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On q-Deformed Supersymmetric Classical Mechanical Models

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Abstract

Based on the idea of quantum groups and paragrassmann variables, we present a generalisation of supersymmetric classical mechanics with a deformation parameter $q = \exp \frac{2\pi i}{k}$ and we work with the $k = 3$ case. The coordinates of the q-superspace are a commuting parameter t and a paragrassmann variable θ, where $\theta^3 = 0$. The generator and covariant derivative are obtained, as well as the action for some possible superfields.

Key-words: Paragrassmann variables; Supersymmetry; Classical mechanics.

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1 Introduction

For the last few years, Quasi Triangular Hopf Algebras or Quantum Groups [1, 2, 3] had attracted a lot of attention from physicists. One of the most interesting features is that such structures can be related to underlying symmetries on spaces where the coordinates are non-commutative [4].

Recently, it has been shown that the creation and anihilation operators of the q-deformed harmonic oscilator [5]

$$a\,a^\dagger - q a^\dagger\,a = q^{-N}, \tag{1}$$

possesses a classical limit where these operators can be understood as coordinates obeying [6]

$$\theta^k = 0, \tag{2}$$

where k is an integer, and the q-factor of the deformation is a prime root of unity, $q^k = 1$. In general, the properties of these coordinates are generalizations of some property associated with Grassmann variables. Promoting these coordinates to functions of a (non-deformed) parameter t, it was shown that it is possible to write down an action for such fields that, when added to the action of a commuting field, it has a symmetry resembling supersymmetry [7], and it was also shown how to do functional integration on a heterotic quantum field theory [8]. The aim of this letter is to show how it is possible to understand the transformations on such fields, and the action invariances, as resulting from a superspace formulation of a classical mechanical model where its coordinates are the Paragrassmann variables (a q-superspace), and the q-supersymmetric multiplets are composed of (in general) non-commuting fields.

In the next section we give a brief review of Paragrassmann variables and also how one can construct coordinates from them. Section 3 is devoted to construct the q-superspace, transformations between its coordinates, and the induced transformations on the q-superfields defined on it. Invariant actions are constructed on section 4, in particular for a free particle and the harmonic oscilator. We leave some final comments to the last section.

2 Paragrassmann Variables and Quermionic Coordinates

We start this section introducing a Paragrassmann variable θ and its derivative, $\frac{\partial}{\partial\theta} \equiv \partial_\theta$ that obeys [9]

$$\theta^k = 0, \qquad {\partial_\theta}^k = 0, \tag{3}$$

for a positive integer k.

If we demand that the action of ∂_θ on θ^n is proportional to θ^{n-1}, it turns out that it becames necessary to deform the Leibniz rule to be

$$\partial_\theta(a\,b) = (\partial_\theta a)b + g(a)(\partial_\theta b), \tag{4}$$

where a, b are arbitrary polynomials in θ, and $g(a)$ is an automorphism of the algebra, satisfying

$$\begin{aligned} g(\alpha\, a + \beta\, b) &= \alpha\, g(a) + \beta\, g(b), \\ g(a\,b) &= g(a)\, g(b), \end{aligned} \tag{5}$$

where α, β are c-numbers.

Choosing $a = \theta$ in (4), we see that ∂_θ and θ must obey a q-deformed commutation (quommutation) relation

$$[\partial_\theta, \theta]_q \equiv \partial_\theta\theta - q\theta\partial_\theta = 1, \tag{6}$$

implying the automorphism for θ

$$g(\theta) = q\,\theta. \tag{7}$$

This derivative, however, is not unique. Indeed, we could change the power 1 in eq. (6) by any other integer, thus for each value of k one can define $k-1$ different derivatives. For the specific case $k = 3$, one can also define another derivative δ_θ [10] that quommutes with θ as

$$[\delta_\theta, \theta]_q \sim \delta_\theta\theta - q^2\theta\delta_\theta = 1, \tag{8}$$

and its Leibniz rule differs from eq.(4) by changing $g(a)$ by $g(g(a))$.

