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SOME COMMENTS ON TWO-DIMENSIONAL MASSLESS QUANTUM CHROMODYNAMICS

by

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Quantum chromodynamics in two space-time dimensions with massless fermions is studied using the path-integral approach. Some aspects of the Roskies gauge are clarified. An effective gluonic action for the model is obtained.

Key-words: Path-integral; Quantum chromodynamics; Two space-time dimensions.

Recently, several attempts have been made to find a complete solution of two-dimensional quantum chromodynamics with massless fermions ([1], [2], [3]), shortly denoted by $(Q.C.D.)_2$. All these approaches were based on integrating out the fermion fields of the model and considering an effective gluonic theory where some interesting phenomena can be easily seen ([2], [3]).

In this comment, we intend to make some clarifying remarks on this effective action for massless $(Q.C.D.)_2$.

We start our analysis by considering the generating functional for the model in a Euclidean space-time R^2 with local gauge group SU(2) (the generalization for the case SU(N) is straightforward)

$$Z[J_{\mu},\eta,\bar{\eta}] = \int \mathcal{D}[G_\mu]\, e^{-\frac{1}{4g^2}\int d^2x\, Tr^{(c)}(F^2_{\mu\nu})(x)}\, e^{\int d^2x \vec{J}_\mu \cdot \vec{G}_\mu}$$

$$\left(\int \mathcal{D}\psi\, D\bar{\psi}\, e^{-\left(\int d^2x(\bar{\psi}\not{D}(G_\mu)\psi + \bar{\eta}\psi + \bar{\psi}\eta)(x)\right)}\right) \qquad (1)$$

where $\not{D}(G_\mu) = i\gamma_\mu(\partial_\mu - iG_\mu)$ denotes the Dirac operator in the presence of the external gauge field G_μ and the tensor field strength is given by $F_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu + [G_\mu, G_\nu]$.

The hermitean γ-matrices we are using satisfy the (Euclidean) relations.

$$\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu} \quad ; \quad \gamma_\mu\gamma_5 = i\varepsilon_{\mu\nu}\gamma_\nu \quad ; \quad \gamma_5 = i\gamma_0\gamma_1$$

$$\varepsilon_{01} = -\varepsilon_{10} \qquad (\mu,\nu = 0,1) \qquad (2)$$

The functional measures in (1) are normalized to unity and the fermion measure $D\psi D\bar{\psi}$ is defined in terms of the eigenvalues of the self-adjoint Dirac operator $\not{D}(G_\mu)$ which insures automatically its gauge invariance.

Our plan to study (1) is to implement a convenient change of variables in the fermionic sector of (1) in order to get an effective generating functional where the fermion fields are decoupled from the gauge field G_μ ([1], [2]). For this analysis, we are going to use a general decomposition of the gauge field G_μ due to Roskies ([1]) and this will be explained in the following.

Roskies in Ref. [1], has shown that for any gauge field configuration $G_\mu(x)$, there is a unique unitary matrix $\Omega(x)$ taking values in SU(2) and a hermitean matrix $V(x) = e^{-\gamma_5\vec{\phi}(x)\vec{J}}$ taking value over the axial gauge group SU(2) (whose Lie algebra is generated by the hermitean generators $\gamma_5\vec{\tau}$, with $\vec{\tau}$ denoting the usual SU(2)-generators) such that:

$$i\gamma_\mu\vec{G}_\mu(x)\vec{J} = \gamma_\mu\partial_\mu(V^{-1}\Omega^{-1})(x)(\Omega V)(x)$$

$$= -\gamma_\mu(V^{-1}\Omega^{-1})(x)\partial_\mu(\Omega V)(x) \tag{3}$$

The proof of the validity of the decomposition (3) can be accomplished by considering the $j = -i\gamma_5$ complexification of space-time, which is denoted by $\mathbb{C}$ ($\mathbb{C} = \{z = (x_0, x_1) \mid z = x_0 + jx_1;\ \bar{z} = x_0 - jx_1$ and $(x_0, x_1) \in \mathbb{R}^2\}$). In this $\mathbb{C}$ space-time, we can re-write the partial differential equation (3) into a single ordinary differential equation ([1]):

$$\hat{G}.\vec{\tau} = -2i(\partial_{\bar{z}}W)W^{-1} \qquad (4)$$

where $\hat{G}$ is the j-complexification of $G_\mu(\hat{G} = G_0 + jG_1)$ and W is an element of $SL(2,\mathbb{C})$ (the associated j-complexification of SU(2)).

The equation (4) is just the equation for a holomorphic. Principal bundle over $\mathbb{C}$, and, as is well known from differential topology, all such bundles are trivial, which means that a unique global solution W for (4) exists.

In order to determine explicitly this solution W, we note that (4) can be easily integrated, leading to the result

$$W\Big((x_0,x_1)\Big) = \mathbb{P}\left\{e^{-2i\int_{-\infty}^{z}(\hat{G}.\vec{\tau})d\bar{z}}\right\} \qquad (5)$$

where $d\bar{z} = dx_0 - jdx_1$ and the SU(2) path-ordered integral in (5) is taken over the (infinite) straight segment joining the $(-\infty)$ point to the $z = (x_0,x_1)$ point.

