Introduction to Dualities in Gauge Theories[1]

Marco A. C. Kneipp[2]

Centro Brasileiro de Pesquisas Físicas (CBPF),

Rua Dr. Xavier Sigaud,150

22290-180 Rio de Janeiro, Brazil

January 2000

Abstract

These notes present a pedagogical introduction to magnetic monopoles, supersymmetry and dualities in gauge theories. They are based on lectures given at the "X Jorge André Swieca Summer School on Particles and Fields".

Key-words: Solitons; Supersymmetry; Dualities.

[1]Published at the "Proceedings of the X Jorge André Swieca Summer School", pages 291-317. Editors: J. C. A. Barata, M. Begalli and R. Rosenfeld, World Scientific Publishing

[2]e-mail:`kneipp@cbpf.br`

Contents

1 Introduction

These notes are based on a series of lectures given at X Jorge André Swieca Summer School on Particles and Fields in 1999. It aims to give a very elementary introduction to dualities in field theories[3]. This subject have been quite a lot activity in the last few years.

[3]We recommned also some other nice review papers [3]-[9].

In summary we could say that dualities relate quantum field theories (or string theories) at different values of the coupling constant. Sometimes it can relate a theory at strong coupling to another (or same) theory at weak coupling. So, it can open the possibility of calculating strong coupling effects by mapping it to a weak coupling problem. Therefore, it can be relevant to understand quark confinement.

Duality was first observed in the end of the last century in Maxwell theory in the vacuum. In order to preserve duality in the presence of matter it is necessary to introduce magnetic monopoles as we will see in section 2. Another example of duality happens between sine-Gordon theory described by Lagrangian

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{m^2}{\beta^2}\left(\cos\beta\phi - 1\right) ,$$

and massive Thirring model described by

$$L = \bar{\psi}\left(i\gamma_\mu\partial^\mu + m\right)\psi - \frac{g}{2}\bar{\psi}\gamma_\mu\psi\bar{\psi}\gamma^\mu\psi .$$

Sine-Gordon has a scalar particle and a soliton, and Thirring model has a spinor particle and a bound state. The two theories are equivalent at the quantum level (or dual) when,

$$\frac{\beta^2\hbar}{4\pi} = \frac{1}{1 + g\hbar/\pi} ,$$

with the soliton and scalar particles of sine-Gordon mapped to the spinor particle and bound state of Thirring respectively. We see that the strong coupling of one theory corresponds to the weak coupling of the other theory. This quantum equivalence was proven by Coleman[1] and Mandelstam[2].

Later, from the spectrum of masses of $SU(2)$ Yang-Mills-Higgs theory was conjectured by Montonen and Olive [21] a duality between a theory with coupling e and another with coupling $e' = 4\pi/e\hbar$, as will be explained in section 3. In this duality, the monopoles of the first theory would be mapped to the gauge particles of the second and vice-versa. However this beautiful idea had some problems. The first was that the mass formula was a classical formula and there was no evidence that it would keep the same form at the quantum level. The second problem was how to have monopoles with spin 1 in order to map to the gauge particles with have spin 1.

We can solve this two problems by adding supersymmetry, more in particular, when we have $N = 4$ supersymmetry, as will by explained in sections 4 and 5. The reason is because, only in this case the monopole would belong to a supermultiplet like the one of the gauge particles, containing a state with spin 1. Moreover, in $N = 4$ Super Yang-Mills(SYM), the β-function vanishes and the mass formula have no quantum correction, solving the second problem. Differently from $N = 4$ SYM which have exact duality, Seiberg and Witten proposed that $N = 2$ SYM would have a kind of residual duality in its effective theory, but unhappily this topic will not be discussed here.

Finally in section 6 we discuss a consistency test for the exact duality conjecture in $N = 4$ SYM proposed by A. Sen. In this quite non trivial test one gets results in agreement with the conjecture.

Although exact duality is not realized in nature, the more realistic theories could be broken versions of the exact theory, having some residue of the duality, which could be important for explaining quark confinement for example.

2 Duality in Maxwell's Theory

2.1 Introduction

Maxwell's equations in the vacuum are given by

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad\qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad\qquad \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = 0$$

These equations are invariant under the electromagnetic duality transformation $(\vec{E}, \vec{B}) \rightarrow (\vec{B}, -\vec{E})$. Indeed, one can enlarge this duality group. In order to see this, it is convenient to write Maxwell equations in a manifestly Lorentz covariant form, by introducing the field-strength $F^{\mu\nu}$given by[4]

$$F^{i0} = E^i\,, \quad F^{ij} = -\epsilon^{ijk} B^k. \tag{1}$$

[4]In these lectures it will be adopted the metric signature $(+,-,-,-)$, $\epsilon^{0123} = 1$, it will be set c=1 and $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

Defining ${}^*F^{\mu\nu} = 1/2\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$, it follows that

$$ {}^*F^{i0} = B^i \,. \tag{2} $$

Then, Maxwell's equations become

$$ \partial_\mu F^{\mu\nu} = 0 \;, \qquad \partial_\mu {}^*F^{\mu\nu} = 0 \tag{3} $$

These two real equations can be combined in a single complex equation

$$ \partial_\mu \left(F^{\mu\nu} + i\, {}^*F^{\mu\nu}\right) = 0 \tag{4} $$

It is easy to see that this equation is invariant under

$$ F^{\mu\nu} + i\, {}^*F^{\mu\nu} \rightarrow e^{i\varphi}\left(F^{\mu\nu} + i\, {}^*F^{\mu\nu}\right) \tag{5} $$

where φ is a constant phase. In terms of the electric and magnetic fields,

$$ E^i + iB^i \rightarrow e^{i\varphi}\left(E^i + iB^i\right) \tag{6} $$

$$ \Rightarrow \left\{ \begin{array}{lcl} E^i & \rightarrow & \cos\varphi E^i - \sin\varphi B^i \\ B^i & \rightarrow & \sin\varphi E^i + \cos\varphi B^i \end{array} \right. $$

Taking $\varphi = -\pi/2$, it gives the previous particular transformation $(\overrightarrow{E}, \overrightarrow{B}) \rightarrow (\overrightarrow{B}, -\overrightarrow{E})$.

This beautiful duality transformation is lost when we consider Maxwell's equation in the presence of matter,

$$ \partial_\mu F^{\mu\nu} = j^\nu_{el} \;, \qquad \partial_\mu {}^*F^{\mu\nu} = 0 \;. \tag{7} $$

These equations are clearly not invariant under (5). In order to restore the symmetry in the presence of matter, Dirac(1931)[10] postulated the existence of particles with magnetic charges and called them magnetic monopoles. In 1969, Schwinger[11] and Zwanziger[12] improved Dirac's idea, considering the possibility of particles having both electric and magnetic charges, and called them dyons. In either case, Maxwell's equations read

$$ \partial_\mu F^{\mu\nu} = j^\nu_{el} \;, \qquad \partial_\mu {}^*F^{\mu\nu} = j^\nu_{mag} \tag{8} $$

where j^ν_{mag} is the magnetic current. As before, these equations can be combined as

$$ \partial_\mu \left(F^{\mu\nu} + i\, {}^*F^{\mu\nu}\right) = \left(j^\nu_{el} + i\, j^\nu_{mag}\right) \tag{9} $$

and it is invariant under (5) if the currents transform as

$$j^{\nu}_{el} + i\, j^{\nu}_{mag} \rightarrow e^{i\varphi}\left(j^{\nu}_{el} + i\, j^{\nu}_{mag}\right) . \tag{10}$$

If the currents result from point particles, each one with electric *and* magnetic charge $(q_a\, ,\, g_a)$, we must have that

$$q_a + i\, g_a \rightarrow e^{i\varphi}\left(q_a + i\, g_a\right) . \tag{11}$$

2.2 Dirac Quantization Condition

In electromagnetism without magnetic monopoles,

$$\partial_\mu {}^{*}F^{\mu\nu} = 0, \tag{12}$$

which implies that $F^{\mu\nu}$ can be written as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{13}$$

where A_μ is a well defined vector function in all spacetime. If A_μ solves (13), then

$$A'_\mu = A_\mu + \partial_\mu \alpha \tag{14}$$

will also satisfy (13)(i.e., it will give the same magnetic field $\overrightarrow{B}$).

The vector potential A_μ plays a central role in the quantum theory: a particle with mass m and electric charge q satisfies the Schrodinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}\left(i\hbar\overrightarrow{\nabla} - q\overrightarrow{A}\right)^2\psi + qA_0\psi \tag{15}$$

In order that this equation to be invariant under (14), the wave-function must transform as

$$\psi' = e^{-\frac{iq\alpha}{\hbar}}\psi \tag{16}$$

Consider now a magnetic monopole at the origin. It will produce a magnetic field

$$\overrightarrow{B}_m = \frac{g}{4\pi r^3}\overrightarrow{r} \quad \Rightarrow \quad \overrightarrow{\nabla}\cdot\overrightarrow{B}_m = g\delta^3(\overrightarrow{r}) \tag{17}$$

Since $\overrightarrow{\nabla} \cdot \overrightarrow{B} \neq 0$ we cannot have an $\overrightarrow{A}$ that is regular *for all* x^μ, satisfying (13) and (17). However, we can use the ambiguity (14) and use vector potential in one part of space and another vector potential in the other part of space.

Let us work in spherical coordinates and take

$$\overrightarrow{A}_N = \frac{g}{4\pi i}\frac{(1-\cos\theta)}{\sin\theta}\widehat{e}_\theta \tag{18}$$

which satisfies $\overrightarrow{\nabla} \times \overrightarrow{A}_N = \overrightarrow{B}_m$. It can be seen that $\overrightarrow{A}_N$ is well defined in all space, except where $\theta = \pi$. Now let us take

$$\overrightarrow{A}_S = -\frac{g}{4\pi i}\frac{(1+\cos\theta)}{\sin\theta}\widehat{e}_\theta \tag{19}$$

which is well defined on all space except $\theta = 0$. Since

$$\overrightarrow{A}_S = \overrightarrow{A}_N + \overrightarrow{\nabla}\alpha \ , \ \ \alpha(\phi) = -\frac{g}{2\pi}\phi \, , \tag{20}$$

in the region where both are well defined, we can conclude that $\overrightarrow{\nabla} \times \overrightarrow{A}_S = \overrightarrow{B}_m$. Therefore $\overrightarrow{B}_m$ can be written in the form (13), with $\overrightarrow{A}_N$ in the north hemisphere and $\overrightarrow{A}_S$ in the south hemisphere. In each hemisphere we shall have a wave-function ψ_N and ψ_S, which will differ by a phase (16). We know from quantum mechanics that wave-functions must single-valued. But from (16) we can conclude that ψ_N and ψ_S can only be simultaneously single-valued if

$$e^{-\frac{iq\alpha(\phi)}{\hbar}} = e^{-\frac{iq\alpha(\phi+2\pi)}{\hbar}} .$$

From (20) we see that this implies that

$$qg = 2\pi n\hbar \quad n \in Z \, . \tag{21}$$

This can be extended (see for instance [3])to the situation with various electric charges and magnetic monopoles, which results in the condition

$$q_i g_j = 2\pi n_{ij}\hbar \quad n_{ij} \in Z \, . \tag{22}$$

This quantization condition has a very important consequence: suppose that at least one magnetic monopole exist in the whole Universe, with a magnetic charge $g = g_0$. Then, condition (22), would imply that all particles would have electric charges of the form

$$q_i = n_i q_0 \ \text{ where } \ q_0 = \frac{2\pi\hbar}{g_0} \, n_i \in Z \, , \tag{23}$$

i.e., the electric charges would be integer multiples of a fundamental charge[5] q_0.

