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On Bosonization Ambiguities of Two Dimensional Quantum Electrodynamics

*S. A. Dias and M. B. Silva Neto**

Centro Brasileiro de Pesquisas Físicas - CBPF
Departamento de Teoria de Campos e Partículas - DCP
Rua Dr. Xavier Sigaud, 150 - Urca
22290-180 - Rio de Janeiro, RJ - Brazil

ABSTRACT

We study bosonization ambiguities in two dimensional quantum eletrodynamics in the presence and in the absence of topologically charged gauge fields. The computation of fermionic correlation functions gives us a mechanism to fix the ambiguities in nontrivial topologies, provided that we do not allow changes of sector as we evaluate functional integrals. This removes an infinite arbitrariness from the theory. In the case of trivial topologies, we find upper and lower bounds for the Jackiw-Rajaraman parameter, corresponding to the limiting cases of regularizations which preserve gauge or chiral symmetry.

Key-words: Bosonization; Schwinger model; Chiral symmetries.

*e-mail: tiao@cbpfsu1.cat.cbpf.br and silvanet@cbpfsu1.cat.cbpf.br

1 Introduction

The study of models in dimensions other than four frequently brings out features that give insights about what happens in higher dimensions. In two dimensions, the procedure known as bosonization [1][2] sometimes gives a gaussian expression to the functional integral, making it possible to obtain the exact solution and then to compare the results with those given by perturbative techniques. This allows the explicit study of characteristics such as charge screening and vacuum structure [3][4][5] which are very important for theories like QCD_4.

A quite intriguing feature of some of these models is the appearance of ambiguities during the regularization of some ill defined quantities. This is the case of QED_2, with or without chiral fermions [6][7] and of the Thirring model [8], where bosonization introduces an arbitrary parameter a which shifts the position of the pole in the photon propagator. This position is a physical observable [9], the mass of the gauge boson, dynamically generated. On the other hand, in the input lagrangean, we have only one parameter to be fixed by "experimental data", the charge e of the fermion. The new parameter is thus completely arbitrary, giving us the impression that we ended up with ill defined predictions after the end of the quantization procedure.

Thus, we see ourselves facing this question: are there physical parameters of a quantum field theory to which we have no classical access? In the case of QED_2 with Dirac fermions (the so called *Schwinger model*) this question is not usually asked because the value $a = 1$ preserves gauge invariance at quantum level, providing the easiest approach to this model. However, if we consider chiral fermions, there is no value for a that does this job, and so the question arises. Considering again the Schwinger model, we can see that there is no intrinsic (physical) reason to consider the particular value $a = 1$. We should study this theory for other values of a in order to decide if different values give different physical implications or not.

In a previous work [10], we have noticed that the computation of correlation functions in nontrivial topology sectors could give a mathematical criterium to fix the value of a for each given sector (except the trivial one). This has raisen the hope that, perhaps with a mix between physical requirements and mathematical skill, one could decide in favour of a fixed value for a. So, we decided to face this question in the context of the Schwinger model, which is very well known for $a = 1$, and to see if this value is favoured by arguments other than gauge invariance.

The paper is organized as follows: in section 1 we briefly review QED_2 in the general case where the gauge field can be given a topological charge; in section 2 we compute the contributions of these nontrivial sectors to correlation functions with general a and give an argument to fix its value in all these sectors; in section 3 we perform the same analysis in the trivial topology sector and find restrictions, based on physical requirements, in the range of values that a can assume. Finally, in section 4, we present our conclusions and some remarks.

2 The model

We will study quantum electrodynamics in two dimensional euclidean space described by the action functional

$$S=\int d^2x\mathcal{L}\left(A_\mu,\overline{\psi},\psi\right)=\int d^2x\left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\overline{\psi}D\psi\right], \tag{1}$$

where D is the Dirac operator

$$D\equiv\gamma^\mu\left(i\partial_\mu+eA_\mu\right). \tag{2}$$

Our γ matrices satisfy

$$\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu},\qquad \gamma_5=i\gamma_0\gamma_1,\qquad \gamma_\mu^\dagger=\gamma_\mu,$$

which implies, in two dimensions,

$$\gamma_\mu\gamma_5=i\epsilon_{\mu\nu}\gamma_\nu,$$

where $\epsilon_{01}=-\epsilon_{10}=1$.

