NOTAS DE FÍSICA

Publicação subvencionada pelo CONSELHO NACIONAL DE PESQUISAS

VOLUME II

Nº 8

EFFECT OF THE FINITE SIZE OF THE NUCLEUS ON μ-PAIR PRODUCTION BY GAMMA RAYS *

by

GEORGE H. RAWITSCHER

CENTRO BRASILEIRO DE PESQUISAS FÍSICAS

Av. Wenceslau Braz, 71

RIO DE JANEIRO

1955

Notas de Física, Vol. II, Nº 8

EFFECT OF THE FINITE SIZE OF THE NUCLEUS ON μ - PAIR PRODUCTION BY GAMMA RAYS *

George H. Rawitscher **
Centro Brasileiro de Pesquisas Físicas
Rio de Janeiro, D.F.

(September 1, 1955)

The cross section for μ-pair production by gamma rays is calculated using the Bethe-Heitler formula taking into account the finite nuclear size. The cross section is shown to be considerably smaller than that for a point charge nucleus.

INTRODUCTION

In recent years, experiments on the pair production of μ-mesons by gamma rays have been performed[1] with the hope of further establishing the nature of the μ-meson. The pair cross section is so small, however, that to date it has been possible to

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* Submitted for publication to The Physical Review.

** At Stanford University where this work was done under a Fellowship of the Conselho Nacional de Pesquisas

determine only the upper limits for its value. These seem to indicate that nuclear forces do not play a significant role in the interaction of μ-mesons with nuclei. Estimates given by Hough[2] and based on purely electromagnetic interaction give a value for the cross section that is about 20 times smaller than the most recently determined upper limit.[1]

Experiments now in progress at Stanford[3] are bringing the upper limit of the pair cross section close to Hough's estimate.[4] These experiments attempt to measure the cross section

$$d^2\sigma/d\Omega\, dE \qquad (1)$$

for obtaining one of the μ-mesons of the pair in a given solid angle $d\Omega$ and with an energy between E and $E + dE$.

It is the purpose of this paper to present the results of a calculation (in Born approximation) of the cross section (1) on the basis of a purely electromagnetic interaction of the μ-meson with the nucleus. This calculation consists in treating the μ-meson as a heavy, spin-1/2, Dirac particle, and the nucleus as a static distribution of protons. The incoherent effects of the individual protons, the excitation of higher nuclear energy states, and nuclear recoil energies are neglected.[5]

Under these assumptions, the differential cross section for pair production is given, in Born approximation, by the Bethe-Heitler formula.[6] Because of the larger rest mass μ of the μ-meson, the μ-pair cross section differs from the electron result by a mass scaling factor $\sim (1/207)^2$, and by a nuclear form factor. The nuclear form factor arises because for μ-pair production the recoil momentum q transferred to the nucleus can be so large that the associated wavelength

$$\lambda\!\!\!^{-}_q = \hbar / q = (1.685 \times 10^{-13}\ \text{cm}) / (q/\mu c),$$

becomes smaller than the nuclear radius. In this case the matrix element describing the interaction of the pair with the electric field of the extended nucleus becomes smaller than the point charge matrix element because the regions of space that most contribute to it have dimensions smaller than $\lambda\!\!\!^{-}_q$.

This reduction of the point charge matrix element, which can also be interpreted as due to the interference of the different coherent regions of the nucleus, is described by a nuclear form factor f given by[2]

$$f\,(q,A) = \int_0^{\infty} \rho\,(r)\, e^{iq \cdot r/h}\, d^3r\ ; \quad \int_0^{\infty} \rho\, d^3r = 1, \tag{2}$$

where ρ is proportional to the charge density of the nucleus of mass number A. The cross section is obtained by multiphying the point charge cross section by f^2.