If one considers the more general case with not just magnetic monopoles but also dyons, it results the Dirac-Schwinger-Zwanziger quantization condition[10][11][12]

$$q_i g_j - q_j g_i = 2\pi n_{ij} \hbar \tag{24}$$

A good property of this more general condition is that it is invariant under the duality transformation (11). This can be easily seen by noting that this condition is the imaginary part of $(q_i + ig_i)(q_j + ig_j)^*$which is manifestly invariant under duality transformation [6].

3 Duality in Yang-Mills Theories

3.1 Yang-Mills Theories

As we know, electromagnetism is just a sector of the Weinberg-Salam Model which is built from Yang-Mills theory with Higgs mechanism. So, a natural question one could raise is if it is the existence of magnetic monopoles and/or dyons in Yang-Mills theories possible. In 1974, 't Hooft[13] and Polyakov[14] independently discovered that Yang-Mills theory with the gauge group $SU(2)$, with scalar fields in the adjoint representation(triplet) admits magnetic monopoles as solutions for the equations of motion. This result was extended to other gauge groups[15]. In order to keep the arguments general, we shall start by considering Yang-Mills with an arbitrary gauge group and the scalar field in the adjoint representation. There are two strong motivations to consider the adjoint representation: the first is because, in this case the gauge group is always broken to a group which contains a $U(1)$ factor. This factor can be identified as the electromagnetic $U(1)$, from which we can define electric and magnetic charges. The second reason is because this $U(1)$ factor is compact (i.e. isomorphic to the circle) and then it can be shown that these theories always possess magnetic monopole solutions. The action for this theory can be written as

$$S \;=\; \int d^4x \left(-\frac{1}{4} G^{\mu\nu} \cdot G_{\mu\nu} + \frac{1}{2} D^\mu \phi \cdot D_\mu \phi - V(\phi) \right) \tag{25}$$

[5]The same is true if one has various magnetic monopoles. For a detailed argument see [3].

[6]Note that Dirac's quantization condition is not invariant under the duality transformation (11), exept for $\phi = -\pi/2$.

$$V(\phi) = \frac{1}{4}\lambda\left(\phi^2 - a^2\right)^2 \tag{26}$$

where $A \cdot B \equiv A_a B_a$ and λ is assumed to be non-negative constant, a is a real number and

$$(D_\mu \phi)_a = \partial_\mu \phi_a - e f_{abc} W_{b\mu} \phi_c \tag{27}$$

$$G_a^{\mu\nu} = \partial^\mu W_a^\nu - \partial^\nu W_a^\mu - e f_{abc} W_b^\mu W_c^\nu \tag{28}$$

The equations of motion are

$$(D_\nu G^{\mu\nu})_a = -e f_{abc} \phi_b \left(D^\mu \phi\right)_c \tag{29}$$

$$(D^\mu D_\mu \phi)_a = -\lambda \phi_a \left(\phi^2 - a^2\right) \tag{30}$$

Further we have the Bianchi identity:

$$D_\mu {}^*G^{\mu\nu} = 0 \tag{31}$$

The associated symmetric energy-momentum tensor is[7]

$$\theta^{\mu\nu} = -G^{\mu\lambda} \cdot G_\lambda^\nu + D^\mu \phi \cdot D^\nu \phi - \eta^{\mu\nu} L \tag{32}$$

Analogously to the Maxwell case, the non-Abelian electric and magnetic fields are defined as

$$G_a^{i0} = E_a^i\,, \quad {}^*G^{i0} = B_a^i \tag{33}$$

Then the total energy can be written as

$$E = \int d^3x \theta^{00} = \int d^3x \left\{ \frac{1}{2}\left[\left(E_a^i\right)^2 + \left(B_a^i\right)^2 + \left(D^0\phi\right)_a^2 + \left(D^i\phi\right)_a^2\right] + V(\phi)\right\} \tag{34}$$

Note that $\theta^{00} \geq 0$ and vanishes if and only if

$$E_a^i = B_a^i = (D^\mu \phi)_a = V(\phi) = 0 \tag{35}$$

A field configuration which satisfies (35) everywhere, and therefore has total energy $E = 0$, is the vacuum configuration. The condition $V(\phi) = 0$ implies that $\phi^2 = a^2$.

[7] The best procedure to arrive to $\theta_{\mu\nu}$ is to couple the theory with a background metric $g_{\mu\nu}$. Then, $\theta_{\mu\nu}$ can be obtained by taking the variation of the action S with respect to $g_{\mu\nu}$, i.e. $\theta_{\mu\nu} \propto \delta S/\delta g_{\mu\nu}|_{g\to \text{flat}}$

We shall see presently that in Yang-Mills theories, the monopoles/dyons appear as solutions of the equations of motion with finite energy. So we shall now analyze the finite energy field configurations. In this case, the field configuration must satisfy (35) asymptotically, as $r \to \infty$. Therefore, *in this limit,* we shall consider $W_\mu^a \to 1/r$ and

$$(\phi)_a^2 = a^2 \ , \quad D^\mu \phi = 0 \tag{36}$$

It is this non-trivial asymptotic configuration for the scalar field that is responsible for the spontaneous symmetry breaking.

We shall define

$$F^{\mu\nu} = \frac{1}{a} G^{\mu\nu} \cdot \phi \tag{37}$$

This tensor has the important property that, at spatial infinity, it satisfies Maxwell's equations in vacuum (3), and therefore it is called the electromagnetic field-strength. To prove this result one has just to use the condition $D^\mu \phi = 0$, the scalar field equations of motion and the Bianchi identity. This result is a consequence from the fact we have mentioned before that spontaneous symmetry breaking of a gauge symmetry by a scalar in the adjoint representation is always[3] of the form $G \to K \otimes U(1)$ where G is the initial gauge symmetry, $K \otimes U(1)$ is the residual symmetry where the $U(1)$ factor is associated to the Maxwell sector of the theory. Given this definition of electromagnetic field-strength, the electric and magnetic charges are defined, as usual, as the surface integral

$$q \;=\; \int d^2 S_i \, E^i = \int d^2 S_i \, F^{i0} = \int d^2 S_i \, \frac{G^{i0} \cdot \phi}{a} \tag{38}$$

$$g \;=\; \int d^2 S_i \, B^i = \int d^2 S_i \, {}^*F^{i0} = \int d^2 S_i \, \frac{{}^*G^{i0} \cdot \phi}{a} \tag{39}$$

3.2 The Bogomol'nyi Bound on the Dyon Masses

Let's calculate the mass of an arbitrary finite energy solution. To do so, we shall first use the fact that in the rest frame $M = E$. Then, from (34), dropping some non-negative terms gives

$$M \;\geq\; \int d^3x \frac{1}{2} \left\{ \left(E_a^i\right)^2 + \left(B_a^i\right)^2 + \left(D^i \phi\right)_a^2 \right\}$$

$$
\begin{aligned}
&= \int d^3x \frac{1}{2}\left\{\left(E_a^i - \sin\alpha D^i\phi_a\right)^2 + \left(B_a^i - \cos\alpha D^i\phi_a\right)^2\right\} + \\
&\quad + \int d^3x \left\{\sin\alpha E_a^i \left(D^i\phi\right)_a + \cos\alpha B_a^i \left(D^i\phi\right)_a\right\} \\
&\geq \int d^3x \left\{\sin\alpha \partial^i \left(E_a^i \phi_a\right) + \cos\alpha \partial^i \left(B^i \phi_a\right)\right\}
\end{aligned}
$$

where α is an arbitrary constant and in the last step we have dropped the first integral, which is non-negative, made integrations by parts, used the equation of motion (29) and the Bianchi identity (31). Using the definition of electric and magnetic charges (38,39) we can rewrite this mass bound as

$$M \geq a\left(q\sin\alpha + g\cos\alpha\right) .$$

The sharpest bound occurs when the right side is a maximum, which happens for $\tan\alpha = q/g$. Plugging this back in the original expression, we find the *BPS bound*[16][17]

$$M \geq a\sqrt{q^2+g^2} = \left|a\left(q+ig\right)\right| . \tag{40}$$

This is a quite important result as will be seen later. It holds for any finite energy solutions of the equations of motion. A natural question one could ask is whether there exist solutions whish saturate the bound. From an inspection at the way we derived the bound we conclude that such kind of solutions (called BPS), with electric and magnetic charges (q,g), must satisfy the following condition throughout the space,

$$
\begin{aligned}
D^0\phi &= 0 \ , \quad V(\phi)=0, && (41)\\
E_a^i &= \sin\alpha \left(D^i\phi\right)_a \ , && (42)\\
B_a^i &= \cos\alpha \left(D^i\phi\right)_a \ , && (43)
\end{aligned}
$$

where $\tan\alpha = q/g$. These conditions are called BPS conditions. The condition $V(\phi)=0$ can only be realized if λ vanishes. However this condition must be understood as a limit $\lambda \to 0$, in order to retain the boundary condition

$$\phi^2 \to a^2 \quad \text{as} \quad r \to \infty ,$$

responsable for the spontaneous breaking of gauge symmetry. Note that $\lambda \to 0$ implies that the scalar field is massless. It is not difficult to prove that the BPS conditions (together with the Bianchi identity) imply the equations of motion (29) and (30) with $\lambda = 0$. Therefore any solution of the BPS condition automatically satisfies the equations of motion.

It is very important to note that, doing the standard calculation of the gauge particle masses, due to the spontaneous symmetry breaking, one obtains that their masses (for the scalar field in the adjoint representation) satisfy

$$M_W = a|q_W| \tag{44}$$

where q_W is gauge particle electric charge. Therefore, the gauge particles satisfy the BPS bound (40). It is also directly observable from the Lagrangian that the Higgs is massless when $\lambda = 0$, and so it satisfy the BPS bound.

3.3 The 't Hooft-Polyakov Monopole

We shall now see that Yang-Mills theories with scalar fields in the adjoint representation possess magnetic monopole solutions. For simplicity, from now on we shall only consider Yang-Mills with the gauge group $SU(2)$. The monopole solutions for other gauge groups were obtained in [15]. Using some symmetry considerations (see [3]), 't Hooft[13] and Polyakov[14] constructed the monopole solution, starting from a radially symmetric ansatz

$$\begin{aligned} \phi_a &= \frac{r^a}{er^2} H(aer) \\ W_a^i &= -\epsilon_{aij} \frac{r^j}{er^2} \left[1 - K(aer)\right] \\ W_a^0 &= 0 \end{aligned} \tag{45}$$

where H and K are some arbitrary functions. Plugging this in the equation of motion, we obtain that

$$\begin{aligned} \xi^2 \frac{d^2 K}{d\xi^2} &= KH^2 + K\left(K^2 - 1\right) \\ \xi^2 \frac{d^2 H}{d\xi^2} &= 2K^2 H + \frac{\lambda}{e^2} H \left(H^2 - \xi^2\right) \end{aligned} \tag{46}$$

where $\xi = ear$. Moreover, putting 't Hooft-Polyakov ansatz in the expression for the total energy (34), it can be obtained from the condition of finite energy that

$$K \to 0 \quad \text{and} \quad H/\xi \to 1 \quad \text{for } \xi \to \infty$$

$$K - 1 \leq O(\xi) \quad \text{and} \quad H \leq O(\xi) \quad \text{for } \xi \to 0 \tag{47}$$

There is no analytical solution for the equations (46) with the boundary conditions (47). But the existence of solutions for these equations have been shown by some numerical studies and proven rigorously by Taubes[18]. On the other hand, if one imposes the BPS conditions (42), it would give rise to first-order differential equations. These equations have closed analytical solutions:

$$K(\xi) = \frac{\xi}{\sinh \xi} \qquad H(\xi) = \frac{\xi}{\tanh \xi} - 1 \tag{48}$$

which are solutions for (47) with $\lambda = 0$.