The generating functional of correlation functions for the Schwinger model is given by

$$Z\left[J^\mu,\overline{\eta},\eta\right]=\int\left[dA_\mu\right]\left[d\overline{\psi}\right]\left[d\psi\right]\exp\left(-S+\langle J^\mu A_\mu\rangle+\langle\overline{\eta}\psi\rangle+\langle\overline{\psi}\eta\rangle\right), \tag{3}$$

where $J^\mu,\overline{\eta}$ and η are the external sources associated with the fields A_μ,ψ and $\overline{\psi}$, respectively. In order to define the functional measure in (3), we write the fermionic fields as linear combinations

$$\psi\left(x\right) = \sum_n a_n\varphi_n\left(x\right)+\sum_i a_{0i}\varphi_{0i}\left(x\right), \tag{4}$$

$$\overline{\psi}\left(x\right) = \sum_n \overline{a}_n\varphi_n^\dagger\left(x\right)+\sum_i \overline{a}_{0i}\varphi_{0i}^\dagger\left(x\right), \tag{5}$$

of the eigenfunctions of D

$$D\left(A_\mu\right)\varphi_n\left(x\right) = \lambda_n\varphi_n\left(x\right), \tag{6}$$

$$D\left(A_\mu\right)\varphi_{0i}\left(x\right) = 0, \tag{7}$$

with a_n, $\overline{a}_n$ and a_{0i}, $\overline{a}_{0i}$ being grassmanian coeficients. Now, the fermionic functional measure is simply

$$\left[d\overline{\psi}\right]\left[d\psi\right]=\prod_n d\overline{a}_n da_n\prod_i d\overline{a}_{0i}da_{0i},$$

such that, after an integration over fermi fields, the fermionic part of the generating functional can be written as

$$Z_F\left[\overline{\eta},\eta\right]\propto \det{}'D.$$

In the above expression, $\det'D$ stands for the product of all nonvanishing eigenvalues of D.

As it is well known [11], the appearance of N zero eigenvalues associated to the Dirac operator, the zero modes, is closely related to the existence of classical configurations, in the gauge field sector, which can be written as [12][13]

$$eA_{\mu}^{(N)} = -\tilde{\partial}_{\mu} f,$$

where the function $f(x)$ behaves, at infinity, as

$$\lim_{x \to \infty} f(x) \simeq -N \ln |x| .$$

These configurations carry a topological charge $Q = N$, where Q is given by

$$Q = \frac{1}{4\pi} \int d^2x \epsilon_{\mu\nu} F_{\mu\nu}.$$

For future purposes, we will define

$$A_{\mu}^{\alpha} = A_{\mu}^{(N)} + \alpha a_{\mu},$$

where α is an interpolating parameter [10] between the fixed configuration $A_{\mu}^{(N)}$ $(\alpha = 0)$ and a general configuration with topological charge N $(\alpha = 1)$. In two dimensions we can always write any configuration of charge N linearly in terms of a fixed one, due to the additive property of the topological charge. The field a_{μ} has vanishing topological charge and can always be written as

$$ea_{\mu} = \partial_{\mu}\rho - \epsilon_{\mu\nu}\partial_{\nu}\phi. \tag{8}$$

The Dirac operator

$$D_{\alpha} = \gamma^{\mu}\left(i\partial_{\mu} + e\left(A_{\mu}^{(N)} + \alpha a_{\mu}\right)\right),$$

has an inverse only if we add a small mass ϵ >0,

$$\left(D_{\alpha} + \epsilon \mathbf{1}\right)^{-1}(x,y) = \mathbf{S}_{\epsilon}^{\alpha}(x,y) + \frac{1}{\epsilon}\mathbf{P}_{0}^{\alpha}(x,y),$$

