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NON RELATIVISTIC EQUATION FOR CHARGED PARTICLES WITH SPIN 3/2 *

J. Tiomno
Centro Brasileiro de Pesquisas Físicas
Rio de Janeiro, D.F.

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Equations for particles with spin higher than I have been proposed by Dirac [1] and Fierz [2]. Rarita and Schwinger[3] considered another type of such equatrions for particles with half integer spin; those equations have been studied by Kusaka [4] and Weinberg [5]. Although the later authors have pointed out some difficulties concerning the supplementary conditions in second quantisation, Caianello[6] has recently reconsidered the R-S equation for spin 3/2 and used it for describing the μ-meson. He has shown that the assumption of spin 3/2 for the μ-meson is consistent with all known experimental results about properties of the μ-meson. This finding makes it important to re-examine carefully the R-S equation for spin 3/2 in order to make further theoretical predictions to be compared with those obtained under the assumption of spin one half [7] and of spin zero for the μ-meson. The comparison with the corresponding experimental results could lead to the determination of the μ-meson spin. One such case might be that of the radiative π→μ decay[9]. This process is being computed, on the assumption of spin 3/2 for the μ-meson.

In the present paper we study the non relativistic approxima-

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tion of the R-S equation for a charged spin 3/2 particle with mass $\underline{m}$ in an electromagnetic field. To the approximation when the derivatives of the field can be neglected it is found that the Hamiltonian is of the usual form for a charged particle but with a magnetic dipole interaction term$(e/3m)$ $(\vec{H} \cdot \vec{M})$, where $\vec{H}$ is the magnetic field and $\vec{M}$ the spin operator for 3/2 spin particle (we use natural units: $n = c = 1$). So we see that the gyromagnetic factor is 2/3 or that the intrinsic magnetic moment of such particles (its higher eigenvalue), is $e/2m$. As this magnetic moment is the same as that for particles with spin 1/2 or 1, it may be expected that the intrinsic magnetic moment of a particle with any spin is given by $e/2m$. This surmise is being examined.

<u>Rarita-Schwinger equation for spin 3/2 particles.</u>

A particle with spin 3/2 is described relativistically by a wave function $\Psi_\alpha^{\mu}(\vec{x}, t)$ where μ is a vector index running from 0 to 3 ($x^0 = t$) and α a spinor index ($\alpha = 1, 2, 3, 4$). The wave function Ψ of such a charged particle satisfies the equation:

$$(\gamma_\sigma \partial_\sigma - m)\Psi^\lambda - \frac{1}{3}(\gamma^\lambda\partial_\sigma + \gamma_\sigma\partial_\lambda)\Psi^\sigma + \frac{1}{3}\gamma^\lambda(\gamma^\sigma\partial_\sigma + m)\gamma_\mu\Psi^\mu = 0 \tag{1}$$

where $\partial_\sigma = \frac{\partial}{\partial x^\sigma} + ieA_\sigma$, A_σ being the electromagnetic potential. Repeated covariant and contravariant indices mean summation from 0 to 3. Passage from covariant to contravariant tensors is made with the help of the metric tensor:

$$g^{\mu\nu} = 0, \ \mu \neq \nu ; \quad g^{11} = g^{22} = g^{33} = - g^{00} = 1 . \tag{2}$$

γ_μ are the Dirac matrices:

$$\gamma^0 = i\beta \ ; \ \vec{\gamma} = i\beta\vec{\alpha} \ ; \ \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\, g^{\mu\nu} \tag{3}$$

Spinor indices are not written, as usual.

The following supplementary conditions are immediate consequences of equation (1):

$$m\,\gamma_\mu\Psi^\mu = -2\ \partial_\mu \tag{4}$$

$$\left[m^2 + i(e/3)\gamma_\mu\gamma_\sigma F^{\sigma\mu}\right]\gamma_\nu\Psi^\nu = -2\,i\,e\,F_{\mu\sigma}\,\gamma^\sigma\,\Psi^\mu \tag{5}$$

The charge-current density is given by

$$j^\mu = e(\bar{\Psi}^\sigma\gamma^\mu\Psi_\sigma - \frac{1}{6}\bar{\Psi}^\sigma\gamma_\sigma\gamma_\rho\gamma^\mu\Psi^\rho - \frac{1}{6}\bar{\Psi}^\rho\gamma^\mu\gamma_\rho\gamma_\sigma\Psi^\sigma) \,. \tag{6}$$

where $\bar{\Psi} = \Psi^*\gamma_0$. It satisfies the conservation equation

$$\frac{\partial\, j^\mu}{\partial\, x^\mu} = 0 \,.$$

Using condition (4) we can write equation (1) as

$$(\gamma^\sigma\,\partial_\sigma - m)\Psi^\mu + \left[\frac{m}{2}\gamma^\mu + \frac{1}{6}(\gamma^\mu\gamma^\sigma - \gamma^\sigma\gamma^\mu)\,\partial_\sigma\right]\gamma^\nu\,\Psi_\nu = 0 \,. \tag{7}$$