However, in order to obtain the value of the magnetic charge of 't Hooft-Polyakov monopole, one need only use the boundary condition at $\xi \to \infty \, (r \to \infty)$ which implies that

$$W_a^i \to -\epsilon_{aij} \frac{r^j}{er^2} \ , \ \phi_a \to a \frac{r^a}{r} \quad \text{for } r \to \infty \tag{49}$$

and therefore

$$G_a^{ij} \to \frac{1}{er^4} \epsilon_{ijk} r^a r^k = \frac{1}{aer^3} \epsilon_{ijk} r^r \phi_a \quad \text{for } r \to \infty \, . \tag{50}$$

So, the asymptotic (Abelian) magnetic field is

$$B^i = \frac{{}^*G^{i0} \cdot \phi}{a} \to -\frac{1}{e} \frac{r^i}{r^3} \tag{51}$$

This expression reveals that the magnetic charge of the 't Hooft-Polyakov monopole is

$$g = -\frac{4\pi}{e}. \tag{52}$$

A natural question one could rise is if the Dirac quantization condition holds here. We shall shortly prove below that the electric charge takes the values[8]

$$q = \begin{cases} \ \ 0, \pm e\hbar \ \text{ for the adjoint repres.} \\ \pm \frac{e\hbar}{2} \ \text{ for the fundamental repres.} \end{cases} \tag{53}$$

[8]Roughly speaking, it comes from the fact that the electromagnetic $U(1)$ embedded in $SU(2)$, is generated by the T_3 element which has eigenvalues $\pm 1, 0$ in the 3 dim. irrep and $\pm 1/2$ in the 2 dim. irrep. On the other hand, by Noether procedure one obtains that $q = e\hbar T_3$ (when the θ angle vanishes).

Therefore the smallest charge which might enter in the theory is $q_0 = e\hbar/2$. Therefore, we conclude that 't Hooft-Polyakov monopole satisfies Dirac quantization condition (22)

$$q_0 g = -2\pi\hbar \tag{54}$$

and g assumes the lowest value compatible with Dirac quantization condition. Note that q_0 is the electric charge of another particle since the 't Hooft-Polyakov is chargeless. We obtained this electrically neutral monopole solution because we imposed the condition $W_0 = 0$. However Julia and Zee[19] obtained a spherically symmetric dyon solution considering 't Hooft-Polyakov ansatz (45) but with

$$W_a^0 = \frac{J(aer)r^a}{er^2}. \tag{55}$$

One can then repeat the same steps as for the 't Hooft-Polyakov monopole. In particular considering the BPS conditions (42) one obtains an analytical solution

$$\phi_a(r) = \frac{aH(\xi \cos\alpha)}{\cos\alpha}\frac{\xi^a}{\xi^2} \tag{56}$$

$$W_a^0(r) = \frac{aH(\xi \cos\alpha)}{\cot\alpha}\frac{\xi^a}{\xi^2} \tag{57}$$

$$W_a^i(r) = -a\epsilon_{aij}\frac{\xi^j}{\xi^2}\left[1 - K\left(\xi\cos\alpha\right)\right] \tag{58}$$

where the functions K and H are defined in (48) and we recover the 't Hooft-Polyakov monopole (satisfying the BPS conditions) by putting $\alpha = 2\pi n$. As we have seen before, the electric charge of this dyon is

$$q = g\tan\alpha = -\frac{4\pi}{e}\tan\alpha \tag{59}$$

This is a classical result. As we shall see later, at the quantum level, the electric charge as $q = n\hbar e/2$ and therefore satisfies the Dirac-Schwinger-Zwanziger quantization condition (24).

The reason why we have never observered a monopole/dyon is because if it existed its mass would be huge in comparison with the particle accelerators we have today. We can give a rough estimative of the monopole mass: from () we see that

$$M_m \geq a\frac{4\pi}{e} = M_W\alpha^{-1} \tag{60}$$

where $M_W = a|q| = ae\hbar$ is the gauge particle mass and $\alpha = e^2\hbar/4\pi$ is the fine structure constant. So if, for example, we consider $M_W = 100GeV$ and $\alpha = 1/137$, we would find $M_m \cong 14TeV$.

3.4 The Topological Origin of the Magnetic Charge

Let's now prove that the Dirac quantization condition holds not just for t'Hooft-Polyakov but for any monopole solution. To do so, we must use that ϕ should satisfy (36) asymptotically. Using the definition for the covariant derivative (27), with $f_{abc} = \epsilon_{abc}$, and remembering the definition of vector product, we can rewrite the condition $D_\mu\phi = 0$ as

$$\overrightarrow{\phi} \times \overrightarrow{W_\mu} = -\frac{1}{e}\partial_\mu \overrightarrow{\phi} \tag{61}$$

Taking a vector product of this equation with $\overrightarrow{\phi}$ and using the identity $u \times (v \times w) = (u \cdot w)v - (u \cdot v)w$ we obtain that the gauge field asymptotically has the form

$$\overrightarrow{W_\mu} = \frac{1}{ea^2}\overrightarrow{\phi} \times \partial_\mu \overrightarrow{\phi} + \frac{1}{a}A_\mu \overrightarrow{\phi} \tag{62}$$

where

$$A_\mu = \frac{\overrightarrow{W_\mu} \cdot \overrightarrow{\phi}}{a} \tag{63}$$

Then,

$$\overrightarrow{G}_{\mu\nu} = \partial_\mu \overrightarrow{W}_\nu - \partial_\nu \overrightarrow{W}_\mu - e\overrightarrow{W}_\mu \times \overrightarrow{W}_\nu = \frac{1}{a}F_{\mu\nu}\overrightarrow{\phi} \tag{64}$$

with

$$F_{\mu\nu} = \frac{1}{a^3 e}\overrightarrow{\phi} \cdot \left(\partial_\mu \overrightarrow{\phi} \times \partial_\nu \overrightarrow{\phi}\right) + \partial_\mu A_\nu - \partial_\nu A_\mu \tag{65}$$

being the electromagnetic field-strength (37). It is interesting to note that asymptotically the only non-zero component of the $\overrightarrow{G}_{\mu\nu}$ is the component in the $\overrightarrow{\phi}$ direction which is the generator of the electromagnetic $U(1)$ and $F_{\mu\nu}$ satisfies Maxwell equations. So, far from the monopole, only the electromagnetic fields survive.

Then, the magnetic charge (39) will be

$$g = \frac{-4\pi N_m}{e} \tag{66}$$

where

$$N_m := \frac{1}{8\pi a^3}\int ds^i \epsilon_{ijk}\overrightarrow{\phi}\cdot\left(\partial^j\overrightarrow{\phi}\times\partial^k\overrightarrow{\phi}\right) \tag{67}$$

So the magnetic charge depends on the asymptotic configuration of $\overrightarrow{\phi}$. This integral is topological under continuous deformations of ϕ. In order to prove this one should consider a new configuration $\phi' = \phi + \delta\phi$, where $\delta\phi$ is infinitesimal deformation and ϕ' also satisfies the asymptotic conditions (36). This implies that

$$D_\mu \delta\overrightarrow{\phi} = 0 \qquad \overrightarrow{\phi}\cdot\delta\overrightarrow{\phi} = 0\,.$$

Using these conditions is possible to prove that $\delta N_m = 0$. This means that N_m is invariant under infinitesimal deformations of ϕ and hence under any deformation achieved by iterating these infinitesimal deformations. Moreover it can be proven N_m can only take integer values (see [20]). Therefore the magnetic charge is topologically conserved and quantized in units of $4\pi/e$, i.e.

$$g = \frac{4\pi}{e}n_m \tag{68}$$

Remembering that the smallest electric charge is $q_0 = e\hbar/2$, then

$$q_0 g = -2\pi n_m \hbar\,, \tag{69}$$

and therefore we see that Dirac's quantization condition holds for any magnetic monopole of this theory.

3.5 The Montonen-Olive Duality Conjecture

In the previous sections we have seen that the mass for the BPS-monopole satisfying the BPS is $m_M = a|g| = 4\pi a/e$. On the other hand, due to the spontaneous symmetry breaking, the gauge particles $W^\mu_\pm = (W^\mu_1 \pm iW^\mu_2)/\sqrt{2}$, which have electric charges $q_{W_\pm} = \pm\hbar e$, will acquire masses $m_{W_\pm} = a\hbar e$ and W_3 remains massless(photon) like the Higgs (remember V=0)

state	elec. char.	mag. char.	mass	spin
Photon	0	0	0	1
Higgs	0	0	0	0
$W_\pm$	$q = \pm e\hbar$	0	$a\|q\| = ae\hbar$	1
$M_\pm$	0	$g = \pm\frac{4\pi}{e}$	$a\|g\| = \frac{4\pi a}{e}$	?

From this table we observe the following features

- all particles saturate the Bogomol'nyi bound $m = |a\,(q + ig)\,|$
- the mass of the BPS-monopole ($W_\pm$)of one theory with coupling constant e is equal to the mass of the $W_\pm$(BPS-monopole) of a *dual* theory with with coupling $e' = 4\pi/e\hbar$.

Based on these observations, Montonen and Olive[21] conjectured that at the quantum level, the monopoles[9] of one theory would be described by the $W_\pm$ particles of the dual theory. Similarly the $W_\pm$'s of the original theory would be the monopoles of the dual theory. For example, the S-matrix element for two monopoles in one theory would be the same as two $W_\pm$'s in the dual theory.

Note that two theories have the same mass spectrum at the classical level. That is the first indication that the two theories can be equivalent. For two theories to be equivalent (or dual) they must have the same S-matrix. So, in particular, they must have the same mass spectrum *at the quantum level* which gives the pole structure for the S-matrix.

So far there is no rigorous proof for this conjecture. What people have done since then was to do some non-trivial tests to check the consistence of the conjecture. The first test was made be Montonen and Olive together with the proposal of the conjecture[21]. This test concerns the fact that a static M_+M_+ pair exert no forces on each other, which was proven by Manton[22]. (This is also a consequence of the existence of static BPS-monopole solutions with magnetic charge 2.) As a consequence of the conjecture, forces between static W_+W_+ pair should not exist. This seems quite strange because the two gauge particles have same electric charges. However the static potential between the W_+W_+ has

[9]From now on we shall only consider BPS-monopoles, but we shall call them just monopoles.

contributions from exchange of photons and of Higgs particles. Using Feynman rules[21], one finds a repulsive potential of modulus q^2/r from the photon exchange and an attractive potential of same modulus from the Higgs exchange, which cancel each other, consistently with the conjecture. The Higgs produces a potential $1/r$ because it is massless in the Prasad-Sommerfield limit.

Note that this duality conjecture is not a symmetry, because it relates a theory with one value of the coupling constant with the same theory with a different coupling constant. There are other duality conjectures, in which not only the value of the coupling constant is different, but the theory as well. One example is the equivalence of sine-Gordon and massive Thirring model in $1+1$dimensions.