Integration over the paragrassmann variable is defined such that

$$\int d\theta\, \theta^n \;\alpha\; \delta_{n,k-1}. \tag{9}$$

It is interesting to notice that, for $k = 2$, $q = -1$, eq.(1) becomes the usual anticommutator, consistent with eqs.(3) and (6), which are the conditions for Grassmann variables. Taking $k \to \infty$, eq.(1) becomes the usual commutator. The meaning of this limit in eq.(3) is that, if we Taylor expand a function of these variables, it will become a series (obviously, if $\theta^k = 0$, k finite, a Taylor expansion will be a polynomial of degree $(k-1)$).

The Paragrassman variables can be promoted to coordinates if we take them to be functions of a (commuting) parameter t. Let us recall that in the Grassmannian case we have two different coordinates: one that behaves like θ (a fermionic coordinate), and another that behaves like θ^0, a bosonic (commuting) coordinate. In the Paragrassmann case, we will have k different types of coordinates, each one corresponding to a power of θ, and again θ^0 being a commuting one. We call the $\psi^{(i)}(t)$ the q-fermionic generalization of the coordinates or, simply, the quermionic coordinates and its label (i) indicates the sector to wich it belongs.

We take two quermions of different sectors to obey the quommutation relation

$$\psi^{(i)}\,\psi^{(j)} = q_{(i,j)}\psi^{(j)}\,\psi^{(i)}, \tag{10}$$

where the parameters $q_{(i,j)}$ are powers of q, $q^k = 1$.

Let us take now the particular case $k = 3$, and to construct an action which extends the supersymmetric point particle through the use of these generalized fields [7]. This generalized particle is described by the coordinates $(x(t), \psi^{(1)}(t), \psi^{(2)}(t))$, in the same way as a supersymmetric point particle is described be the coordinates $(x(t), \psi(t))$. The action involving the quermions is given by

$$S = \int dt(\frac{1}{2}\dot{x}^2 - qC^{(s)^2}\dot{\psi}^{(2)}\psi^{(1)}), \tag{11}$$

where we choose the mass equal to one. We choose the second term in (11) in such a way that it resembles the classical fermionic equation of motion. The cocycle–type factor $C^{(s)^2}$ is required because when we multiply two objects of different sectors, $A^{(r)}B^{(s)}$, it must behave like an object of the sector $(r+s)\,\mathrm{mod}\,3$. However, if this factor is not inserted this product would not quommute correctly with $A^{(r)}$ or $B^{(s)}$. Underlying this point is the fact that differently from the fermionic case we choose that equal fields at equal points commute.

This cocycle–type factor $C^{(s)}$ actually behaves like a sector–counter, that is,

$$C^{(s)}A^{(i)} = q^i A^{(i)}C^{(s)}. \tag{12}$$

Finally, with the choice $[\psi^{(1)}, \psi^{(2)}]_q = 0$, taking all the fields as real, the second term in the action, eq. (11), becomes real and is a representative of the zeroth sector.

Doing the transformation (the variations of a field will be written as Δ to one not be confused with the derivative δ),

$$\begin{aligned} \Delta x &= qC^{(s)}\epsilon^{(1)}\psi^{(2)}, \\ \Delta\psi^{(1)} &= q^2 C^{(s)^2}\epsilon^{(1)}\dot{x}, \\ \Delta\psi^{(2)} &= \pm q\epsilon^{(1)}\psi^{(1)}, \end{aligned} \tag{13}$$

on the action (11) we get

$$\Delta S = \pm \int dt \frac{d}{dt}(\epsilon^{(1)}\psi^{(1)^2}), \tag{14}$$

where we used $[\epsilon^{(1)}, \psi^{(1)}]_q = [\psi^{(2)}, \epsilon^{(1)}]_q = 0$. Such transformation is similar to a supersymmetric transformation: the parameter $\epsilon^{(1)}$ is non-commuting one, the action transforms as a total derivative, and one of the fields, $\psi^{(1)}$, transforms as a total derivative, which can be taken as indicating that $\psi^{(1)}$ is the highest term in a θ–expansion of some superfield. One could also write transformations among the fields with a parameter belonging to the sector-two. However, it can be shown that this transformation is not a symmetry of the action (11) [7].