Now we introduce a few operators acting on the vector indices of the wave function which are helpful for writing these equations in complete matrix notation, without using explicite vector indices for the wave function. They are the operators $t^{\rho\sigma}$ whose $\mu\,,\nu$ matrix elements are

$$(t^{\rho\sigma})_{\mu\nu} = g^\rho_\mu g^\sigma_\nu \,, \; (\mu, \nu, \rho, \sigma = 0, 1, 2, 3) \tag{8}$$

and the metric matrix:

$$(G)_{\mu,\nu} = g_{\mu\nu} \tag{9}$$

They do satisfy the relations

$$t^{\rho\sigma}\,t^{\mu\nu} = t^{\rho\nu}\,g^{\sigma\mu} \,, \tag{10}$$

$$G\,t^{\rho\sigma\dagger}tG = t^{\sigma\rho} \,, \tag{11}$$

$$t^\mu{}_\mu \equiv t^{\mu\nu}\,g_{\mu\nu} = 1 \,. \tag{12}$$

It is also convenient to introduce the operators:

$$m^{\mu\nu} = \frac{1}{i}(t^{\mu\nu} - t^{\nu\mu}) = -m^{\nu\mu} \tag{13a}$$

$$\gamma^{\mu\nu} = \frac{1}{2i}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = -\gamma^{\nu\mu} \,. \tag{13b}$$

We give here the matrices for $m^{\mu\nu}$ in the representation where the wave function Ψ^{μ} is written as

$$\Psi = \begin{pmatrix} \psi^0 \\ \psi^1 \\ \psi^2 \\ \psi^3 \end{pmatrix} \tag{14a}$$

They are:

$$m^{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ; \quad m^{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} ; \quad m^{31} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

$$m^{01} = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ; \quad m^{02} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ; \quad m^{03} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \tag{14b}$$

These matrices commute with the Dirac matrices because they act on different indices of the wave function.

In this notation we can write equation (7), (4), (5) and (6), respectively, as:

$$(\gamma^{\sigma}\,\partial_{\sigma} - m)\,\Psi + \left(\frac{i}{3}\,\gamma_{\rho\sigma}\,\partial^{\sigma} + \frac{m}{2}\,\gamma_{\rho}\right)t^{\rho\lambda}\,\gamma_{\lambda}\Psi = 0\,, \tag{7a}$$

$$m t^{\rho\sigma}\,\gamma_{\sigma}\Psi = -\,2\,t^{\rho\sigma}\,\partial_{\sigma}\Psi\,, \tag{4a}$$

$$\left(m^2 + \frac{e}{3}\,\gamma_{\mu\nu}\,F^{\mu\nu}\right)t^{\rho\sigma}\gamma_{\sigma}\Psi = -\,2\,i\,e\,F_{\sigma\lambda}\,\gamma^{\lambda}\,t^{\rho\sigma}\Psi\,, \tag{5a}$$

$$j^{\mu} = e\left(\bar{\Psi}G\gamma^{\mu}\Psi - \frac{1}{6}\,\bar{\Psi}G\gamma^{\mu}t^{\rho\sigma}\gamma_{\rho}\gamma_{\sigma}\Psi - \frac{1}{6}\,\bar{\Psi}G\,t^{\rho\sigma}\,\gamma_{\rho}\gamma_{\sigma}\gamma^{\mu}\Psi\right). \tag{6a}$$

Infinitesimal operators of the Lorentz group.

In the coordinates x^{μ} are submitted to an infinitesimal Lorentz transformation,

$$x^{\mu} \longrightarrow x'^{\mu} = x^{\mu} + d\Theta^{\mu}{}_{\nu}\,x^{\nu} \tag{15}$$

determined by the infinitesimal "angles" $d\Theta^{\mu}{}_{\nu}$, then a transformation of the wave function

$$\psi \longrightarrow \psi' = S\,\psi \tag{16}$$

where

$$S = 1 + \frac{i}{2} M^{\mu\nu}\, d\Theta_{\mu\nu} \tag{17}$$

makes the expressions (4a), (5a), (6a) and (7a) transform in a covariant way if

$$M^{\mu\nu} = m^{\mu\nu} + \frac{1}{2}\gamma^{\mu\nu} \;; \tag{18}$$

this is a consequence of the commutation relations of the Dirac matrices [10] and of the commutation relation

$$\left[m^{\rho\sigma},\ t^{\mu\nu}\right] = \frac{1}{i}\left(g^{\sigma\mu}t^{\rho\nu} + g^{\nu\sigma}t^{\mu\rho} - g^{\nu\rho}t^{\mu\sigma} - g^{\rho\mu}t^{\sigma\nu}\right). \tag{19}$$