3.6 The Witten Effect

The Yang-Mills-Higgs action (25) is not the most general we can construct. We can add to it the topological θ term,

$$S_\theta = \int d^4x \frac{\theta \hbar e^2}{32\pi^2} \widetilde{G}^{\mu\nu} \cdot G_{\mu\nu} \tag{70}$$

which breaks CP symmetry. Since the θ term is a total derivative, it will not change the equations of motion and since it doesn't depend on the metric, it will not contribute to the energy-momentum tensor. However, as has been pointed out by Witten[23], it will change the spectrum of electric charges. Let us see how it happens: under a general infinitesimal gauge transformation $f = e^{i\varepsilon(x)}$,

$$\begin{aligned} \delta\phi &= [i\varepsilon, \phi] \\ \delta W_\mu &= -\frac{1}{e} D_\mu \varepsilon \end{aligned}$$

Consider the particular transformation

$$f = e^{i\alpha\widehat{\phi}} \quad \text{where} \quad \widehat{\phi} = \frac{\phi}{|\phi|}, \tag{71}$$

and α is a infinitesimal global parameter. Clearly, under this transformation the Higgs field is left invariant while

$$\delta W_\mu = -\frac{1}{e}\alpha D_\mu \widehat{\phi} \, .$$

Then, the Noether charge which generates this transformation is

$$N = \frac{1}{\alpha}\int d^3x \left\{ \left(-G^{0i} + \frac{\theta \hbar e^2}{8\pi^2}\widetilde{G}^{0i} \right) \cdot \left(-\frac{1}{e}\alpha D_i \widehat{\phi} \right) \right\} .$$

Using the equations of motion, the Bianchi identity and the definitions of electric and magnetic charges (38)(39), one finds that

$$N = \frac{1}{e}q - \frac{\theta \hbar e}{8\pi^2} g \, . \tag{72}$$

At the quantum level the operator

$$\mathcal{G}_\beta = e^{i\beta N/\hbar} \tag{73}$$

acts on the asymptotic states $|W_\mu, \phi >$ as[24]

$$\mathcal{G}_\beta |W_\mu, \phi >= |W'_\mu, \phi' > \tag{74}$$

where

$$W'_\mu = f W_\mu f^{-1} + ief\partial_\mu f^{-1} \;\text{ with }\; f = e^{i\beta\widehat{\phi}} \tag{75}$$

$$\phi' = f\phi f^{-1} = \phi \tag{76}$$

Since the transformation acts on asymptotic states, ϕ satisfy the vacuum conditions: $\phi^2 = a^2$ and $D_\mu \phi = 0$. Then it is possible to show at $R \to \infty$ and $\beta = 2\pi$

$$f = e^{2\pi i \widehat{\phi}} = e^{2\pi i T_3} \tag{77}$$

and $W'_\mu = W_\mu$ (remembering that $[T_3, T_\pm] = \pm T_\pm$). Therefore,

$$e^{2\pi i N/\hbar} |W_\mu, \phi >= |W_\mu, \phi > \tag{78}$$

for any asymptotic state $|W_\mu, \phi >$, which implies that $e^{2\pi i N/\hbar} = 1$ and therefore using (72) it can be concluded that

$$\frac{1}{e}q - \frac{\theta \hbar e}{8\pi^2} g = n_e \hbar \, , \tag{79}$$

where n_e is an arbitrary integer. Using the magnetic charge quantization condition $g = 4\pi n_m/e$, we arrive out the electric charge quantization condition

$$q = \hbar e\left(n_e + \frac{\theta}{2\pi}n_m\right) \quad n_e,\, n_m \in Z \tag{80}$$

Note that if θ vanishes, we recover the standard quantization condition $q = n_e\hbar e$ that we have mentioned in (53), which holds for the fields in the adjoint representation. If there exists a field ψ, say, in the fundamental representation, under a gauge transformation $\psi' = f\psi$. If we take $\beta = 2\pi$, (77) continues to be true but now T_3 is the diagonal Pauli matrix with eigenvalues $\pm 1/2$, and therefore ψ is not invariant. In order to be invariant, we must consider $\beta = 4\pi$ instead, which results the charge quantization

$$q = \frac{\hbar e}{2}\left(n_e + \frac{\theta}{\pi}n_m\right) \quad n_e,\, n_m \in Z \tag{81}$$

and we obtain (53) for the fundamental representation when $\theta = 0$. A natural question one could ask is if the Dirac-Schwinger-Zwanziger quantization condition, which has been obtained for Maxwell theory, holds for Yang-Mills theories. It is direct to check that it does. In order to see this, consider two dyons with electric and magnetic charges (q_a, g_a) and (q_b, g_b), which satisfy the electric and magnetic charges quantization conditions (81) and (68) and therefore

$$q_a g_b - q_b g_a = 2\pi\hbar\left(n_e^a n_m^b - n_m^a n_e^b\right) = 2\pi\hbar Z \tag{82}$$

3.7 The SL(2,Z) Duality Conjecture

For simplicity, let us consider that all fields are in the adjoint representation. Then using the electric and magnetic charge quantization conditions[10],

$$q + ig = q_0\left(n_e + \tau n_m\right) \;\; \text{where} \;\; q_0 = \hbar e\,, \;\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{\hbar e^2}\,. \tag{83}$$

Since the numbers n_e and n_m can only take integer values, the set of *possible* states of the theory form a lattice in the $a(q + ig)$ plane. Note that we have an infinite number of

[10]If there are fields in the fundamental, using the charge quatization condition (81), it convenient to define $q_0 = \hbar e/2$ and $\tau = (\theta/\pi) + i(8\pi/he^2)$

possible states. The photon is associated with the point ($n_m = 0,\, n_e = 0$), the $W^{\pm}_{\mu}$gauge particle with $(0, \pm 1)$ and the 't Hooft-Polyakov (anti)monopole $M^{\pm}$ with $(\pm 1, 0)$. The existence of other states have been proven by A.Sen [25] as we shall explain later. Defining

$$\sqrt{u} = aq_0\,, \tag{84}$$

$\sqrt{u}\tau$ and $\sqrt{u}$ form basis vectors for the lattice. The ratio of these vectors gives τ. Note that τ contains all information about the couplings of the theory: from its real and imaginary part we obtain $(e,\, \theta)$. If we rescale $W_\mu \to eW_\mu$ and $\phi \to e\phi$, the Yang-Mills action can be rewritten in the form

$$\frac{S}{\hbar} = -\int d^4x \frac{\tau}{64\pi i}\left[\left(G_{\mu\nu} + i\widetilde{G}_{\mu\nu}\right)^2 - 4D_\mu\phi \cdot D^\mu\phi\right] + hc \tag{85}$$

where the τ dependence is shown explicitly. Therefore, different τ correspond to theories with different couplings.

The mass formula (40) for a BPS state $|n_m,\, n_e;\, \tau >$, for a theory with coupling τ, can be put in the form

$$M(n_m,\, n_e,\, \tau) = |a(q + ig)| = \left| \begin{pmatrix} n_m & n_e \end{pmatrix} \begin{pmatrix} \sqrt{u}\tau \\ \sqrt{u} \end{pmatrix} \right| . \tag{86}$$

In the original Montonen-Olive conjecture, the authors considered just the existence of the gauge particles $W^{\pm}$, γ^0 and the (anti)monopoles $M^{\pm}$ and based their conjecture in the observation that their masses fulfilled the relations

$$M(0, \pm 1; \tau) = M(\mp 1, 0; -1/\tau) \tag{87}$$

$$M(\pm 1, 0; \tau) = M(0, \mp 1; -1/\tau) \tag{88}$$

$$M(0, 0; \tau) = M(0, 0; -1/\tau) \tag{89}$$

where they considered $\theta = 0$ and so $\tau = i(4\pi/\hbar e^2)$. Therefore, each state of the theory with coupling τ can be mapped to a state of the theory with coupling $-1/\tau$, such that the mass of the two states is the same.And the two theories have the same mass spectrum at the classical level. That was the first indication that the two theories can be equivalent.

In the Montonen-Olive conjecture just the existence of gauge particles and monopoles was considered. Let us now extend this conjecture, considering the existence of an arbitrary subset of the states of the lattice.

Like in the Montonen-Olive conjecture, we want to map each state $|n_m, n_e; \tau>$ of a theory with coupling τ, to a state $|n'_m, n'_e; \tau'>$ of a theory with coupling τ', such that $M(n_e, n_m, \tau) = M(n'_e, n'_m; \tau')$. From (86) we obtain

$$\left| \begin{pmatrix} n_m & n_e \end{pmatrix} \begin{pmatrix} \sqrt{u}\tau \\ \sqrt{u} \end{pmatrix} \right| = \left| \begin{pmatrix} n'_m & n'_e \end{pmatrix} \begin{pmatrix} \sqrt{u'}\tau' \\ \sqrt{u'} \end{pmatrix} \right| \tag{90}$$

In order to this condition to be true, we can take

$$\left\{ \begin{array}{rcl} \begin{pmatrix} \sqrt{u'}\,\tau' \\ \sqrt{u'} \end{pmatrix} & = & e^{i\gamma}\, M \begin{pmatrix} \sqrt{u}\tau \\ \sqrt{u} \end{pmatrix} \\ \begin{pmatrix} n'_m & n'_e \end{pmatrix} & = & \begin{pmatrix} n_m & n_e \end{pmatrix} M^{-1} \end{array} \right. \qquad M := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{91}$$

where $\gamma \in R$. Since n_m, n_e, n'_m, $n'_e \in Z$, we must have that all entries in M and M^{-1} are integers and therefore A, B, , C , $D \in Z$ and $\det M = AD - BC = \pm 1$. From (91) we obtain that

$$\tau' = \frac{A\tau + B}{C\tau + D} \qquad u' = e^{2i\gamma}\,(C\tau + D)^2\, u \tag{92}$$

We can then obtain that

$$Im\,\tau' = \frac{(AD - BC)}{|C\tau + D|^2} Im\,\tau$$

Now, since the coupling e is a real number, we must have that $Im\,\tau$, $Im\,\tau' > 0$ which implies that $AD - BC > 0$ and therefore $\det M = 1$. So $M \in$SL(2,Z). However not all transformations are allowed. The possible transformtations depend on the subset of states we take and quantum numbers of the states (like spin, representation under gauge group, etc.) which may forbid some of the maps. The set of all possible transformations form a subgroup of SL(2,Z) which will shall denote by Γ. So we can conclude saying that we extended Montonen-Olive conjecture, relating an infinite number of theories with different coupling constants but with the same mass spectrum *at the classical level* and the theories are related by transformations which form a subgroup of SL(2,Z). Clearly we would like to have the same *quantum mass spectrum.*

Notice that, from (91),

$$a'(q' + ig') = \begin{pmatrix} n'_m & n'_e \end{pmatrix} \begin{pmatrix} \sqrt{u'}\tau' \\ \sqrt{u'} \end{pmatrix} = e^{i\gamma} \begin{pmatrix} n_m & n_e \end{pmatrix} \begin{pmatrix} \sqrt{u}\tau \\ \sqrt{u} \end{pmatrix} = e^{i\gamma} a(q + ig)$$

and we can identify $e^{i\gamma}$ with the phase which appear in Maxwell's duality.

From the Montone-Olive conjecture or its generalization one could raise the following questions:

1. The gauge particles have spin 1. Do the monopole (or dyons) also have spin1?? Clearly one can only map states with the same number of degrees of freedom.

2. The mass formula holds at classical level. Will the quantum corrections destroy it?

These two questions can be answered with the introduction of supersymmetry as will be explained in the next sections. Before that it would be interesting to say that it was proved by Witten[26] and Verlinde [27](see also [28] for a review) using path integral approach showed that Maxwell theory has a $SL(2,Z)$ duality (or a subgroup of it) at *the quantum level.*

4 Supersymmetry.

4.1 Representations without central charge.

There are many good references on supersymmetry. Here we shall give just a very brief review on some points of supersymmetry which will be useful for us. For further details we recommend [29][30].