3 The q-Superspace and q-Superfields

We now begin to contruct a q-superspace formulation that will recover the results concerning the quermionic coordinates presented in the last section. As previously stated, we will consider in detail only the $k = 3$ case. Some of the ideas discussed here and in the next section had been discussed also in refs. [11, 12].

The q-superspace coordinates are $(t;\theta)$, where t is a c-number to be identified with time and θ is a paragrassmannian variable obeying $\theta^3 = 0$, and both are taken as real parameters.

Let us now to introduce transformations between these coordinates that are translations on the q-superspace. We write them as

$$\begin{aligned} \theta' &= \theta + q^2\, C^{(0)}\, \varepsilon, \\ t' &= t + qC^{(0)}\theta^2\varepsilon, \end{aligned} \tag{15}$$

where ε is an infinitesimal constant that, by homogenity, is in the same sector than θ. The cocycle-like factor $C^{(0)}$ is necessary because the transformed coordinates (t', θ') must behave under quommutation relation in the same way than (t, θ). The translation in the q-superspace fixes the mass dimensions of θ and ε to be $-\frac{1}{3}$.

Defining the quommutator to be

$$[A, B]_q \equiv A\,B - q\,B\,A, \tag{16}$$

and choosing

$$[\theta, \varepsilon]_{q^2} = 0, \tag{17}$$

we fix

$$[C^{(0)}, \theta]_{q^2} = [C^{(0)}, \varepsilon]_{q^2} = 0. \tag{18}$$

It is after determining these quommutation relations that we set the q factors in (15) to preserve the reality conditon for the coordinates. We could choose q instead of q^2 in (17) (i. e., take $[\varepsilon, \theta]_q = 0$). With this choice we necessarilly have to change $q \leftrightarrow q^2$ in eqs. (15) and (18). In fact, there is no significative difference between these two cases.

After introducing the q-superspace (t, θ), our next step is to write down a function of these variables. As in the supersymmetric case, let us expand this function in a Taylor series on θ, and this expansion is a polynomial of degree 2 (for the generic case $\theta^k = 0$, the polynomial goes up to the order $(k-1)$).

$$X(t; \theta) = x(t) + q^2\ \theta\ C^{(2)}\psi^{(2)}(t) + q^2\ \theta^2\ C^{(1)}\psi^{(1)}(t). \tag{19}$$

The coordinate $x(t)$ is a commuting function, so by homogenity $X(t, \theta)$ also has this property, playing the same rôle of a scalar field. The $\psi^{(i)}(t)$ are the q-supersymmetric partners of the coordinate $x(t)$, and their dimensions are $[\psi^{(j)}] = -\frac{j}{3}$. We take their quommutators to be

$$\begin{aligned}
[\psi^{(1)}, \psi^{(2)}]_{q^2} = 0 \quad &, \quad [C^{(1)}, C^{(2)}]_{q^2} = 0, \\
[\theta, \psi^{(j)}]_{q^{2j}} = 0 \quad &, \quad [\varepsilon, \psi^{(j)}]_{q^j} = 0, \\
[\psi^{(1)}, C^{(1)}]_q = 0, \quad &, \quad [\psi^{(1)}, C^{(2)}] = 0, \\
[\psi^{(2)}, C^{(1)}]_q = 0 \quad &, \quad [\psi^{(2)}, C^{(2)}]_q = 0.
\end{aligned} \tag{20}$$

With this choice we garantee that X is indeed a zero-sector field, and it is real.

The infinitesimal coordinate transformations (15) induce a variation on the q-superfield $X(\theta, t)$ in the form

$$X(t', \theta') - X(t, \theta) = \Delta\,X = q^2\ C^{(0)}\ \varepsilon\ Q\,X. \tag{21}$$

We can get the operatorial realization of the q-supersymmetric generator transformation, Q, by Taylor expanding the l.r.s. of this equation. We should, however, take care to define a q-Taylor expansion, since we have two derivatives with respect to θ. Choosing the factors to keep the reality condition we have

$$X(\theta', t') = X(\theta, t) + \Delta\theta \left(q^2 \frac{\partial X}{\partial \theta} + q \frac{\delta X}{\delta\, \theta} \right) + \Delta t\, \frac{\partial X}{\partial t}. \tag{22}$$

With this expansion, and using eq.(15). Q becomes

$$Q = \theta^2\ \partial_t + q^2 \partial_\theta + q\ \delta_\theta \tag{23}$$

We notice that the generator is in θ^2 sector, and its canonical dimension is $[Q] = \frac{1}{3}$.