In electron-pair formation, the deviation of the nuclear form factor from unity is never appreciable, because most of the contributions to the integrated cross section come from small recoil momenta which are much smaller for electrons than for μ-pairs; for example, the smallest possible values for q near threshold are

$\sim (0.96 \times 10^{-2})\ \mu c$ for electron pairs, and $2.0\ \mu c$ for meson pairs. This corresponds to $\lambda\!\!\!^{-}_q$ equal to 1.93×10^{-11} cm and 0.93×10^{-13} cm, respectively, and to f^2 for $A = 27$ equal to ≈ 1 and 0.014, respectively.[7] As the photon energy increases, the minimum possible recoil momentum decreases, and the influence of the form factor becomes less pronounced.

The actual steps of the calculation of the cross section

(1) are indicated in the Appendix. The procedure consists in integrating the Bethe-Heitler formula over the variables of one of the mesons of the pair. This is most conveniently accomplished by choosing q as one of the two variables of integration since the form factor depends on only q and A. The form factor used is that corresponding to a uniform charge distribution of radius

$$R_0 = 1.20 \times 10^{-13} \times A^{1/3} \text{ cm.} \tag{3}$$

The integration over q was performed numerically with the aid of an IBM-Card-Programmed Computer to an accuracy of 5 percent. To this 5-percent inaccuracy must be added the error due to the use of the Born approximation, which is $\sim (Z/137)^2 (c/v)^2$.[8,9]

The error due to neglecting the excitation of the nucleus can be estimated from a sum rule over all the excited states of the nucleus, which gives

$$Z^2 f^2 \sigma_p + Z \sigma_p (1 - f^2) \tag{4}$$

for an upper limit of the cross section. The first term is the elastic coherent contribution calculated in this paper, and the second represents an upper limit on the inelastic contribution. The relative importance of the second increases as f decreases, and the cross section approaches that of Z incoherent protons, $Z \sigma_p$.

RESULTS

In order to bring out clearly the effect of the form factor, a factor T is defined by the relation

$$\frac{d^2\sigma}{d\Omega\, dE} = \left(\frac{m}{\mu}\right)^2 \times \frac{1}{2\pi} \times \bar{\phi} \times \frac{1}{\mu c^2} T, \tag{5}$$

where m is the electron rest mass; μ is taken equal to 207m, and μc^2 = 105.77 Mev; and

$$\bar{\phi} = (Z^2/137)r_0^2 = Z^2 \times 5.793 \times 10^{-28} cm^2,$$

where r_0 is the classical electron radius. Figure 1 shows T plotted as a function of the incoming photon energy k for a fixed angle θ of 10° between the meson and the photon. The families of curves starting at the threshold k = 312.6 Mev and k = 463.4 Mev correspond to fixed meson kinetic energies of 101.0 and 251.9 Mev, respectively. Each curve of a family corresponds to a value of the mass number A, as indicated in the figure. For comparison, the curve corresponding to a point charge nucleus is also drawn. The ordinate, when multiplied by

$$\phi \times (1/207)^2 \times (1/2\pi) \times (1/105.77)$$

$$= Z^2 \times 2.0346 \times 10^{-35} \text{ cm}^2/\text{Mev} \qquad (6)$$

from Eq. (5), represents the cross section (1) in cm^2/sterad/Mev.

Figure 2 shows the cross section (1) in cm^2/sterad/Mev/Q, where Q is the effective photon; this cross section per photon is obtained as follows: Eq. (5) is integrated over the bremsstrahlung spectrum $N(k) = N(k,k_{max},Z)$ from 0 to k_{max} = 500 Mev,[10] and divided by

$$(1/k_{max.}) \int_0^{*} k\, N(k)\, dk .$$

The quantity $N(k,k_{max},Z)$ is the thin-target spectrum[11,12] taken for Z = 13. The angle θ is again taken to be 10°. The ordinate shows the resulting cross section <u>vs</u> the atomic charge Z at the abscissa. The three curves indicated are calculated for meson kinetic energies T_μ of 101.0, 201.7 and 251.9 Mev, respectively.