The supersymmetry algebra have the relations[11]

$$\left\{Q^i_\alpha, Q^{j\dagger}_\beta\right\} = 2\sigma^\mu_{\alpha\beta}P_\mu\delta^{ij}\,, \tag{93}$$

$$\left\{Q^i_\alpha, Q^j_\beta\right\} = 0\,, \tag{94}$$

$$\left\{Q^{i\dagger}_\alpha, Q^{j\dagger}_\beta\right\} = 0\,, \tag{95}$$

where $\sigma^\mu = \{1, \overrightarrow{\sigma}\}$are the Pauli matrices, $\alpha\,\beta = 1,2$ and $i,j = 1,2,...,N$ with N being the number of supersymmetries. Moreover we have the commutation relation

$$\left[Q^i_\alpha, P^\mu\right] = 0 \quad\Rightarrow\quad \left[Q^i_\alpha, P^\mu P_\mu\right] = 0\,.$$

[11] For simplicity we shall not use dotted and undotted indices.

Since $P^\mu P_\mu$ gives the mass of the state, this last commutation relation implies that all states in a supersymmetry representation have the same mass. Let us consider the massless and massive representations separately:

1. Massless representations

For a massless state we can always choose a referential frame such that $P_\mu = (E, 0, 0, E)$ where E is the state energy. Then the commutation relation (93) will assume the form

$$\left\{Q^i_\alpha, Q^{j\dagger}_\beta\right\} = 4 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\beta} \delta^{ij} . \tag{96}$$

Remember that in a unitary representation, all states $|\psi>$ must be such that $|| \, |\psi> \, || \geq 0$ and $|| \, |\psi> \, || = 0$ if and only if $|\psi> \equiv 0$. Now taking $\alpha = 2 = \beta$ and $i = j$ in (96) we obtain

$$0 = <\psi| \left\{Q^i_2, Q^{i\dagger}_2\right\} |\psi> = || \, Q^{i\dagger}_2 |\psi> \, ||^2 + || \, Q^i_2 |\psi> \, ||^2 \tag{97}$$

which implies that $Q^{i\dagger}_2|\psi> = 0 = Q^i_2|\psi>$ for any state $|\psi>$, considering a unitary representation. So for massless representations

$$Q^i_2 = 0 = Q^{i\dagger}_2 \tag{98}$$

Then, we define the generators

$$a^i = \frac{1}{2\sqrt{E}} Q^i_1 \ , \quad a^{i\dagger} = \frac{1}{2\sqrt{E}} Q^{i\dagger}_1 \, , \tag{99}$$

satisfying the anticommutation relations

$$\left\{a^i, a^{j\dagger}\right\} = \delta^{ij} \ , \ \left\{a^i, a^j\right\} = 0 = \left\{a^{i\dagger}, a^{j\dagger}\right\} \ , \tag{100}$$

which form a Clifford algebra with $2N$ generators. It is well known that a Clifford algebra with n generators has a unique unitary irreducible representation of dimension $2^{n/2}$. So a massless irrep has dimension 2^N. Let us analyze this irrep: let J_3 be the helicity operator which satisfies the commutation relation

$$\left[J_3, a^{i\dagger}\right] = -\frac{1}{2} a^{i\dagger} . \tag{101}$$

If $J_3|\lambda>=\lambda|\lambda>$, then $J_3 a^{i\dagger}|\lambda>=(\lambda-1/2)a^{i\dagger}|\lambda>$ and therefore $a^{i\dagger}|\lambda>\propto|\lambda-1/2>$. So a massless irrep built from a highest state $|\lambda>$, such that $a^i|\lambda>=0$ for $i=1,...,N$ has the structure shown in table below[12], where we can see explicitly that the total number of states in a massless irrep is 2^N.

States	#of states	helicity
$\vert\lambda>$	1	λ
$a^{i\dagger}\vert\lambda>$	N	$\lambda-1/2$
$a^{i_1\dagger}a^{i_2\dagger}\vert\lambda>,\ i_1>i_2$	$\begin{pmatrix} N \\ 2 \end{pmatrix}$	$\lambda-1$
$\vdots$	$\vdots$	$\vdots$
$a^{i_1\dagger}a^{i_2\dagger}\cdots a^{i_N\dagger}\vert\lambda>,\ i_1>i_2>\cdots>i_N$	1	$\lambda-N/2$

Thefore we shall have

N=1: $|\lambda>, |\lambda-1/2>$,

N=2: $|\lambda>, 2\,|\lambda-1/2>, |\lambda-1>$,

N=4: $|\lambda>, 4\,|\lambda-1/2>, 6\,|\lambda-1>, 4\,|\lambda-3/2>, |\lambda-2>$.

Usually these irrep's are not CPT invariant. For these cases, we must add the CPT conjugate with opposite helicities.

The CPT invariant massless representations with maximal helicity 1 or less will be the following:

[12]Remember that $a^\dagger_{i_1}a^\dagger_{i_2}|\lambda>=-a^\dagger_{i_2}a^\dagger_{i_1}|\lambda>$ for $i_1\neq i_2$ so these two states should be count as the same state.

N=1:

J_3		
1		x
1/2	x	x
0	x o	
-1/2	o	o
-1		o

We used "o" for the irrep CPT conjugate to the one with "x". The two irreps form a CPT invariant supermultiplet. We see that for N=1 there are two possible on-shell supermultiplets with maximal helicity 1: in the first column we have 1 Majorana spinor and 2 real scalars and in the second column 1 vector and 1 Majorana spinor.

N=2:

J_3		
1		x
1/2	x o	xx
0	xx oo	x o
-1/2	x o	oo
-1		o

For N=2 there is the hypermultiplet in the first column and the vector supermultiplet in the second column. For the hypermultiplet we have put two irrep not because of CPT symmetry, since each irrep by itself is CPT invariant, but because of $SU(2)_R$ symmetry. The hypermultiplet is composed of 2 Majorana spinors and 4 real scalars. The vector supermultiplet contains 1 vector 2 Majorana spinors and 2 real scalars

N=4

J_3	
1	x
1/2	xxxx
0	xxxxxx
-1/2	xxxx
-1	x

For N=4 there is only one possibility which is the vector supermultiplet which is a CPT self-conjugate irrep. It contains 1 vector, 4 Majorana spinors and 6 real scalars. It is not difficult to see that, for N¿4 there is no supermultiplet with maximal helicity 1 or smaller for the massless case. One could ask why we didn't talk about N=3. It is straightforward to check that if one tries to construct a CPT invariant supermultiplet it will result the N=4 vector supermultiplet. And since it has the same field content, the dynamics will be governed by the same Lagrangian and therefore the N=3 supersymmetry will be enhanced to N=4.

2. Massive representations

For a massive state we can always choose a rest frame such that $P_\mu = (M, 0, 0, 0)$, where M is the state mass. In this frame the commutation relation (93) will take the form

$$\left\{Q^i_\alpha, Q^{j\dagger}_\beta\right\} = 2M \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha\beta} \delta^{ij} \, . \tag{102}$$

We define the generators

$$a^i_\alpha = \frac{1}{\sqrt{2M}} Q^i_\alpha \ , \ \ a^{i\dagger}_\alpha = \frac{1}{\sqrt{2M}} Q^{i\dagger}_\alpha \, , \tag{103}$$

satisfying the anticommutation relations

$$\left\{a^i_1, a^{j\dagger}_1\right\} = \delta_{ij} \ , \ \left\{a^i_2, a^{j\dagger}_2\right\} = \delta^{ij} \, , \tag{104}$$

and the other anticommutators vanishing. This is a Clifford algebra with $4N$ generators which has a unique unitary irreducible representation of dimension 2^{2N}. Therefore we shall have

	$N=1$	$N=2$	$N=4$
$p^2=0$	2	4	16
$p^2>0$	4	16	256

But then we arrived at a paradox: consider massless N=4 super Yang-Mills. For this case we must use the vector supermultiplet which has 16 states. But now, suppose we perform a spontaneous symmetry breaking of the gauge symmetry. In this case the theory will become massive and then we would need to pass from 16 massless states to 256 massive states (some of them with spin graeter than 1). But the Higgs mechanism preserves the number of states and helicities. How can we solve this paradox? The solution is in the next subsection[31].

4.2 Representations with central charges.

There exist the possibility to add central chargers to (94). The must general form for these anticommutators in 3+1 dimensions is

$$\left\{Q^i_\alpha, Q^{j\dagger}_\beta\right\} = \sigma^\mu_{\alpha\beta} P_\mu \delta^{ij} + \sigma^\mu_{\alpha\beta} Z^{ij}_\mu \,, \;\; \left(Z^{ii}_\mu = 0\right) \tag{105}$$

$$\left\{Q^i_\alpha, Q^j_\beta\right\} = \epsilon_{\alpha\beta} Z^{[ij]} + \sigma^{\mu\nu}_{\alpha\beta} Z^{(ij)}_{\mu\nu} \tag{106}$$

One can check directly that the total number of generators in the rhs is $8N^2+2N$, which is the same dimension as the lhs, which is a symmetric $4N \times 4N$ matrix. The central charges Z^{ij}_μ and $Z^{(ij)}_{\mu\nu}$ are related to topological charges for strings and domain walls. They are important but since we are dealing just with monopoles, we shall consider these central charges equal to zero. In this case, since $Z^{[ij]}$is antisymmetric, we can rotate the

Q^i_α unitarity in such a way that $Z^{[ij]}$ takes the form

$$Z^{[ij]} = \begin{pmatrix} 0 & z_1 & & & & & & \\ -z_1 & 0 & & & & & & \\ & & 0 & z_2 & & & & \\ & & -z_2 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & z_{N/2} \\ & & & & & & -z_{N/2} & 0 \end{pmatrix}$$

where the z_i can be chosen to be real. To simplify the discussion we have assumed that N is even, but one should have in mind that if N is odd, there will of course be a zero 1×1block in the above normal form. Clearly we can only have $Z^{[ij]} \neq 0$ for $N \geq 2$.

1. Massless representations.

Once more we we can take $P_\mu = (E, 0, 0, E)$. As before, from the first anticommutator (105) (with $Z^{ij}_\mu = 0$) we obtain the condition (98) $Q^i_2|\psi>= 0 = Q^{i\dagger}_2|\psi>$. Then, from the second anticommutator (106) (with $Z^{ij}_{\mu\nu} = 0$) it results

$$0 =<\psi|\{Q^i_1, Q^j_2\}|\psi>= 2Z^{ij} <\psi|\psi> \ .$$

As $<\psi|\psi>> 0$ for any $|\psi>\neq 0$ it implies that

$$Z^{ij} = 0$$

for the massless representation, and we return to the situation of the previous subsection without central charge.