where $P_0^{\alpha}(x,y)$ is the projector on the subspace generated by the zero modes of D_{α}

$$\mathbf{P}_{0}^{\alpha}(x,y) = \sum_{i=1}^{|N|} \varphi_{0i}^{\alpha}(x)\, \varphi_{0i}^{\alpha\dagger}(y), \tag{9}$$

and $\mathbf{S}_{\epsilon}^{\alpha}(x,y)$ inverts $D_{\alpha} + \epsilon\mathbf{1}$ in the rest of the space and is formally given by

$$\mathbf{S}_{\epsilon}^{\alpha}(x,y) = \sum_{n \neq 0} \frac{\varphi_{n}^{\alpha}(x)\, \varphi_{n}^{\alpha\dagger}(y)}{\lambda_{n}^{\alpha} + \epsilon}.$$

In the limit $\epsilon \to 0$, $\mathbf{S}_{\epsilon}^{\alpha}$ is regular and can be expressed as [14],

$$\mathbf{S}^{\alpha}(x,y) = G^{\alpha}(x,y) - \int d^2z G^{\alpha}(x,z)\, \mathbf{P}_{0}^{\alpha}(z,y) - \int d^2z \mathbf{P}_{0}^{\alpha}(x,z)\, G^{\alpha}(z,y), \tag{10}$$

where $G^{\alpha}(x,y)$ is the fermionic Green function. We can see that $\mathbf{S}^{\alpha}$ satisfies

$$D_{\alpha}\mathbf{S}^{\alpha}(x,y)=\delta(x-y)-\mathbf{P}_0^{\alpha}(x,y)=\mathbf{S}^{\alpha}(x,y)D_{\alpha}. \tag{11}$$

In the sector associated with topological charge N, we shift the fermions by

$$\begin{aligned}\psi(x) &\rightarrow \psi(x)-\int d^2y\mathbf{S}(x,y)\eta(y),\\ \overline{\psi}(x) &\rightarrow \overline{\psi}(x)-\int d^2y\overline{\eta}(y)\mathbf{S}(y,x),\end{aligned}$$

where $\mathbf{S}(x,y)=\mathbf{S}^{\alpha=1}(x,y)$, to obtain

$$\begin{aligned}Z[J^{\mu},\overline{\eta},\eta] &= \int[dA_{\mu}]\left[d\overline{\psi}\right][d\psi]\exp\langle\overline{\eta}\mathbf{S}\eta\rangle\\ &\quad\times\exp\left(-S+\langle J^{\mu}A\mu\rangle+\langle\overline{\eta}\mathbf{P_0}\psi\rangle+\langle\overline{\psi}\mathbf{P_0}\eta\rangle\right).\end{aligned}$$

At this point, it must be stressed the role played by the external sources. They are responsible for preventing Z from vanishing, by picking up the zero modes explicitly.

Now we can bosonize the theory in this sector, performing the change of variables

$$\begin{aligned}\psi &\rightarrow \exp(-i\rho+\phi\gamma_5)\psi,\\ \overline{\psi} &\rightarrow \overline{\psi}\exp(i\rho+\phi\gamma_5),\end{aligned}$$

where ρ and ϕ were already defined in (8). Taking into account the Fujikawa jacobian, we will end up with [15]

$$\begin{aligned}Z[J^{\mu},\overline{\eta},\eta] &= \sum_N\int[da_{\mu}]\Im\left(a_{\mu},A_{\mu}^{(N)}\right)\exp\left(\left\langle\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right\rangle+\langle J^{\mu}A\mu\rangle\right)\\ &\quad\times\exp\left(\left\langle\overline{\eta}'\mathbf{S}^{(\mathbf{N})}\eta'\right\rangle\right)\det{}'D^{(N)}\prod_{i=1}^{|N|}\left\langle\overline{\eta}'\varphi_{0i}^{(N)}\right\rangle\left\langle\varphi_{0i}^{(N)\dagger}\eta'\right\rangle,\end{aligned} \tag{12}$$