This last commutation relation is obtained from (8), (10) and (13). In the demonstration of the covariance of expressions of the type (6a), we have furthermore, to use the properties:

$$\beta\gamma^{\mu\nu\dagger}\beta = \gamma^{\mu\nu}\ ,\ G\, m^{\mu\nu\dagger}\, G = m^{\mu\nu}. \tag{20}$$

So we see that the $M^{\mu\nu}$ given by (18) are the infinitesimal operators of the Lorentz Group in this representation. Thus the 3-vector $\vec{M}$, whose components are

$$M^1 = M^{23},\quad M^2 = M^{31},\quad M^3 = M^{12}, \tag{21}$$

is the operator for the intrinsic angular momentum (spin operator) for the particle with spin 3/2.

Non relativistic approximation.

The non relativistic approximation for the R-S equation should involve only the space components of ψ and only two spinor components. Besides this, a supplementary condition should reduce these six components to four independent ones.

In order to obtain this result we proceed by a method which is essentially that of Foldy and Wouthuysen[11] in order to eliminate the "small" spinor components. The time component of ψ is eliminated by the supplementary condition[12]. In the F-W method the Hamiltonian is expressed as a power series in 1/m. If we wish to obtain the non relativistic approximation up to a given power of 1/m the odd Dirac matrices appearing in terms of lower power are eliminated by successive canonical transformations. Here we shall keep the terms only up to the first power of 1/m. For this reason the supplementary conditions (5) can be written as:

$$\gamma_{\nu}\psi^{\nu} \cong -\frac{2\,ie}{m^2}\, F_{\mu\sigma}\gamma^{\sigma}\psi^{\mu} \cong -\frac{2\,ie}{m^2}\,(F_{jk}\,\gamma^{k} + F_{k0}\,\beta\,\gamma^{k}{}_{j})\,\psi^{j} \qquad (22)$$

(latin indices run from 1 to 3).

We start with equation (7), but in view of (22) and of the condition

$$|\partial_{i}\psi| \ll m\,|\psi|, \quad i = 1, 2, 3, \qquad (23)$$

we write to the considered approximation:

$$(\gamma^{\sigma}\partial_{\sigma} - m)\,\psi^{i} + \gamma^{i}\left(\frac{m}{2} + \frac{1}{3}\gamma^{0}\,\partial_{0}\right)\gamma_{\nu}\,\psi^{\nu} - 0\ . \qquad (24)$$

Now we have to terms in 1/m:

$$\gamma^{0}\,\partial_{0}\,\gamma_{\nu}\psi^{\nu} = \frac{2\,ie}{m}\,(F_{jk}\,\gamma^{k} - F_{ko}\,\beta\,\gamma^{k}{}_{j})\,\psi^{j}\ . \qquad (25)$$

Taking (25) and (22) successively in (24), we obtain

$$(\gamma^{\sigma}\,\partial_{\sigma} - m)\,\psi^{i} - \frac{ie\gamma^{i}}{3m}\,(F_{jk}\,\gamma^{k} + 5F_{ko}\ \gamma^{k}{}_{j})\,\psi^{j} = 0\ .$$

Here we use a matrix notation similar to (8), (14) but <u>with the exclusion of the fourth rows and columns of the vectors and operators</u> (this should not be confusing in the same way as when in the passage from Dirac's to Pauli's equation the same notation is kept for the two component wave functions as for the 2 x 2 matrices $\vec{\sigma}$). So the <u>3-vector</u> wave function ψ satisfies the equation:

$$i\,\frac{\partial\psi}{\partial t} = H_{0}\psi = \Big[m\beta + eA_{0} + \vec{\alpha}\cdot(\vec{p} + e\vec{A}) + \frac{e\beta}{3m}\,\vec{H}\cdot\vec{m} - \frac{e\beta}{3m}\,\gamma^{ik}F^{j}{}_{k}t_{ij} + \frac{5ie}{3m}\,F_{ko}\gamma^{i}\gamma^{kj}t_{ij}\Big]\,\psi. \qquad (26)$$