2. Massive representations.

Let us take the rest frame and for simplicity consider $N = 2$ where $Z^{[ij]} = \epsilon^{ij} z$. Then,

$$\left\{Q^i_\alpha, Q^j_\beta\right\} = 2\epsilon_{\alpha\beta}\epsilon^{ij} z \, .$$

Defining

$$a^{\pm}_{\alpha} = \frac{1}{2}\left(Q^1_{\alpha} \pm \epsilon_{\alpha\beta} Q^{2\dagger}_{\beta}\right)$$

which satisfy the anticommutation relation

$$\left\{a^{\pm}_{\alpha}, \left(a^{\pm}_{\beta}\right)^{\dagger}\right\} = \delta_{\alpha\beta}\left(M \pm z\right)\,, \tag{107}$$

where the other anticommutators vanish. Taking $\alpha = \beta$ and putting inside a "sandwich" of states it results that,

$$(M \pm z)\,||\,|\psi>\,||^2 =<\psi|\{a^{\pm}_{\alpha}, a^{\pm\dagger}_{\alpha}\}|\psi>=||\,a^{\pm\dagger}_{\alpha}|\psi>||^2 + ||\,a^{\pm}_{\alpha}|\psi>||^2 \geq 0$$

Since $|\psi>$ is an arbitrary state, we must have $M \pm z \geq 0$ or

$$M \geq |z|\,. \tag{108}$$

In the case of $N > 2$ one obtains (see for instance [29]) that $M \geq |z_r|$, $r = 1, ...N/2$. For simplicity let us consider that all z_r are equal. Then we recover (108) for a generic N. Now we have two possibilities: if $M > |z|$, (107) will be a Clifford algebra with $4N$ generators with a unique unitary irrep of dimension 2^{2N}. It is called long representation. On the other hand, if $M = |z|$, the anticommutator of $a^{i-}_{\alpha}(a^{i+}_{\alpha})$ vanishes for z positive (negative). Then we can repeat the same argument of the massless representation and conclude that we have a Clifford algebra with $2N$ generators with a unique unitary irrep of dimension 2^N, which is called a short representation. In summary, we have that

1. Massless case $\Rightarrow$ irrep's dimension $= 2^N$
2. Massive case
   (a) $M > |z|$ $\Rightarrow$ irrep's dimension $= 2^{2N}$
   (b) $M = |z|$ $\Rightarrow$ irrep's dimension$= 2^N$

Therefore the paradox can be solved: after symmetry breaking, the massless representations which "become" massive must satisfy $M = |z|$ in order to continue with the same dimension. Let us now calculate the central charge for $N = 2, 4$ super Yang-Mills.

5 Super Yang-Mills and central charges.

5.1 Extended Supersymmetry

Consider in a Minkowski space of dimension D the action

$$S=\int d^dx\,\mathrm{Tr}\left\{-\frac{1}{4}G_{MN}G^{MN}+\frac{i}{2}\overline{\lambda}\Gamma^M D_M\lambda\right\} \tag{109}$$

for an arbitrary compact gauge group where λ is a Majorana spinor in the adjoint representation, Γ^M are the gamma matrices and the indices $M,N=0,1,\ldots,D-1$. Consider the supersymmetry transformations

$$\begin{aligned}\delta W_M &= \frac{i}{2}\left[\overline{\lambda}\Gamma_M\alpha-\overline{\alpha}\Gamma_M\lambda\right]\,, &(110)\\ \delta\lambda &= \frac{1}{2}\Gamma_{RS}G^{RS}\alpha\,, &(111)\\ \delta\overline{\lambda} &= -\frac{1}{2}\overline{\alpha}\Gamma_{RS}G^{RS}\,, &(112)\end{aligned}$$

where α is a Majorana spinor parameter and $\Gamma_{RS}=1/2[\Gamma_R,\Gamma_S]$. One can show that the above action is invariant (up to a total derivative) to these transformations, by using that[13]

$$\Gamma^M\Gamma^{RS}=\left[g^{MR}\Gamma^S-g^{MS}\Gamma^R-\frac{(-1)^{D/2}}{(D-3)!}\epsilon^{MRSN_1\cdots N_{D-3}}\Gamma_{D+1}\Gamma_{N_1}\ldots\Gamma_{N_{D-3}}\right]\,,$$

$$\Gamma^{RS}\Gamma^M=\left[-g^{MR}\Gamma^S+g^{MS}\Gamma^R-\frac{(-1)^{D/2}}{(D-3)!}\epsilon^{MRSN_1\cdots N_{D-3}}\Gamma_{D+1}\Gamma_{N_1}\ldots\Gamma_{N_{D-3}}\right]$$

and the identity

$$\mathrm{Tr}\left\{\overline{\lambda}\Gamma_M\left[\lambda,\delta W^M\right]\right\}=0$$

which holds for

1. D=3, if λ is a Majorana spinor,
2. D=4, if λ is a Majorana spinor,
3. D=6, if λ is a Weyl spinor,

[13] where $\Gamma_{D+1}\equiv\Gamma_0\Gamma_1\ldots\Gamma_{D-1}$.

4. D=10, if λ is a Majorana-Weyl spinor.

Therefore, for these 4 cases only, the action (109) will invariant under $N=1$ supersymmetry. There is a simple way to understand this result by noting that the number of on-shell bosonic degrees of freedom, which is equal to $D-2$, is equal to the fermionic degrees of freedom, which is equal to $x2^{[D/2]}$ where $x=1/2$ if λ is a Majorana or Weyl spinor and $x=1/4$ if λ is a Majorana-Weyl spinor. The notation $[D/2]$, means the greater integer inside $D/2$. Only for the above 4 cases, these two numbers coincide. Note that this equal number of bosonic and fermionic degrees of freedom is a necessary but not sufficient condition in order to have supersymmetry.

As a consequence of this invariance one obtain the supercurrent

$$J^M = \frac{1}{2} i \Gamma_{RS} \Gamma^M \mathrm{Tr}\left\{ G^{RS} \lambda \right\} , \tag{113}$$

and from it we obtain the supercharges Q which are spinors satisfying the same conditions as λ in the four cases cited above.

From the $N=1$ supersymmetric actions in $D=6$ and $D=10$, we can write down the actions for super Yang-Mills with $N=2$ and $N=4$ supersymmetries in $D=4$ dimensions, using so-called dimensional reduction. Let us divide the D dimensional space-time components x^M in a part x^μ, with $\mu=0,1,2,3$ being the four dimensional space-time indices, and a part x^r, with the indices r,s running over the compactified $D-4$ dimensions. We also consider that the fields don't depend on these compactified coordinates. The gauge field components W_r transform as scalars under the four dimensional space-time Lorentz transformations. Therefore we shall define $\phi_r \equiv W_r$ (with the lower index!). Then the dimensional reduction of G_{MN} gives

$$G_{MN} = \left\{ \begin{array}{lcl} G_{\mu\nu} & = & \partial_\mu W_\nu - \partial_\nu W_\mu + ie\left[W_\mu, W_\nu\right] \\ G_{\mu r} & = & \partial_\mu \phi_r + ie\left[W_\mu, \phi_r\right] = D_\mu \phi_r \\ G_{rs} & = & ie\left[\phi_r, \phi_s\right] \end{array} \right. \tag{114}$$

Then the compactification of the bosonic part of the Lagrangian results in

$$L = \mathrm{Tr}\left\{ -\frac{1}{4} G_{MN} G^{MN} \right\} = \mathrm{Tr}\left\{ -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \frac{1}{2} D_\mu \phi_r D^\mu \phi_r - \frac{e^2}{4}\left[\phi_r, \phi_s\right]\left[\phi_r, \phi_s\right] \right\}$$

where we can see that the pure Yang-Mills Lagrangian in D dimensions under dimensional reduction will result in an Yang-Mills-Higgs theory in four dimensions with $D-4$ scalar fields in the adjoint representation of the gauge group. The fermionic part must be analyzed in each dimension separately. We refer to [32] for more details. The important thing to have in mind is that the supercharge Q_α in $D=6$ is a Weyl spinor with 8 real components which, under dimensional reduction, will result on 2 Majorana spinor supercharges Q^i_α, $i=1,2$, with 4 real components each. That is the reason why we obtain $N=2$ in $D=4$ from dimensional reduction of $N=1$ in $D=6$. Similarly the supercharge in $D=10$ is a Majorana-Weyl spinor with 16 real components which, under dimensional reduction, will result in 4 Majorana spinor supercharges Q^i_α , $i=1,2,3,4$, generating the $N=4$ supersymmetry.

We shall now calculate the algebra of supercharges for $N=1$ in $D=6$ and $D=10$ and then, using dimension reduction, obtain $N=2$ and $N=4$ algebras for $D=4$. The strategy is the following:

1. Use the fact that under a generic symmetry transformation of a quantity $\mathcal{O}$

$$\delta_\alpha \mathcal{O} = i\hbar\left[\alpha Q, \mathcal{O}\right]$$

where Q is an arbitrary charge (or symmetry generator) and α is the transformation parameter. In particular, for supersymmetry

$$\delta\epsilon\overline{Q}_\alpha = i\hbar\left[\overline{\epsilon_\beta}Q_\beta, \overline{Q_\alpha}\right] \tag{115}$$

where $\overline{\epsilon_\beta}$ is the supersymmetry transformation parameter. Then we calculate explicitly the supersymmetry transformation of the supercharge and write it as

$$\delta_\epsilon\overline{Q_\alpha} = \overline{\epsilon_\beta}X_{\beta\alpha}. \tag{116}$$

So, from (115) and (116), remembering that ϵ_β has a fermionic character, we conclude that

$$\left\{\overline{Q_\alpha}, Q_\beta\right\} = \frac{1}{i\hbar}X_{\beta\alpha}. \tag{117}$$

Let us now calculate $X_{\beta\alpha}$. From supercurrent expression (113), doing a supersymmetry transformation[33]

$$\delta_\epsilon \overline{J^A} = \overline{\epsilon}\left\{2iT^{AB}\Gamma_B - \frac{1}{4}i\mathrm{Tr}\left(G_{BC}G_{DE}\right)\Gamma^{ABCDE}\right\} + \text{fermions}\,,$$

where we have used the equation of motion, $\Gamma^{ABCDE} = 1/5!\Gamma^{[A}\ldots\Gamma^{E]}$ and T^{AB} is the energy-momentum tensor

$$T^{AB} = \mathrm{Tr}\left(G^{AC}G_C^B + \frac{1}{4}\eta^{AB}G^{CD}G_{CD} + \frac{1}{2}i\overline{\lambda}\Gamma^A D^B\lambda\right)\,.$$

Now using that

$$\begin{aligned}
\Gamma^{\alpha_1\cdots\alpha_k} &= \frac{(-1)^{k\frac{(k-1)}{2}+D\frac{(D-1)}{2}}}{(D-k)!}\epsilon^{\alpha_1\cdots\alpha_D}\Gamma_{\alpha_{k+1}\cdots\alpha_D}\Gamma_{D+1}\,,\\
\overline{\epsilon}\Gamma_{D+1} &= \overline{\epsilon}\,,
\end{aligned}$$

where the last equality is due to the fact that $\overline{\epsilon}$ is a Weyl spinor. Let us now analyze each dimension separately:

1. $D = 6$:

$$\begin{aligned}
\delta_\epsilon\overline{J^A} &= 2i\overline{\epsilon}\left(T^{AF} + \theta^{AF}\right)\Gamma_F\,,\\
\theta^{AF} &\equiv \frac{1}{8}\epsilon^{ABCDEF}\mathrm{Tr}\left(G_{BC}G_{DE}\right)\,,
\end{aligned}$$

where we used $\overline{\epsilon}\Gamma^{ABCDE} = \overline{\epsilon}\epsilon^{ABCDEF}\Gamma_F$ in $D = 6$. Note that T^{AB} is symmetric whereas θ^{AB} is antisymmetric. Moreover θ^{AB} is conserved without use of the equations of motion and it doesn't depend on the metric. So we can say that θ^{AB} is a topological current. Since $\delta\overline{Q_\alpha} = \int d^5x\delta\overline{J^0_\alpha}$, using (116) and (117), we obtain that

$$\left\{\overline{Q_\alpha}, Q_\beta\right\} = \frac{1}{\hbar}2\Gamma_N^{\alpha\beta}\left(P^N + Z^N\right) \tag{118}$$

with

$$\begin{aligned}
P^N &= \int d^5xT^{0N}\,,\\
Z^N &= \int d^5x\theta^{0N}\,.
\end{aligned}$$

So we see that $N=1$ super Yang-Mills in $D=6$ has a central charge which is a vector. In reality, it is not very rigorous to say that it is a central charge the of supersymmetry algebra since it has a non trivial commutation relation with the Lorentz generators.