By introducing the axial gauge field

$$*\vec{G}_\mu.\vec{\tau} = (\varepsilon_{\mu\alpha}\vec{G}_\alpha).\vec{\tau} \qquad (6)$$

we can re-write Eq. (5) in the more transparent form:

$$W\Big((x_0,x_1)\Big) = \mathbb{P}\left\{e^{\left(-2i\int_{(-\infty)}^{(x_0,x_1)}(\vec{G}_\mu.\vec{\tau}dx_\mu - 2\gamma_5\int_{(-\infty)}^{(x_0,x_1)}(*G_\mu.\vec{\tau})dx_\mu\right)}\right\} \qquad (7)$$

where, again, the SU(2) path ordered integral in (7) is taken over the straight segment joining the $(-\infty)$ point to the (x_0,x_1) point.

Continuing our study, we can see that the Dirac operator $\not{D}(G_\mu)$ can be re-written in the suitable form ([3], [5])

$$\not{D}(G_\mu) = (\Omega V^{-1})(x)(i\gamma_\mu \partial_\mu)(V^{-1}\Omega^{-1})(x) \qquad (8)$$

Here, the matrices $\Omega(x)$ and $V(x)$ are respectively the unitary and hermitean factors of the $SL(2,\mathbb{C})$ Wu-Yang factor (7).

In order to decouple the fermion fields from the gauge field, we follow Ref. ([1]) by making the variable change.

$$\begin{aligned} \psi(x) &= (\Omega . V)(x)\chi(x) \\ \bar{\psi}(x) &= \bar{\chi}(x)(V.\Omega^{-1})(x) \end{aligned} \qquad (9)$$

which yields the fermionic generating functional

$$\tilde{Z}[\eta,\bar{\eta}] = \int \mathcal{D}[\chi]\mathcal{D}[\bar{\chi}] J[G_\mu] e^{-(\int d^2x (\bar{\chi}(i\gamma_\mu \partial_\mu)\chi + \bar{\eta}(\Omega . V)\chi + \bar{\chi} V \Omega^{-1} \eta)(x)} \qquad (10)$$

where the quantum aspect of the variable change (9) is taken into account by considering the associated jacobian $J[G_\mu]$ ([1], [2]).

Now, it is important to note that the jacobian $J[G_\mu]$ is given by the ratio

$$J[G_\mu] = \frac{\mathrm{DET}(\not{D}(G_\mu))}{\mathrm{DET}(i\gamma_\mu \partial_\mu)} \qquad (11)$$

In order to evaluate the functional fermionic determinant in (11), we introduce a family of gauge fields $G_\mu^{(\sigma)}$ $(0 \le \sigma \le 1)$ interpolating continuously the zero field configuration $G_\mu^{(\sigma=0)} \equiv 0$ to the considered configuration $G_\mu^{(\sigma=1)} = G_\mu$ in (11) and defined by the relation (see Eq. (3)).

$$i\gamma_\mu G_\mu^{(\sigma)} = -\gamma_\mu (e^{-\sigma\gamma_5 \vec{\phi}.\vec{\tau}}\Omega^{-1})\partial_\mu (\Omega e^{\sigma\gamma_5 \vec{\phi}.\vec{\tau}}) \tag{12}$$

We note that we are assuming implicitly that we are computing the Jacobian $J[G_\mu]$ in the trivial topological sector of the manifold of the gauge fields configurations, since G_μ is in the same homotopical class of the zero field configuration. As a consequence of this fact, we do not taken into account the zero-modes of the operator $\not{D}(G_\mu)$ in what follows.

Now, it seems important to remark that to evaluate $DET(\not{D}(G_\mu^{(\sigma)}))$ we can consider solely the "reduced" operator.

$$\tilde{\not{D}}(G_\mu^{(\sigma)}) = e^{\sigma\gamma_5 \vec{\phi}.\vec{J}}(i\gamma_\mu\partial_\mu)e^{\sigma\gamma_5 \vec{\phi}.\vec{J}} \tag{13}$$

since $\tilde{\not{D}}(G_\mu^{(\sigma)})$ is related to $\not{D}(G_\mu^{(\sigma)})$ by a similarity transformation defined by the unitary matrix $\Omega(x)$ (see Eq. (8)). This result is directly related to the gauge invariance of the Jacobian $J[G_\mu]$, i.e. only the axial SU(2) matrix $V(x)$ contributes to $J[G_\mu]$.