Figure 3 shows the dependence of the factor

$$T' = \int_0^{500} T\, N(k)\, dk \Bigg/ \left(\frac{1}{k_{max.}}\right) \int_0^{500} kN(k)\, dk \qquad (7)$$

on θ for different elements, where T is defined in Eq.(5), and where $N(k,k_{max},Z)$ has the same meaning as for the calculation of Fig. 2. When multiplied by Eq. (6), the ordinate is a cross section in cm^2 /sterad/Mev/Q . This graph shows the marked increase of the form-factor effect with the angle. The point-charge case is again indicated for comparison.

For a check of the method, the point charge curve for T as a function of θ was integrated over the solid angle $d\Omega$ of the μ-meson. The result was in satisfactory agreement with the known values of $\phi(E)\, dE$.[13]

The author wishes to express his great appreciation for the orientation and encouragement received from Dr. D. R. Yennie throughout this work. He is also indebted to Professors W. K. H. Panofsky and L. I. Schiff for many valuable discussions.

APPENDIX

The Born approximation for the pair-production differential cross section is given by the Bethe-Heitler formula,[14] and can be written in the form

$$d\sigma = -\frac{d\Omega_+}{2\pi}\, dE_+\, d\Omega_-\, \frac{1}{2\pi}\, Z \left(\frac{1}{207}\right)^2 \alpha \qquad (8)$$

$$\frac{P_+P_-}{k^3}\, \frac{1}{q^4} \left\{ H \right\}\, F^2(q,A) \;,$$

where k, E_+ and E_- are the energies of the incident photon and the emerging positive and negative mesons, respectively, measured in units of the meson rest energy μc^2; $\vec{p}_+$, $\vec{p}_-$ and q are the positive and negative meson momenta and the nuclear recoil momentum, respectively, measured in units of μc ; $d\Omega_\pm$ refers to the solid angle into which the $\mu^\pm$-meson is emitted; $\{H\}$ represents the expression appearing in brackets in the formula given by Heitler[14] — it is a complicated function of the now-dimensionless meson and photon variables; f(q,A) is the form factor given by Eq. (2) describing the effect of the finite size of the nucleus on the pair-production cross section. Taking for ρ a uniform charge density for $r < R_0$ and zero for $r \geq R_0$, where the value of R_0 is given by Eq. (3), the expression for the form factor as given by (2) becomes

$$f(q,A) = (3/q'^3)(\sin q' - q'\cos q'),$$

where $q' = q(\mu c/\hbar)R_0 = q \times 0.6433 \times A^{1/3}$. For a point charge nucleus, or for zero recoil momentum, f(q,A) is equal to unity; otherwise, it is less than unity.

The cross section (1) is now obtained by integrating Eq. (8) over the variables of one of the mesons, say the negative one, constrained by energy momentum conservation relations that can be written in the form

$$k = E_+ + E_-,$$
$$\vec{q} = \vec{a} - \vec{p}_-, \tag{9}$$

where $\vec{a}$ is the fixed vector given by $\vec{a} = \vec{k} - \vec{p}_+$. Equation (9) shows that for a given value of E_+ the magnitude of p_- is fixed, and for each of its directions the corresponding value of q is determined.

The recoil momentum can very between $q_{min} = a - p_-$ and

$q_{max} = a + p_-$, where $\vec{q}$ connects a point O to any point on a sphere of radius p_- centered at $O + \vec{a}$. A new coordinate system is now chosen with the z axis in the direction of $\vec{a}$, and $\vec{p}_+$ and $\vec{k}$ in the x,z plane. In this system $\vec{q}$ and $\vec{p}_-$ have the same azimuthal angle φ_-', and the polar angle θ_-' of $\vec{p}_-$ can be expressed entirely in terms of the magnitude of q, by making use of the relation

$$a^2 - ap_- \cos \theta_-' = k(E_+ - p_+ \cos \theta_+) + (q^2/2) ; \quad (10)$$

$\theta_\pm$ and $\varphi_\pm$ represent the polar and azimuthal coordinates of the direction of emergence of the $\mu^\pm$-meson, measured in a frame in which the photon beam is in the z direction.