2. $D=10$:

$$\begin{aligned} \delta_\epsilon \overline{J^A} &= 2i\overline{\epsilon}\left(T^{AF}\Gamma_F+\theta^{ABCDEF}\Gamma_{BCDEF}\right) \\ \theta^{A_1\cdots A_6} &\equiv \frac{1}{5!}\epsilon^{A_1\cdots A_{10}}\mathrm{Tr}\left(G_{A_7A_8}G_{A_9A_{10}}\right) , \end{aligned}$$

where $\theta^{A_1\cdots A_6}$ is also conserved without use of the equations of motion and doesn't depend on the metric. So $\theta^{A_1\cdots A_6}$ is a topological current. Then we obtain

$$\left\{\overline{Q_\alpha},Q_\beta\right\}=\frac{1}{\hbar}2\left(P^N\Gamma_N^{\alpha\beta}+Z^{ABCDE}\Gamma_{ABCDE}^{\alpha\beta}\right) \tag{119}$$

with

$$\begin{aligned} P^N &= \int d^5xT^{0N} , \\ Z^{ABCDE} &= \int d^5x\theta^{0ABCDE} . \end{aligned}$$

So we see that $N=1$ super Yang-Mills in $D=10$ has a central charge which is a 5-form(since it is completely antisymmetric).

Doing the dimensional reduction either from $D=6$ or $D=10$ to $D=4$ we find that the central charge for $N=2$ and $N=4$ super Yang-Mills is[31]

$$z=a\left(q+ig\right) .$$

Therefore, the mass of the short representation is

$$M=|z|=|a\left(q+ig\right)| .$$

However, we have seen before in (40) that, monopoles, dyons and gauge particles satisfy this mass formula exactly. So they must belong to short representations.

We have seen that $N = 4$ has a unique short representation. Therefore, we conclude that the gauge particles, the monopoles and dyons must belong to vector supermultiplets with the same spin content, having 1 vector, 4 Majorana spinors and 6 scalars. That gives an affirmative answer for the first question if duality transformations are maps preserving the number of degrees of freedom. So the Montonen-Olive (or the more general $SL(2, Z)$) conjecture is in good shape, at least for $N = 4$ super Yang-Mills.

The fact that the monopole in $N = 4$ belongs to the vector supermultiplet was confirmed explicitly by Osborn[33] using semiclassical methods.

For $N = 2$ the situation is a little more complicated since there are two short representations. The gauge particles belong to a vector supermultiplet and the quarks to a hypermultiplet. It was shown by Osborn[33] that the monopoles belong to a hypermultiplet, like the quarks. So, the original Montonen-Olive conjecture, of a mapping between gauge particles and monopoles, cannot work in $N = 2$ super Yang-Mills since they belong to different supermultiplets with different spin content. However, it was proposed by Seiberg and Witten[34][35] that for $N = 2$ one should think about a duality between monopoles and quarks. In reality, they consider $N = 2$ $SU(2)$ Super Yang-Mills with N_f quarks, with $N_f = 0, ..., 4$. They proposed that the duality transformations would be a subgroup of $SL(2, Z)$ and that this subgroup would depend on the number N_f of quarks. Moreover this duality would hold not for the original theory (also called microscopic), but for the effective theory in which one considers just the massless modes. We will not discuss this topic, and continue to analyse the original Monotonen-Olive proposal or its $SL(2, Z)$ extension.

5.2 The β-function and the quantum corrections to Super Yang-Mills

The second question we have raised about the duality conjecture was concerning to the quantum corrections to the mass formula. We have found the mass formula (86) which holds at the classical level and which was the key ingredient for the formulation of the conjecture. The question is if the quantum corrections spoil this mass formula. In order to answer this question we must use the β -function which for a generic Yang-Mills theory, for an arbitrary gauge group is given by

$$\beta \sim \left(-\frac{11}{6}\chi_{\mathrm{Gauge}} + \frac{F}{3}\chi_{\mathrm{Weyl}} + \frac{S}{6}\chi_{\mathrm{scalars}}\right) e^3 + O(e^5) \tag{120}$$

where χ is the Dynkin index of the representation, F is the number of Weyl spinors and S is the number of complex scalars.

Consider $N = 2$ $SU(N_c)$ Super Yang-Mills with N_F quarks in the fundamental representation. Remember that $\chi = 1$ for the adjoint representation and $\chi = 1/2$ for the fundamental representation of $SU(N_c)$ and that the gauge particle is a vector supermultiplet with 1 vector, 2 Weyl spinors and 1 complex scalar, and the quarks are in a hypermultiplet with 2 Weyl spinors and 2 complex scalars. Then, using (120) it results that

$$\beta \sim \left(-N_c + \frac{N_F}{2}\right) e^3 \,. \tag{121}$$

This result is exact! For $N = 2$ (and $N = 4$) the β -function receives just one-loop contributions(in the pertubative expanssion). The reason for this is because at the quantum level

$$\theta^{\mu}_{\mu} \sim \frac{\beta(e)}{e^3}\left[\mathrm{Tr}\,(G_{\mu\nu}G^{\mu\nu}) + \mathrm{susy} \quad \mathrm{terms}\right] . \tag{122}$$

But θ^{μ}_{μ} and $\partial_\mu J^{\mu}_5$ belong to the same supermultiplet in $N = 2$ where, where J^{μ}_5 is the chiral current. Since $\partial_\mu J^{\mu}_5$ receives only one loop quantum contributions from the chiral anomaly, the same must be true for θ^{μ}_{μ} and consequently for $\beta(e)$ using (122).

From (121) we see that $\beta \equiv 0$ only for $N_F = 2N_c$ and in particular for $N_F = 4$ for $SU(2)$. Therefore,it is only in these cases the coupling constant e doesn't receive quantum corrections. The same is also true for θ since in these cases the chiral anomaly vanishes. That couplings do not receive quantum corrections implies that the mass also will not receive quantum corrections and therefore the mass formula (86) also holds at the quantum level.

For $N = 4$ Super Yang-Mills there is just the vector supermultiplet and it is straightforward to prove that

$$\beta \equiv 0 \,.$$

Like in $N = 2$, this result is exact and there is no quantum correction to the coupling constants e and θ. So, once more the mass formula (86) also holds at the quantum level. And therefore the duality conjecture remains in good shape for $N = 4$ Super Yang-Mills(SYM).

6 Sen's test for duality conjecture.

In this section we shall submit the duality conjecture to another test which was proposed by A. Sen[25]. Let us now assume that $SL(2, Z)$ duality is true for $N = 4$, note the consequences and check if they are true.

Let us take the duality transformation given in (91). The gauge particle with ($n_m = 0$, $n_e = 1$) is mapped to a state with ($n_m = -C$, $n_e = A$). From the condition that $AD - BC = 1$ is implied that A and C are coprime numbers, that is, they don't have a common factor (other than ± 1). Indeed, if n were a common factor: $A = nA'$ and $C = nC'$ for integers A' and C' we would have that $n(A'D - BC') = 1$. But $(A'D - BC')$ is an integer, which forces $n = \pm 1$. A BPS state $(n_m\, n_e)$ with n_m and n_e coprimes is stable. This states are indicated in fig. 1.1 with $\times$. So if $SL(2, Z)$ duality is true in $N = 4$ SYM we can say the following[25]: "The existence of the gauge particle W^+ with ($n_m = 0$, $n_e = 1$) in $N = 4$ SYM with coupling τ implies the existence of stable particles $(n_m\, n_e)$, with n_e and n_m coprime with coupling τ'. We assume that W^+ exist for all values of τ in the complex upper plane. In that case the other states $(n_m\, n_e)$ must also exist for all values of τ in the upper plane. Since W^+ belongs to a vector supermultiplet with dimension 16, it implies that $(n_m\, n_e)$ with n_e and n_m coprime must also belong to a vector supermultiplet of dimension 16." Indeed, the existence of some of these states was proven by A. Sen. Let us now see the argument.

The set of *static* monopoles with fixed magnetic charge $g = 4\pi n_m/e$ form a multiparameter family $(W_\mu(x^\mu, z^a)$, $\phi(x^\mu, z^a))$ of solutions of the equations of motion. This space of solutions form a manifold called *moduli space* $\mathcal{M}_{n_m}$ with the parameters z^a being the coordinates of this space. For the case $n_m = 1$,

$$\mathcal{M}_1 = R^3 \times S^1 \tag{123}$$

where R^3 corresponds to the position of the monopole center of mass and the momentum associated with the coordinate on S^1 corresponds to the monopole electric charge. For a generic n_m the moduli space have the general form[36]

$$\mathcal{M}_{n_m} = R^3 \times \frac{S^1 \times \mathcal{M}^0_{n_m}}{Z_{n_m}}, \tag{124}$$

where R^3 and S^1 is like before, Z_{n_m} is the cyclic group of n_m elements and $\mathcal{M}^0_{n_m}$ is the "rest" of the manifold. $\mathcal{M}_{n_m}$ has dimension $4n_m$ and we can define a metric

$$g_{ab} = -\int d^3x \mathrm{Tr}\left(\frac{\partial W_\mu}{\partial z^a}\frac{\partial W^\mu}{\partial z^b} + \frac{\partial \phi}{\partial z^a}\frac{\partial \phi}{\partial z^b}\right). \tag{125}$$

Manton[37][38] showed that the dynamics of slowly moving (or low energy) BPS monopoles with total magnetic charge g correspond to the geodesic motion of a particle on $\mathcal{M}_{n_m}$ given by the effective action

$$\begin{aligned} S_{eff} &= \int dt\left(\frac{1}{2}g_{ab}\dot{z}^a\dot{z}^b + a|g|\right) &(126)\\ &= \int dt\left\{\frac{1}{2}\left(a|g|\dot{R}_i^2 + \frac{4\pi}{e^3 a}\dot{\chi}^2 + g^0_{ab}\dot{z}^a_0\dot{z}^b_0\right) + a|g|\right\}, &(127)\end{aligned}$$

where g^0_{ab} is the metric on $\mathcal{M}^0_{n_m}$ and the coordinates R_i, χ and z^a_0 are the coordinates associated to R^3, S^1 and $\mathcal{M}^0_{n_m}$ respectively.