A straightforward calculation shows that

$$Q^3 = -\partial_t. \tag{24}$$

This means that the q-supersymmetric transformations are the cubic roots of time translations.

The variation of the component fields of X can be read off from eq.(21), comparing the powers of θ

$$\begin{aligned} \Delta x &= -\, q\ C^{(0)}\, C^{(2)}\ \varepsilon\ \psi^{(2)}, \\ \Delta\psi^{(1)} &= C^{(1)^2}\, C^{(0)}\, \varepsilon \dot{x}, \\ \Delta\psi^{(2)} &= q^2\, C^{(2)^2}\, C^{(0)}\, C^{(1)}\, \varepsilon \psi^{(1)}. \end{aligned} \tag{25}$$

As in the usual supersymmetric case, this is a cyclical transformation among the fields.

Having written down the q-superspace transformations and the variations on the q-superfield, let us now construct a q-covariant derivarive, D, that is, a differential operator that obeys

$$\begin{aligned} [D\,,\, Q]_q &= 0, \\ D\,(\Delta X) &= \Delta\,(DX). \end{aligned} \tag{26}$$

Choosing D to belong to the same sector as Q, they will be proportional

$$D = qC^{(0)^2} \left(\theta^2\ \partial_t + q^2 \partial_\theta + q\ \delta_\theta \right). \tag{27}$$

As in the supersymmetric case, the component fields can be defined by projecting the superfield on different sectors, using the covariant derivatives on $\theta = 0$.

$$\begin{aligned} X|_{\theta=0} &= x, \\ DX|_{\theta=0} &= -q\, C^{(0)^2}\, C^{(2)} \psi^{(2)}, \\ D^2 X|_{\theta=0} &= -q^2\, C^{(0)}\, C^{(1)}\, \psi^{(1)}. \end{aligned} \tag{28}$$

From now on, we will neglect the subscript $\theta = 0$.

We also notice some relations between powers of D and Q, that will be useful later

$$\begin{aligned} D\,.\,| &= q^2\,C^{(0)^2}\,Q\,.\,|, \\ D^2\,.\,| &= q^2\,C^{(0)}\,Q^2\,.\,|, \\ D^3\,.\,| &= -\,\partial_t\,.\,|. \end{aligned} \tag{29}$$

Besides the above defined bosonic superfield, we can also construct sectors one and two superfields. There θ expansion can be taken to be

$$\Lambda^{(1)} = \lambda^{(1)} + q^2\theta M^{(1)} A + q\theta^2 M^{(2)}\lambda^{(2)}, \tag{30}$$

and

$$\Xi^{(2)}(t) = \xi^{(2)} + \theta L^{(1)}\xi^{(1)} + q^2\theta^2 L^{(2)} F\ , \tag{31}$$

where the supercripts indicates the sectors which the fields belongs, and A and F are bosonic fields.

The dimension of the q-superfield $\Xi^{(2)}$ is taken as $\frac{2}{3}$, its bosonic component F being dimensionless and, as we will see later, behaving as an auxiliar field. We can not, however, take the dimension of the q-superfield $\Lambda^{(1)}$ as $\frac{1}{3}$, since this would imply a negative dimension for the component field $\lambda^{(2)}$. Thus we take its dimension as $\frac{4}{3}$. This, however, will bring different equations of motion for its quermionic components, as we will see in the next section.

