calculated by means of formulae (18), (19), (20), (21), once are known the directions of the vectors $\vec{s}_n$ and the parameter $\frac{\lambda}{2}$, which may be considered as specific parameters for the case just studied.

We shall notice that equation (21), implies in some restrictions for the variation intervals of the angles γ_n and the value of the parameter λ Indeed, the following conditions must be fulfilled:

a) All the angles γ_n will be greater than $\frac{\pi}{2}$:

$$\cos \gamma_n < 0$$

b) $$1 + \lambda^2 + 4 \sin^2 \gamma_n \prod_{n'} \frac{\cos \gamma_{n'}}{\sin^2 \gamma_{n'}} \geq 0 \qquad (22)$$

holding for each value of n. But if it holds for the larger angle, it also holds for the others. This condition follows from the fact that we must have necessarily $s_n \leq 1$.

2) Special Cases: the spin is along a principal axis of the tensor, or vanishes.

These states constitute a very important class. The result of simple reactions in which both the initial and final states contain only two particles, as shown by Lakin and Wolfenstein, belong to this class. Indeed, if neither initial particles is polarized and one reaction product is a spin one particle, such as a deuteron, then the plane of scattering is a plane of symmetry**; hence the axial-vectors $\langle \vec{s} \rangle$ and $\langle Q_{ij} \rangle$ $\langle s_j \rangle$ are normal to his plane.***

Also, this class includes all degenerate states of a system.

The fundamental property of these special states, introduced in equation (11), gives:

$$\langle Q_{ij} \rangle \langle s_j \rangle = -\left(\rho_n' - \frac{1}{3} \right) \langle s_i \rangle \qquad (23)$$

that is the principal value Q of the second rank tensor in the spin direction is:

$$Q = - \frac{2P}{3} \cos \frac{\omega + 2 n \pi}{3} \qquad (24)$$

then:

$$\cos \omega = - \cos \left(3 \cos^{-1} \frac{3Q}{2P} \right)$$

which shows that in these cases the percentages of polarization are directly related to P and Q. The restrictive conditions for Q are:

$$|Q| \leq \frac{2P}{3}, \text{ and if } P > \frac{1}{2} : \left| Q + \frac{1}{6} \right| \geq \frac{1}{\sqrt{3}} \sqrt{P^2 - \frac{1}{4}} \qquad (25)$$

Equation (23) also can be written:

$$Q_o^{ij} s_o^j = - s_o^i \cos \frac{\omega + 2 n \pi}{3}$$

thus taking into account (16) we have:

$$\vec{s}_n = 0 , \vec{s}_{n'} + \vec{s}_{n''} = 0$$

that is, one state has spin zero and the others have symmetrical spin vectors. Furthermore the principal directions of each tensor Q_n^{ij} coincide with those of Q_o^{ij}. It is easy to see that in such cases, knowing either of the non zero spin states, we are able to find the other ones. The required parameters to characterize the three pure states are then the four specific parameters $\psi, \theta, \varphi, \alpha$ for any of the non zero spin states, already mentioned.

The zero spin state is a linearly polarized one, while the others are in general elliptically polarized; the two ellipses are equal but oriented in opposite senses, and their axes are rotated through a right angle to each other.

Similar situation occurs with light beams since only states are found which have the spin directed along the propagation vector. The amount of linear polarization in the direction of propagation is just zero.

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APPENDIX A

Determination of the expectation values of s_i and Q_{ij} for the eigenstates $|\chi_n\rangle$

The density matrix may be written:

$$= \sum_n \rho'_n \rho_n \tag{1,A}$$

where

$$\rho_n = |\chi_n\rangle\langle\chi_n|$$

These matrices have the following property:

$$\rho_n \rho_{n'} = \delta_{nn'} \rho_n \tag{2,A}$$

Summing over the index n, we get the completeness condition:

$$\sum \rho_n = I \tag{3,A}$$

Squaring equation (1,A) and using (2,A) we find

$$\rho^2 = \sum \rho'^2 \rho_n \tag{4,A}$$

From equations (1,A), (2,A), (4,A), we may find ρ_n in terms of powers of . The solutions of this system of equations are:

$$\rho_n = \frac{(\rho - \rho'_{n'})(\rho - \rho'_{n''})}{(\rho'_n - \rho'_{n'})(\rho'_n - \rho'_{n''})}$$