In order to evaluate $DET(\not{D}(G_\mu^{(\sigma)}))$ we proceed as in ([3], [5]). Using the proper-time method to define the functional determinant and making use of the relation:

$$\frac{d}{d\sigma}\not{D}(G_\mu^{(\sigma)}) = \gamma^5\vec{\phi}.\vec{\tau}\,\not{D}(G_\mu^{(\sigma)}) + \not{D}(G_\mu^{(\sigma)})\gamma^5\vec{\phi}.\vec{\tau} \tag{14}$$

we get the following ordinary differential equation

$$\frac{d}{d\sigma}\left(\mathrm{LOG\ DET}(\not{D}(G_\mu^{(\sigma)}))^2\right) = \lim_{\varepsilon\to 0^+} 4\int d^2x \mathrm{Tr}^{(C,D)}$$

$$(\gamma^5(\vec{\phi}.\vec{\tau})(x)\langle x|e^{-\varepsilon(\not{D}(G_\mu^{(\sigma)})^2)}|x\rangle \tag{15}$$

where $\mathrm{Tr}^{(C,D)}$ denotes the trace over the Dirac and the color indices.

The asymptotic expansion of the operator

$$\langle x|e^{-\varepsilon(\not{D}(G_\mu^{(\sigma)}))^2}|x\rangle = \langle|e^{-\varepsilon\left\{(-(\partial_\mu - iG_\mu^{(\sigma)})^2 + \frac{i}{2}\varepsilon_{\mu\nu}\gamma_5 F_{\mu\nu}(-iG_\mu^{(\sigma)})\right\}}|x\rangle$$

is tabulated ([6]):

$$\lim_{\varepsilon\to 0^+}\langle x|e^{-\varepsilon(\not{D}(G_\mu^{(\sigma)})^2)}|x\rangle$$

$$= \lim_{\varepsilon\to 0^+}\frac{1}{4\pi\varepsilon}(1 + \varepsilon(\frac{iG_{\mu\nu}\gamma_5}{2} F_{\mu\nu}(-iG_\mu^{(\sigma)})))(x) \tag{16}$$

Substituting (16) into (15), we get the result

$$J[G_\mu] = e^{\frac{i}{2\pi}\varepsilon_{\mu\nu}\left\{\int_0^1 d\sigma\left(\int d^2x \mathrm{Tr}^{(c)}(\vec{\phi}.\vec{\tau}F_{\mu\nu}(-iG_\mu^{(\sigma)}))(x)\right)\right\}} \tag{17}$$

We remark that the result (17) coincides with the result obtained by Roskies ([1]), and so the σ integration can be done explicitly producing the expression:

$$J[G_\mu] = e^{-\frac{\varepsilon_{\mu\nu}}{\pi}\left\{\int d^2x(\vec{\phi}.\vec{F}_{\mu\nu}(G_\mu)(\frac{1}{|\vec{\phi}|.\mathrm{TANH}|\vec{\phi}|} - \frac{1}{\mathrm{SINH}^2|\vec{\phi}|}))(x)\right\}} \tag{18}$$

We also note that by considering the vector and axial components of the gauge field $G_\mu^{(\sigma)}$ ($G_\mu^{(\sigma)} = iV_\mu^{(\sigma)} + \varepsilon_{\mu\nu}A_\nu^{(\sigma)}$), we re-obtain the result established in Refs. ([3]) and ([5]):

Finally, the effective generating functional for the model where the fermion fields are decoupled from the gauge fields can be written

$$Z[\bar{J}_\mu,\beta,\bar{\beta}] = \int \mathcal{D}[G_\mu] e^{-\frac{1}{4g^2}\int Tr^{(c)}(F_{\mu\nu}^2)(x)d^2x} J[G_\mu]$$

$$e^{\int(\vec{J}_\mu\cdot\vec{G}_\mu)(x)d^2x} \left(\int \mathcal{D}\chi\mathcal{D}\bar{\chi} e^{-\{\int d^2x(\bar{\chi}(i\gamma_\mu\partial_\mu)\chi}\right.$$

$$\left. + \bar{\eta}(\Omega V)\chi + \bar{\chi}(V.\Omega^{-1})\eta)(x)\}\right) \quad (19)$$

We remark that we have to fix a gauge in (19). This gauge is not necessarily the Roskie's gauge ([1], [2]), which choice, will imply to consider $\Omega(x) = \mathbb{1}$ in (19).

From (18) we see that the analysis of the fermionic correlation functions are reduced to the computation of the interaction among the SL(2,$\mathbb{C}$) Wu-Yang factors (7) with the quantum average defined by the local effective gluonic action

$$S^{EFF}[G_\mu] = -\frac{1}{4g^2}\int d^2x\, Tr^{(C)}(F_{\mu\nu}^2)(x) + \ell g J[G_\mu] \quad (20)$$

For instance, the two point fermionic correlation function is given by

$$<\psi(x)\bar{\psi}(y)> = \frac{1}{2\pi}\gamma_\mu\frac{(X_\mu-Y_\mu)}{|X-y|^2}<W(x)W^{-1}(y)>_{EFF} \quad (21)$$

where $\langle\ \rangle_{EFF}$ is the quantum average defined by the action (20); $W((z1)) = W((z_0,z_1))$ is given by Eq. (7) and $\frac{1}{2\pi}\gamma_\mu\frac{(x_\mu-y_\mu)}{|x-y|^2}$ is the free fermion propagator.

In a forthcoming paper, we use the effective gluonic action (20) to analyse several phenomena in the two-dimensional massless quantum chromodynamics in the approach of Ref. [7].

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