Changing the variables of integration from θ_- and φ_- to q and φ_-', we can then write

$$d\Omega_- = d(\cos \theta_-') \, d\varphi_-' = (q/ap_-) \, dq \, d\varphi_-' .$$

The quantity $\{H\}$ is now expressed as a function of the new variables with the help of relations of the type (10), and the integration over φ_-' is performed analytically:

$$d^2\sigma = \frac{d\Omega_+}{2\pi} dE_+ \, \bar{\phi} \left(\frac{1}{207}\right)^2 \times \frac{p_+}{ak^3} \int_{q_{min.}}^{q_{max.}} R \frac{2}{q^3} \, dq \times f^2(q) \quad (11)$$

where

$$R = \frac{1}{4\pi} \int_0^{2\pi} \{H\} \, d\varphi_-'$$

$$= M + N(q^2/2) + \frac{2a/\alpha}{X^{1/2}} \left\{ \frac{q^2}{2} \left(\frac{q^2}{2} + n \right) + m + \frac{\alpha\beta\mu^2}{2} \, \frac{(q^2/2) \, [(q^2/2) + p] + s}{X} \right\} , \quad (12)$$

and where

$$X = (q^2/2)\,[\,(q^2/2) + r\,] + s\,,$$
$$r = 2(E_+p_+\cos\theta_+ - p_+^{\,2})\,,$$
$$s = \alpha^2 p_+^{\,2},$$
$$a = |\vec{k} - \vec{p}_+|\,,$$
$$\alpha = E_+ - p_+\cos\theta_+\,,$$
$$\beta = k - p_+\cos\theta_+\,,$$

$$p = -2E_+^{\,2} + k\alpha\,[\,1 - (a^2/k\beta)\,]\,,$$
$$g = -2E_+^{\,2}k\alpha\,[\,1 - (a^2/k\beta)\,]\,,$$

$$m = (k^2\alpha^2/2) + 2E_+E_-\,\mu^2\,,$$
$$n = -k^2 - p_+^{\,2} + E_+E_- + kp_+\cos\theta_+\,,$$
$$M = (2E_-^{\,2}p_+^{\,2}\sin^2\theta_+/\alpha^2) + 2E_+^{\,2} + k^2[\,1 + (k\beta/a^2)\,] - (4E_+E_-\beta/\alpha)\,,$$
$$N = 1 - (p_+^{\,2}\sin^2\theta_+/\alpha^2) + (k^2\beta/a^2\alpha) - (2\beta/\alpha)\,.$$

For the case of a point nucleus, R/q^3 expresses the probability of occurrence of each of the different possible recoil momenta that contribute to the integral cross section. For an extended nucleus R/q^3 is multiplied by the form factor, which therefore depresses the contributions of the higher momenta. For example, take $k = 4.9$, $p_+ = 2.2$, and $\theta_+ = 10^\circ$. Then q varies between 0.48 and 5.02, which, for the case of $A = 27$, corresponds to a (form factor)^2 of 0.84 and 0.03, respectively. The quantity $(2p_+/ak^3)\,(R/q^3)$ goes from 0.27 as $q = 0.48$, to a sharp maximum of 2.9 at $q \sim 0.75$ at which $f^2\,(0.75) = 0.65$, and then falls off passing through the values 1.46, 0.4, 0.1 and ~ 0 for q equal to 1, 2, 3, and 5, respectively. For these values of q, f^2 is equal to 0.47, 0.01, 0.008, and ~ 0, respectively.

The final integration over $q_\perp$ was performed numerically for various values of p_+ and θ_+ suitable for the Stanford experiment, and for various values of k and A. The results are shown in Figs. 1-3.