We assume that the fields $\xi^{(j)}$ have the same behaviour as $\psi^{(j)}$ with respect the quommutations relations with each other, with θ and with ε. The relations with the cocycles are then fixed to be

$$\begin{aligned} [\xi^{(j)}, L^{(i)}]_{q^{j\delta_{i,j}}} &= 0, \\ [L^{(1)}, L^{(2)}]_q &= 0. \end{aligned} \tag{32}$$

The Leibnix rule for the covariant derivative D is fixed by the particular Leibniz rule that each of the differential operators appearing on it obey. Considering the q-superfields introduced above, the action of D on a product of two of them is of the form

$$D(AB) = (DA)\,B + h(A)\,(DB) + 3\,\theta\,h^2(A_1)\,B_1, \tag{33}$$

where A_1 and B_1 are the coefficients of the linear term in the θ-expansion of the respective q-superfield. We notice that, because of the integration rule eq.(9), this last term will not contribute on an integration over the Paragrassmann variable, and integration by parts is allowed. The factor $h(A)$ is simply a power of q times the q-superfield A, obtained by the rule

$$C^{(0)^2}\,\theta^2\,A \;=\; h(A)\,C^{(0)^2}\,\theta^2. \tag{34}$$

4 Examples of Superactions

In this section, we are going to make a general discussion about actions that are functions of the q-superfields introduced in the last section and give the same examples of them.

In general, an action for a generic superfield Φ must be of the form

$$S = \int dt d\theta \mathcal{P}(\Phi, \dot{\Phi}, D\Phi, D^2\Phi), \tag{35}$$

where the polynomial $\mathcal{P}$ in Φ and its covariant derivatives must behave like under quommutation relations like θ^2, belonging to the sector two (since $\int d\theta = {\partial_\theta}^2$, and S is scalar), and since the measure has mass dimension $\frac{-1}{3}$ and S is dimensionless, its dimension must be $\frac{1}{3}$.

By comparing the expression for the covariant derivative and the θ-integration, we notice the relation

$$\int d\theta = q^2 C^{(0)^2} D^2|. \tag{36}$$

Let us now do a transformation on the action

$$\Delta S = \int dt d\theta\, \Delta\mathcal{P}(\Phi, \dot{\Phi}, D\Phi, D^2\Phi) \tag{37}$$

since the Jacobian is one. Since $\mathcal{P}$ is a superfield, its variation is of the form of eq.(21). Using this fact and eq.(36), we arrive at the conclusion that S transforms in a total derivative

$$\Delta S = -\,q\,C^{(0)}\,\varepsilon\,\int dt\,\partial_t \mathcal{P}. \tag{38}$$

and the transformations eq.(15) induces symmetries on the action.

Let us now write an action of the q-superfields X, $\Lambda^{(1)}$ and $\Xi^{(2)}$ defined in section 3, and compute their equations of motion. We begin with the bosonic superfield X. Its quadratic action is

$$S_X = -\frac{m}{2}\int dt d\theta\; C^{(0)}(D^2X)(D^2X) \tag{39}$$

where m is a commuting mass parameter. By explicit computation of its θ integral, or by use of eq.(36), this action can be read off in components as

$$S_X = m\int dt\left(\frac{1}{2}\dot{x}^2 + q^2 C^{(2)} C^{(1)} \dot{\psi}^{(2)}\psi^{(1)}\right) \tag{40}$$

The equation of motion

$$D\dot{X} = 0 \tag{41}$$

gives in components $\ddot{x} = \dot{\psi}^{(j)} = 0$ $(j = 1, 2)$. Computing its q-supersymmetric variation, we obtain

$$\Delta S_X = qC^{(0)}\epsilon \int dt \partial_t (C^{(1)^2} \psi^{(1)^2}), \tag{42}$$

We notice that the action given by eq.(39), its variation eq.(42) and the variation of the component fields eq.(25) are, up to factors, equal to eqs.(11), (14) and (13) respectivelly. Thus we see that the q-superfield X describes the dynamics of a free particle and its associated quermionic partners.