Now if, according to Foldy and Wouthuysen, we make the transformation generated by the Hermitian operator

$$S = -\frac{i\beta}{2m}\vec{\alpha}\cdot(\vec{p} + e\vec{A})$$

we obtain (again calling ψ the new wave function),

$$i\frac{\partial\psi}{\partial t} = \Big[m\beta + eA_o + \frac{\beta}{2m}(\vec{p} + e\vec{A})^2 + \frac{e\beta}{2m}\vec{\sigma}\cdot\vec{H} + \frac{e\beta}{3m}\vec{H}\cdot\vec{m} - \\ - \frac{e\beta}{3m}\gamma^{ik}F^j{}_k t_{ij} + \frac{5ie}{3m}F_{ko}\gamma^i\gamma^{kj}t_{ij} - \frac{ie\beta}{32m}\vec{\alpha}\cdot\vec{E}\Big]\psi \qquad (27)$$

The last two terms in (27) are eliminated (to the present approximation) by the transformation

$$\psi \longrightarrow e^{-iT}\psi$$

$$T = \frac{e\beta}{6m^2}F_{ko}\gamma^i\gamma^{ko}t_{ij} - \frac{e}{4m}\vec{\alpha}\cdot\vec{E}$$

So we get:

$$i\frac{\partial\psi}{\partial t} = \Big[m\beta + eA_o + \frac{\beta}{2m}(\vec{p} + e\vec{A})^2 + \frac{e\beta}{3m}\vec{H}\cdot\vec{M} + Q^i\sigma^j t_{ij}\Big]\psi \qquad (28)$$

In (28) the following identity was used:

$$\gamma^{ik}F^j{}_k t_{ij} = \vec{\sigma}\cdot\vec{H} - H_i\sigma_j t^{ij}$$

which is easily verified.

Now equation (28) is not Hermitian because $Q^i\sigma^j t_{ij}$ is a non Hermitian operator. It is easily seen that the supplementary condition

$$t^{ij}\sigma_j\psi = 0 \qquad (29)$$

or equivalently

$$\sigma_i\psi^i = 0 \qquad (29a)$$

is compatible with equation (28). So we impose it.

Finally, taking

$$\psi = e^{imt}\varphi$$

and imposing the condition

$$\beta\varphi = \varphi$$

consistent with equation (28) we obtain the Schrodinger equation for 3/2 spin particles:

$$i\,\frac{\partial\varphi}{\partial t} = H_o\varphi = \left[e\;A_o + \frac{(\vec{p} + e\vec{A})^2}{2m} + \frac{e}{3m}\;\vec{H}\cdot\vec{M}\right]\;\varphi\;. \tag{30}$$

where φ has two spinor and three vector components and

$$\vec{M} = \vec{m} + \frac{1}{2}\;\vec{\sigma}\;, \tag{31}$$

$\vec{\sigma}$ being given by the Pauli matrices (acting on the spinor components of φ) and $\vec{m}$ being given by the 3 x 3 matrices:

$$m_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}\;;\quad m_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}\;;\quad m_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\;.$$

They satisfy the commutation relations

$$[m_i,\; m_j] = im_k$$

for i, j, k, a cyclic permutation of 1, 2, 3. As a consequence the operator for intrinsic angular momentum $\vec{M}$ satisfies the relations:

$$[M_i,\; M_j] = i\;M_k$$

The general procedure for obtaining the Schrodinger equation to higher order approximation is more complicated and is the subject of another paper[13]. In that paper it is shown that charged particles with spin 3/2 have also an intrinsic electric quadrupole moment.

Regarding the meaning of the supplementary condition (29), we should say the following:

a) Being a spinor equation, (29a) allows us to eliminate two of the six components of the spinor-vector (2 x 3) wave function φ_r^i ($i = 1, 2, 3$; $r = 1, 2$). So we have only four independent components as should be the case for spin 3/2 particles.

b) We can easily show that

$$t^{ij}\;\sigma_i\;\sigma_j = 1 - \vec{m}\cdot\vec{\sigma}$$

So condition (29) can be written:

$$\vec{m}\cdot\vec{\sigma}\,\varphi = \varphi$$

Now, as

$$\vec{M}^2 = \vec{m}^2 + \frac{1}{4}\;\vec{\sigma}^2 + \vec{\sigma}\cdot\vec{m} = \frac{1}{4} + \frac{1}{4}\;\vec{\sigma}\cdot\vec{m}$$

we have

$$M^2 \varphi = \frac{1}{4}(1 + \vec{\sigma} \cdot \vec{m})\varphi = \frac{15}{4} \varphi . \qquad (33)$$

Equation (33) expresses the fact that the eigenvalues of the operator M^2 are 15/4, as should be for the square of spin angular moment for spin 3/2 particles.

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