1. Feld, Julian, Odian, Osborne and Wattenberg, Phys. Rev. 96, 1386 (1954); further references are given in this paper.
2. P. V. C. Hough, Phys. Rev. 74, 80 (1948); the approximations used in this paper are not applicable for the energy region near threshold for which the present calculations are performed.
3. Masek, Lazarus and Panofsky, Bull. Am. Phys. Soc., Washington Meeting, April 28, 1955, Abstract S1; W. K. H. Panofsky (private communication).
4. As suggested in Ref. 2, an estimate of the effect of the nuclear form factor can be obtained by multiplying the integrated point charge result by f^2 taken at the most probable value. For the ranges of the variable considered in this paper, this procedure gives a result too large by a factor of 2.
5. For example, the recoil energies occurring for a 500 - Mev photon vary between 0.1 and 16 Mev for beryllium, and between 0.05 and 5 Mev for aluminum; the most probable values are approximately 0.6 and 0.2 Mev. respectively. In the case of a single proton, the minimum recoil energy is 1.2 Mev; the most probable values occur near 6 Mev.
6. See Eq. (8) in the Appendix.
7. More precisely, $f^2 = 1 - (6.9 \times 10^{-5})$ for electron pairs and A = 27.
8. G. K. Horton and E. Phibbs, Phys. Rev. 96, 1066 (1954); H. A. Bethe and L. C. Maximon, Phys. Rev. 93, 768 (1954).
9. For the case of electron-pair production, the actual correction to the Born approximation is found to be close to $0.29 \times (Z/137)^2$, according to J. L.

Lawson, Phys. Rev. 75, 433 (1949); DeWire, Ashkin and Beach, Phys. Rev. 83, 505 (1951); and A. I. Berman, Phys. Rev. 90, 215 (1953).

10. The author is indebted to the staff of the High-Energy Physics Laboratory, Stanford University, for furnishing graphs of $N(k, k_{max}, Z)$, and to Mrs. I.E. Machein for performing the graphical integrations.

11. H. A. Bethe and J. Ashkin, *Experimental Nuclear Physics*, Vol. I, edited by E. Segrè (John Wiley and Sons, Inc., New York, 1953), Eqs.(56)-(58), p. 260.

12. Taking dk/k as the bremsstrahlung spectrum, the results are 20-30 percent too high.

13. Bethe and Ashkin, *op. cit.*, Fig. 38, p. 328.

14. W. Heitler, *Quantum Theory of Radiation*, 2nd ed. (Clarendon Press, Oxford, 1954), p. 257, Eq. (6).

FIGURE CAPTIONS

FIG. 1. Plot of T *vs* k. The factor T is proportional to the differential cross section $d\sigma/d\Omega\,dE$, as defined by Eq.(5), and k is the incident photon energy measured in Mev. The two families of curves starting at k = 312.6 and 463.4 Mev correspond to kinetic energies of the observed meson of 101.0 and 251.9 Mev, respectively. The different curves in each family correspond to nuclei of different mass numbers, as indicated. The angle between the observed μ-meson and the photon beam is 10° in each case.

FIG. 2. The differential cross section $d\sigma/d\Omega\,dE$ folded into the bremsstrahlung spectrum of maximum photon energy 500 Mev, as described in the text, plotted *vs* the charge number Z. The ordinate is in units cm /sterad/Mev/ . In each curve, T_{μ} represents the kinetic energy in Mev of the fixed outcoming meson whose direction with respect to the incoming photon beam is 10° in each case.

FIG. 3. Dependence of T' on the angle Θ of the observed meson relative to the incoming photon; T' is the factor T folded into the bremsstrahlung spectrum of maximum energy 500 Mev, as described in the text, where T is defined by Eq. (5). The three curves shown refer to the same kinetic energy T_{μ} = 201.7 Mev of the observed μ-meson; the sizes of the recoiling nuclei correspond to zero radius, and to the sizes obtained from Eq. (3) by taking the mass number A equal to 27 and 63, respectively.