This result was generalised by Gauntlett[39] and Blum[40] for monopoles in $N = 4$ Super Yang-Mills. In this case the dynamics of slowly moving monopoles with total magnetic charge g correspond to the geodesic motion of a superparticle on $\mathcal{M}_{n_m}$ given by the effective action

$$S_{\text{eff}} = S_0 + S_{int}$$

with

$$\begin{aligned} S_0 &= \int dt\left[\frac{1}{2}\sum_{a=1}^{4}\left(\dot{x}_a^2 + i\overline{\eta}_a\gamma^o\partial_0\eta_a\right) + a|g|\right], &(128)\\ S_{int} &= \int dt\frac{1}{2}\left[g^0_{ab}\left(\dot{z}^a_0\dot{z}^b_0 + i\overline{\lambda^a}\gamma^0 D_0\lambda^b + \frac{1}{6}R_{abcd}\overline{\lambda^a}\lambda^b\overline{\lambda^c}\lambda^d\right)\right], &(129)\end{aligned}$$

where η^a, λ^a are 2 component Majorana spinors, $D_0\lambda^b = \partial_0\lambda^b + \Gamma^b_{ac}\partial_0 z^a_0\,\lambda^c$, Γ^b_{ac} is the Christoffel symbol and R_{abcd} is the Riemann curvature tensor on $\mathcal{M}^0_{n_m}$. S_0 is the part of S_{eff} which depends on the bosonic and fermionic coordinates associated to $R^3 \times S^1$, and for simplicity we haven't written the coefficients explicitly. From S_{eff} we obtain the Hamiltonian

$$H = H_0 + H_{int}$$

with

$$H_0 = \frac{1}{2a|g|} P_i^2 + \frac{ae^3}{8\pi n_m} \Pi^2 + a|g| \, ,$$

where $P_i = a|g|\dot{R}_i$, $i = 1, 2, 3$, $\Pi = \frac{4\pi n_m}{e^3 a}\dot{\chi}$. Note that H_0 (and therefore H) doesn't depend on η^a. The quantum theory of a low energy monopole is equivalent to the quantum theory of this super particle. So, let us proceed with the canonical quantization. One can combine the 8 real spinors η^a_α in 4 complex spinors a^a_α, $a^{a\dagger}_\alpha$ with $a = 1, 2$ and $\alpha = 1, 2$, which satisfy

$$\left\{ a^a_\alpha, a^{a\dagger}_\alpha \right\} = i\hbar \delta^{ab} \delta_{\alpha\beta} \, ,$$

which is a Clifford algebra with 8 generators. It has an irrep of 16 states as we have seen in section 4.1. Since H doesn't depend on η^a (or a^a_α, $a^{a\dagger}_\alpha$), if

$$H|\psi >= E|\psi >$$

$$\Rightarrow H \left(a^{a_1 \dagger}_{\alpha_1} a^{a_2 \dagger}_{\alpha_2} \cdots |\psi > \right) = E \left(a^{a_1 \dagger}_{\alpha_1} a^{a_2 \dagger}_{\alpha_2} \cdots |\psi > \right) \, ,$$

and therefore each eigenstate of H has a 16 fold degeneracy.

Since χ is a periodic coordinate, in the quantum theory, Π must be quantized, i.e., $\Pi = n_e \hbar$ which is interpreted as the total electric charge[37]. Then, in the rest frame, $P_i = 0$,

$$\begin{aligned} H_0 &= \frac{ae^3}{8\pi n_m} (n_e \hbar)^2 + a \frac{4\pi n_m}{e} \\ &= a \frac{4\pi n_m}{e} \left(1 + \frac{(n_e \hbar e^2)^2}{32\pi^2 n_m^2} \right) \\ &\cong a \left[(n_e \hbar e)^2 + \left(\frac{4\pi n_m}{e} \right)^2 \right]^{1/2} = a\sqrt{q^2 + g^2}, \end{aligned}$$

where in the last line we considered the low energy approximation $\hbar e^2 \cong 0$. Therefore on a eigenstate $|\psi >$,

$$H_0 |\psi >= a\sqrt{q^2 + g^2} |\psi > \, .$$

But, in the rest frame

$$H|\psi>=M|\psi>=a\sqrt{q^2+g^2}|\psi> .$$

So we can conclude that

$$H_{int}|\psi>=0\,,$$

and the wave function must have the form

$$\psi=e^{(i\overrightarrow{p}\cdot\overrightarrow{R}+i\Pi\chi)/\hbar}\phi(z_0)=e^{i\frac{\overrightarrow{p}\cdot\overrightarrow{R}}{\hbar}+in_e\chi}\phi(z_0)\,,$$

where the first factor is eigenfunction of H_0 and the second factor is an eigenfunction of H_{int}. Under Z_{n_m}, $\chi\rightarrow\chi+\frac{2\pi}{n_m}$ and

$$\phi(z_0)\rightarrow e^{-2\pi i\frac{n_e}{n_m}}\phi(z_0)\,. \tag{130}$$

What is the form of $\phi(z_0)$? A long time ago [41] Witten have analised the problem of a super particle moving on a manifold with the dynamics governed by H_{int} and proved that

$$H_{int}=\left\{Q,Q^\dagger\right\}\,,$$

where Q, $Q^\dagger$ are the supercharges for the superparticle problem. They satisfy $Q^2=0=Q^{2\dagger}$, i.e., they are nilpotent. Therefore Q and $Q^\dagger$ act as exterior derivatives on $\mathcal{M}^0_{n_m}$ .Then we substitute the condition $H_{int}\phi=\left\{Q,Q^\dagger\right\}\phi=0$, to

$$\left\{d,d^\dagger\right\}\phi=\left(dd^\dagger+d^\dagger d\right)\phi=\Box\phi=0\,,$$

and conclude that ϕ is a harmonic form on $\mathcal{M}^0_{n_m}$.

In summary we obtained that the quantum theory of slowing moving (or low energy) monopoles (or dyon) with magnetic and electric numbers (n_m,n_e) is equivalent to the quantum theory of a super particle moving on a manifold $\mathcal{M}_{n_m}$. The wave function has the form

$$\psi=e^{i\frac{\overrightarrow{P}\cdot\overrightarrow{R}}{\hbar}+in_e\chi}\phi(z_0)\,,$$

where $\phi(z_0)$ under Z_{n_m} transformation (130) and is a harmonic form on $\mathcal{M}^0_{n_m}$ which has dimension $4(n_m-1)$. Moreover, since we want the final state to have a 16-fold degeneracy, and since the quantization of H_0 already gives rise to this degeneracy, we need to have a unique $\phi(z_0)$ for which n_e and n_m coprime. Now, given a harmonic p form on $\mathcal{M}^0_{n_m}$ we can always construct a harmonic $4(n_m - 1) - p$ form on $\mathcal{M}^0_{n_m}$ by taking the Hodge dual. This would violate the condition that $\phi(z_0)$ should be the unique harmonic form on $\mathcal{M}^0_{n_m}$. The only exception is the case when it is an (anti-)self-dual $2(n_m - 1)$ form.

Therefore $SL(2, Z)$ duality requires that for every integer n_e for which n_e and n_m are coprime, the space $\mathcal{M}^0_{n_m}$ must contain a normalizable (anti-) self-dual harmonic $2(n_m - 1)$ form $\phi(z_0)$ which pick-up a phase $e^{-2\pi i n_e/n_m}$ under the action of Z_{n_m} transformations.

The metric of $\mathcal{M}^{(0)}_2$ is known[36][38]. So these harmonic forms were constructed explicitly by A.Sen for $n_m = 2$ and n_e odd confirming the duality conjecture. On the other hand, the metric on the other $\mathcal{M}^0_{n_m}$ are not known. However, Segal and Selby[42], considering some topological assumptions on these manifolds, claimed a proof of existence of these harmonic forms whenever n_e and n_m are coprimes.

Until now all tests that have been done are consistent with the duality conjecture for $N = 4$. However a rigorous proof is still lacking. There are also some generalizations of the original conjecture for $N = 2$, $N = 1$ SYM, for gauge groups other than $SU(2)$ and for superstring theories.

Acknowledgments

I would like to thank the X Jorge André Swieca Summer School Organizing Committee for the invitation to give lectures in this so nice school, to D. Olive for many discussions, for P. Brockill, J.A. Helayel-Neto and M. S. Sarandy for reading the manuscript and many suggestions, and FAPERJ for financial support.

References

[1] S. Coleman, *Phys. Rev.* **D11** (1975) 2088.

[2] S. Mandelstam, *Phys. Rev.* **D11** (1975) 3026.

[3] P. Goddard and D.I. Olive, *Rep. Prog. Phys.,* **41** (1978) 1357.

[4] D.I. Olive, "*Exact electromagnetic duality*", hep-th/9508089.

[5] P. Di Vecchia, *"Duality in N=2,4 supersymmetric gauge theories"*, hep-th/9803026.

[6] J.A. Harvey, "*Magnetic Monopoles, Duality and Supersymmetry*", hep-th/9603086.

[7] A. Bilal, "*Duality in N=2 SUSY SU(2) Yang-Mills Theory: a pedagogical introduction to the work of Seiberg-Witten*", hep-th/9601007.

[8] L. Alvarez-Gaumme and S.F. Hassan, "*Introduction to S-duality in N=2 Supersymmetric Gauge Theories*", hep-th/9701069.

[9] M. Peskin, "*Duality in Supersymmetric Gauge Theories*", hep-th/9702094.

[10] P.A.M. Dirac, *Proc. Roy. Soc.* **A33** (1931) 60.

[11] J. Schwinger, *Science* **165** (1969) 757.

[12] D. Zwanziger, *Phys. Rev.* **176** (1968) 1489.

[13] G. 't Hooft, *Nucl. Phys.* **B79** (1974) 276.

[14] A. Polyakov, *JETP Lett.* **20** (1974) 194.

[15] E.J. Weinberg, *Nucl. Phys.* **B167**(1980) 500 and *Nucl. Phys.* **B203** (1974) 445.

[16] E.B. Bogomolny, *Sov. J. Nucl. Phys.* **24** (1976) 449.

[17] M.K. Prasad and C.M. Sommerfield, *Phys. Rev. Lett.* **35** (1975) 760.

[18] C.H. Taubes, *Comm. Math. Phys.* **91** (1983) 473.

[19] B. Julia and A. Zee, *Phys. Rev.* **D11** (1975) 2227.

[20] J. Arafune, P.G.O. Freund, C.J. Goebel, *J. Math. Phys.* **16** (1975) 433.

[21] C. Montonen and D.I. Olive, *Phys. Lett.* **72B** (1977) 117.

[22] N.S. Manton, *Nucl. Phys.* **B126**(1977) 525.

[23] E. Witten, *Phys. Lett.* **86B** (1979) 283.

[24] Jackiw in "Current Algebra and Anomalies" ed. S.B. Treiman et al..World Scientific Publishing Co., Singapore, 1985.

[25] A.Sen, *Phys. Lett.* **329B** (1994) 217.

[26] E. Witten, "*On S-duality in Abelian gauge theory*" hep-th/9505186.

[27] E. Verlinde, *Nucl. Phys.* **B455** (1995) 211.

[28] G. Thompson, "*New results in Topological Field Theory an Abelian Gauge Theory*", hep-th/9511038.

[29] M.F. Sohnius, *Phys. Rep.* **128** (1985), 39.

[30] P. West, *"Introduction to supersymmetry and supergravity"*, Word Scientific 1990.

[31] E. Witten and D.I. Olive, *Phys. Lett.* **78B** (1978) 97.

[32] P.A. Zizzi, *Nucl. Phys.* **B228** (1983) 229 and *Phys. Lett.* **149B** (1984) 333.

[33] H. Osborn, *Phys. Lett.* **83B** (1979) 321.

[34] N. Seiberg and E. Witten, *Nucl. Phys.* **B426** (1994) 19.

[35] N. Seiberg and E. Witten, *Nucl. Phys.* **B431** (1994) 484.

[36] M.F. Atiyah and N.J. Hitchin, *Phys. Lett.* **107B** (1985) 21; *"The geometry and dynamics of magnetic monopole"*, Princeton University Press, 1988.

[37] N.S. Manton, *Phys. Lett.* **110B** (1982) 54.

[38] G.W. Gibbons and N.S. Manton, *Nucl. Phys.* **B274** (1986) 183.

[39] J.P. Gauntlett, *Nucl. Phys.* **B411** (1994) 433.

[40] J. Blum, *Phys. Lett.* **333B** (1994) 92.

[41] E. Witten, *Nucl. Phys.* **B202** (1982) 253.

[42] G. Segal and A. Selby, *Comm. Math. Phys.* **177** (1996) 775.