The quadratic action for the q-superfield $\Lambda^{(1)}$ is

$$S_\Lambda = -\frac{m}{2} \int dt d\theta \, M^{(1)} \, (\dot{\Lambda}^{(1)})^2. \tag{43}$$

By covenience the mass parameter was taken to be the same as in the X action. In components, the action turns out to be

$$S_\Lambda = \frac{m}{2} \int dt \, (\dot{A}^2 + 2q^2 M^{(1)} M^{(2)} \dot{\lambda}^{(2)} \dot{\lambda}^{(1)}). \tag{44}$$

Its interesting to notice that the equation of motion for $\Lambda^{(1)}$,

$$\ddot{\Lambda}^{(1)} = 0, \tag{45}$$

gives in component $\ddot{A} = \ddot{\lambda}^{(i)} = 0$. Thus this q-superfield also represents a free particle, but its quermionic partners obey an equation of motion that is of second order in the time derivative, whereas in the case of q-superfield X it is of first order. The q-supersymmetric variation of the S_Λ is

$$\Delta S_\Lambda = \varepsilon \, C^{(0)} \, M^{(1)} \int dt \frac{\partial (\dot{\Lambda}^{(1)})^2}{\partial t}, \tag{46}$$

We now consider the quadratic action for the q-superfield $\Xi^{(2)}$. It is

$$S_\Xi = m \int dt d\theta \; L^{(2)} C^{(0)^2} \, (D\Xi^{(2)})^2, \tag{47}$$

In component fields, the action reads

$$S_\Xi = \int dt \, [-2 \, L^{(2)} L^{(1)} \dot{\xi^{(2)}} \xi^{(1)} + F^2]. \tag{48}$$

The equation of motion for $\Xi^{(2)}$ is

$$D^2 \Xi^{(2)} = 0, \tag{49}$$

giving $F = \dot{\xi}^{(j)} = 0$, meaning, as it was anticipated, that the bosonic coordinate F is an auxiliary one. The variation of S_Ξ is

$$\Delta S_\Xi = -C^{(0)}\epsilon L^{(2)} {L^{(1)}}^2 \int dt\, \partial_t\, {\xi^{(1)}}^2, \tag{50}$$

The superfields X and $\Xi^{(2)}$ can have a quadratic action with a mixed term

$$S = m\omega \int dt d\theta q^2 {L^{(2)}}^2\, X\Xi^{(2)} \tag{51}$$

where ω has a mass^{-1} dimension, that reads in components

$$S = m\omega \int dt\, Fx \tag{52}$$

Summing up the actions (39), (47) and (51)

$$S_{HO} = S_X + S_\Xi + S_{X\Xi} \tag{53}$$

we see that the its bosonic part is

$$S_{HO} = \int dt\, m(\frac{1}{2}\dot{x}^2 + \frac{1}{2}F^2 + \omega F x) \tag{54}$$

Computing the equation of motion of the auxiliary field F and reintroducing it in the action, it becomes

$$S_x = \int dt \left[\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega x\right] \tag{55}$$

which is the action for the harmonic oscilator.

5 Conclusions

In this letter, we presented a generalisation of some supersymmetric classical mechanical models based on a deformation parameter q, where the non-commuting coordinate is nilpotent of order 3. Translations on the q-supespace induces transformations on the fields, and we were able to construct actions of these fields that are, up to total derivatives, invariant under such transformations.

Although there is a great paralellism between the usual supersymmetric case and the formulation we discussed here, there are some differences, the most important one is the needness to introduce cocycle-like factors to correct statistical behaviour of the fields. We expect that such cocycle-like factors should be some functions of the fields, that would not participates in the dynamics. One should conjecture that the needness to introduce such factors is related to the fact that the identity operator of an algebra must also be deformed when one goes from the non-deformed algebra to the deformed one. For instance, the q^{-N} factor appearing in the r.h.s. of eq.(1) can be seen as the deformation parameter of the identity operator of Heisenberg algebra. Such factors are also important for consistency conditions on the algebra, like associativity condition [15]. Thus one should search for factors of the form q exponentiated to some operator as possible solutions for the cocycle-like factors.

Another possibility would be take the parameters appearing an the formulae not as commuting ones, but obeying some quommutation relations with the operators. This has been done, for instance, in ref. [16] where the mass parameter was taken to be non-commuting to ensure unitarity of the time evolution operator.

It should be interesting to study this formulation from a field theoretical point of view, in particular in the $(2+1)$-dimensional case. Also, we could ask if such fields are representations of some q-deformed algebra, either a q-Poincaré or a q-Clifford one.

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