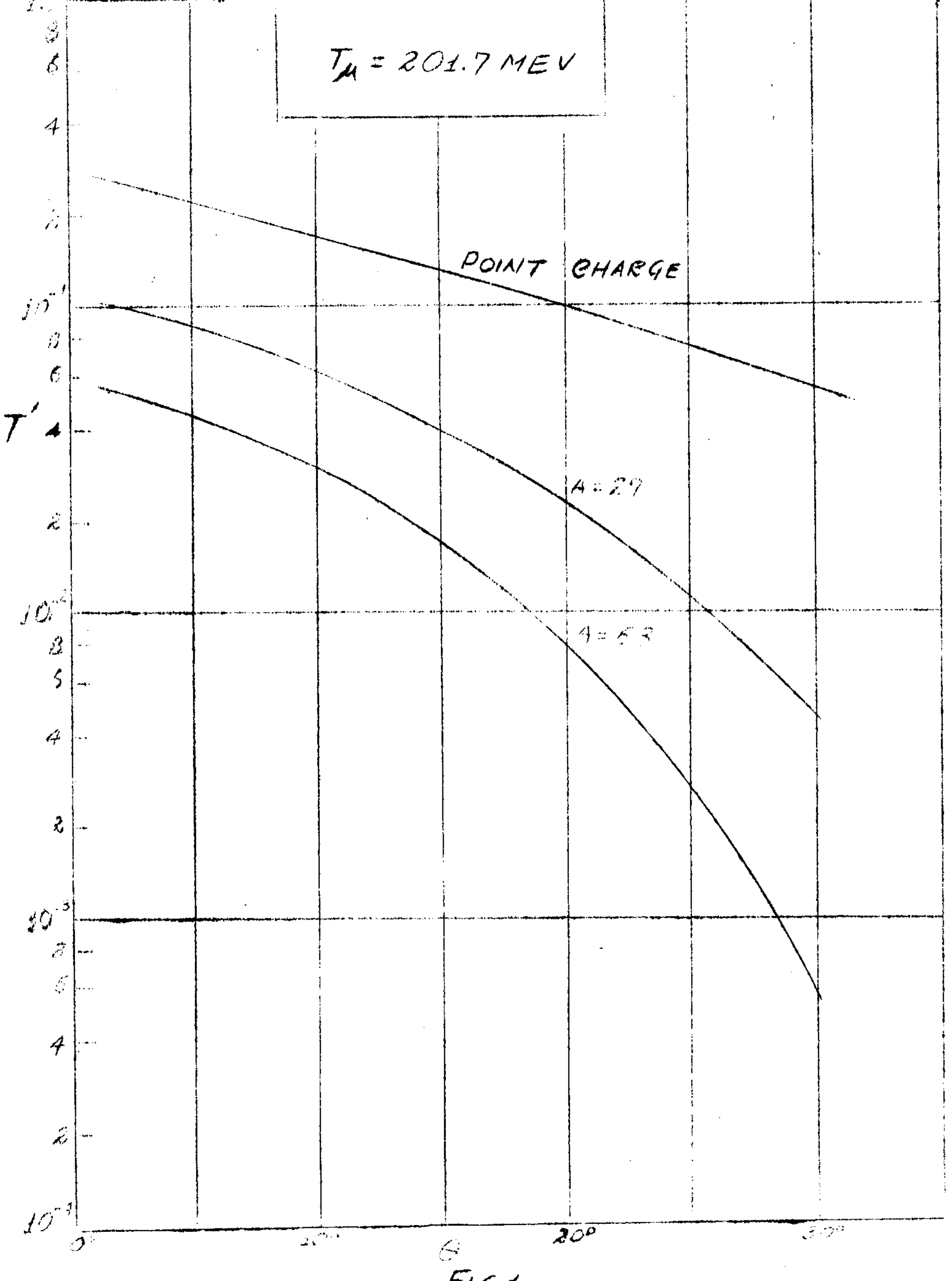
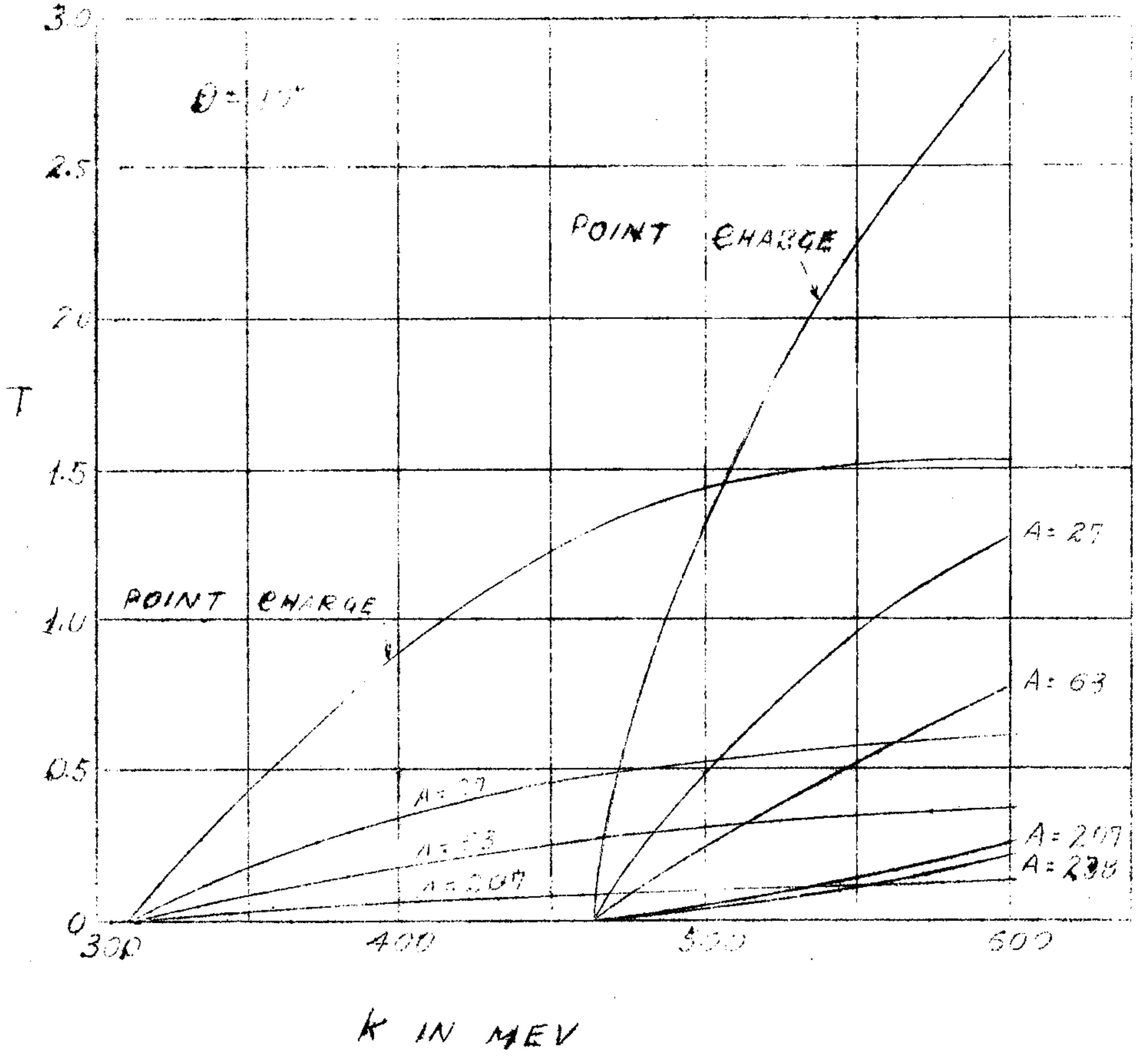
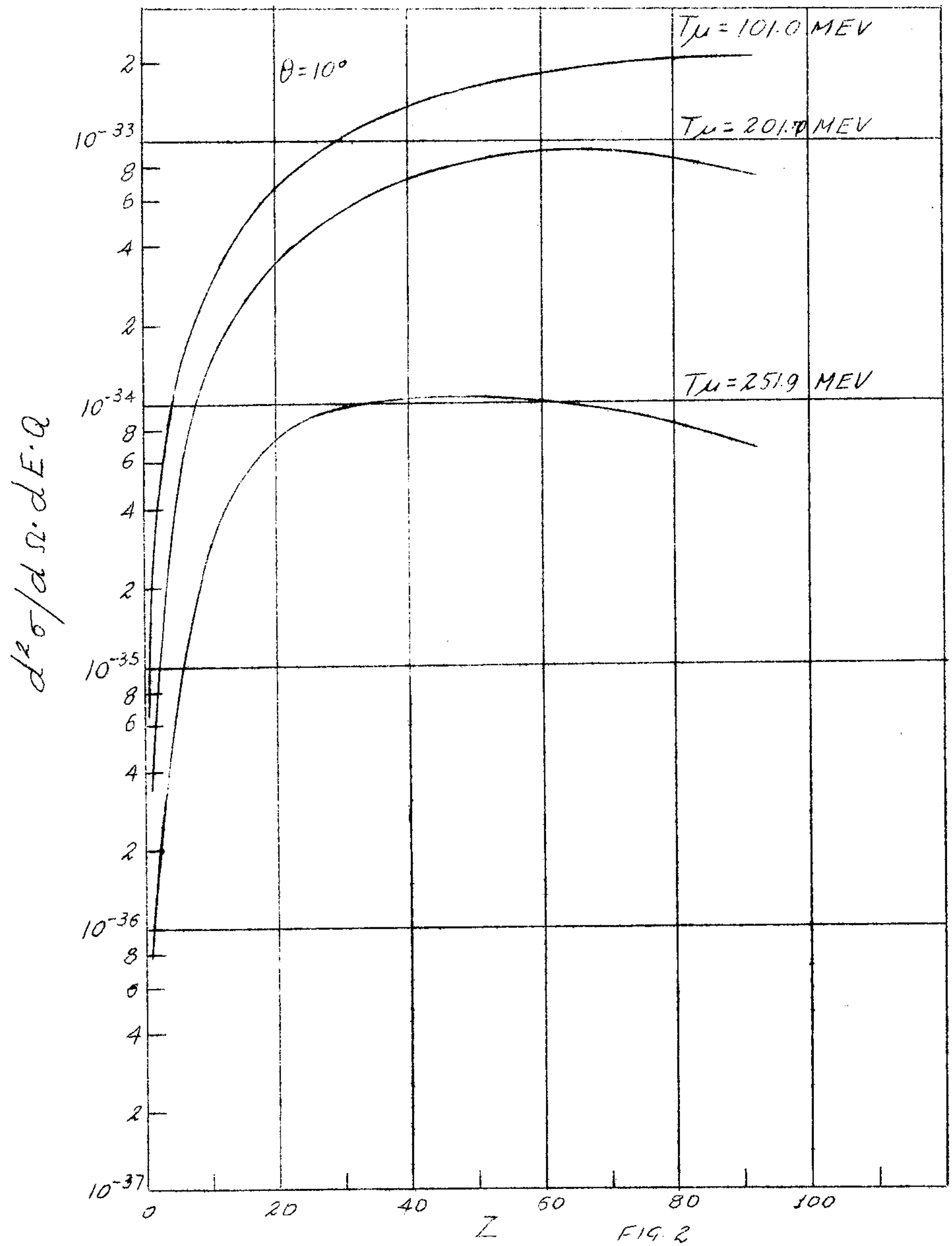


FIG. 1



θ = 11°
T
POINT CHARGE
POINT CHARGE
A = 27
A = 63
A = 207
A = 238
0
0.5
1.0
1.5
2.0
2.5
3.0
300
400
500
600
K IN MEV

Fig. 3



$T_\mu = 101.0$ MEV
$\theta = 10°$
$T_\mu = 201.7$ MEV
$T_\mu = 251.9$ MEV
$d^2\sigma / d\Omega \cdot dE \cdot Q$
10^{-33}
10^{-34}
10^{-35}
10^{-36}
10^{-37}
0
20
40
60
80
100
Z

FIG. 2