Now we use (7) to get

$$\rho_n = \frac{I}{3} + \frac{4 \sin \frac{\omega + 2 n \pi}{3}}{3 \sin \omega} \left\{ \left[\frac{3}{2 P}\left(\rho - \frac{I}{3}\right) \right]^2 + \frac{3}{2 P}\left(\rho - \frac{I}{3}\right) \cos \frac{\omega + 2 n \pi}{3} - \frac{I}{2} \right\} \tag{5,A}$$

Using representation (10) this equation can be put into the form | cf. (11) |:

$$\langle \psi_i | \rho_n | \psi_j \rangle = - \left(Q^{ij}_n + \frac{i}{2} \epsilon_{ijk} s^k_n \right)$$

where s_n^i and Q_n^{ij}, the expectation values of s_i and Q_{ij} for the eigenstates $|\chi_n\rangle$ are given by:

$$\left.\begin{aligned} s_n^i &= \frac{4\sin\frac{\omega+2n\pi}{3}}{3\sin\omega}\left\{Q_o^{ij}\,s_o^j + s_o^i\cos\frac{\omega+2n\pi}{3}\right\} \\ Q_n^{ij} &= -\frac{4\sin\frac{\omega+2n\pi}{3}}{3\sin\omega}\left\{Q_o^{ik}\,Q_o^{kj} + \frac{1}{4}\left(s_o^k\,s_o^k\,\delta^{ij} - s_o^i\,s_o^j\right) - \right. \\ &\qquad \left. - Q_o^{ij}\cos\frac{\omega+2n\pi}{3} - \frac{1}{2}\,\delta^{ij}\right\} \end{aligned}\right\} \quad (6,A)$$

with $s_o^i = \frac{3}{2P}\langle s_i\rangle$ and $Q_o^{ij} = \frac{3}{2P}\langle Q_{ij}\rangle$

It is easy to see that:

$$\sum_n s_n^i = 0 \quad , \quad \sum Q_n^{ij} = 0 \qquad (7,A).$$

which are consequence of the completeness condition added to the fact that each of these operators has trace zero.

APPENDIX B

Complementary relations to the determination of the polarization ellipses by specific parameters.

Proceeding in a similar fashion as was done to get (19) we can find, from (16) and (14):

$$Q_n^{ij}\frac{s^i_{n'}s^j_{n'}}{s^2_{n'}} = \frac{1}{2}\left(\sqrt{1-s_n^2}\cos 2\phi_n - \frac{1}{3}\right)\left|\frac{\vec{s}_n \wedge \vec{s}_{n'}}{s_n\, s_{n'}}\right|^2 +$$

$$+ \quad \frac{1}{3}\left(\frac{\vec{s}_n\cdot\vec{s}_{n'}}{s_n\, s_{n'}}\right)$$

$$= \frac{1}{2}\left(\sqrt{1-s_n^2}\cos 2\phi_n - 1\right)\left|\frac{\vec{s}_n \wedge \vec{s}_{n'}}{s_n\, s_{n'}}\right| + \frac{1}{3}$$

Fixing n' and summing for all values of the index n, we get:

$$\sum_n\left(1 - \sqrt{1-s_n^2}\cos 2\phi_n\right)\left|\frac{\vec{s}_n \wedge \vec{s}_{n'}}{s_n\, s_{n'}}\right|^2 = 2$$

a system of equations which enables us to determine $\sqrt{1-s_n^2}\cos 2\phi_n$. Solving this system we find:

$$\sqrt{1-s_n^2}\cos 2\phi_n = 1 - 4\frac{\sin 2\gamma_n}{\sum_{n'}\sin 2\gamma_{n'}} \qquad (1,B)$$

where γ_n is the angle between $s_{n'}$ and $s_{n''}$. Eliminating $2\phi_n$, between (19) and (1,B) we obtain the main result:

$$\left(1+\lambda^2\right)s_n^2 + 4\sin^2\gamma_n \prod_{n'}\frac{\cos\gamma_{n'}}{\sin^2\gamma_{n'}} = 0. \qquad (2,B)$$

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* Q_n^x, Q_n^y, Q_n^z, represents the principal values of Q_n^{ij}.

** This argument doesn't apply to photons, since then the interaction with matter is linear in the field components.

*** The pseudo-scalar character of the spin is a consequence of the comutation rules:

$$s_i s_j - s_j s_i = i \ \varepsilon_{ijk} s_k$$

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