where

$$\begin{aligned}\eta' &= \exp(i\rho+\phi\gamma_5)\eta,\\ \bar{\eta}' &= \bar{\eta}\exp(-i\rho+\phi\gamma_5),\end{aligned} \tag{13}$$

and

$$\Im\left(a_{\mu},A_{\mu}^{(N)}\right)\equiv\frac{\det' D}{\det' D^{(N)}}.$$

To compute this ratio, we can make use of the formal relation

$$\det D=\exp Tr\ln D,$$

where, instead of D, we use [16]

$$\det{}'D_{\alpha}=\lim_{\epsilon\rightarrow 0^+}\frac{\det(D_{\alpha}+\epsilon\mathbf{1})}{\epsilon^{|N|}}.$$

Now,

$$\frac{d}{d\alpha}\det{}'D_\alpha = \lim_{\epsilon\to 0^+}\frac{\det\left(D_\alpha+\epsilon\mathbf{1}\right)}{\epsilon^{|N|}}Tr\left[\left(D_\alpha+\epsilon\mathbf{1}\right)^{-1}\frac{dD_\alpha}{d\alpha}\right], \tag{14}$$

or, in the limit $\epsilon\to 0^+$

$$\frac{d}{d\alpha}\ln\det{}'D_\alpha = Tr\left[\mathbf{S}^\alpha\frac{dD_\alpha}{d\alpha}\right].$$

Writing $D\equiv D_{\alpha=1}$ and $D^{(N)}\equiv D_{\alpha=0}$, we do the integration in α to obtain

$$\ln\frac{\det{}'D}{\det{}'D^{(N)}} = \int_0^1 d\alpha Tr\left[\mathbf{S}^\alpha\frac{dD_\alpha}{d\alpha}\right]. \tag{15}$$

The computation of this trace requires the use of some regularization procedure. Using, for example, the point-splitting regularization [7][17], we obtain

$$\ln\frac{\det{}'D}{\det{}'D^{(N)}} = -\Gamma\left[a_\mu\right]-\overline{\Gamma}\left[A_\mu^{(N)},a_\mu\right]+\ln\det\left|\left\langle\varphi_{0i}^{(N)\dagger}\exp\left(2\phi\gamma_5\right)\varphi_{0j}^{(N)}\right\rangle\right|, \tag{16}$$

where

$$\begin{aligned}\Gamma\left[a_\mu\right] &= \frac{e^2}{4\pi}\int d^2x a_\mu\left(a\left(N\right)\delta_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{\Box}\right)a_\nu, \\ \overline{\Gamma}\left[A_\mu^{(N)},a_\mu\right] &= \frac{e^2}{2\pi}\int d^2x a_\mu\left(a\left(N\right)\delta_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{\Box}\right)A_\nu^{(N)},\end{aligned} \tag{17}$$

and we have written the zero modes of D_α in terms of the ones of $D^{(N)}$,

$$\varphi_{0i}^\alpha = \exp\left(\alpha\left(i\rho+\phi\gamma_5\right)\right)\sum_{j=1}^N D_{ij}\varphi_{0j}^{(N)},$$

where the D_{ij} are introduced to insure the orthogonality of the φ_{0i}^α. One should notice the presence of the parameters $a\left(N\right)$ in the results above, that come from the path ordered exponential, put for occasionally keep gauge invariance ($a=1$). We remark that $a\left(N\right)$ can be chosen independently in each topological sector, what gives an infinite degree of arbitrariness to the theory.

Actually, it is more convenient to express the generating functional in terms of the original (non orthonormal) set of eingenfunctions of $D^{(N)}$, obtained by directly solving

$$D^{(N)}\Phi_{0i}^{(N)} = 0,$$

where the $\Phi_{0i}^{(N)}$, in terms of the light-cone variables, is given by [12][13]

$$\Phi_{0i}^{(N)} = \begin{cases} z^{i-1}\exp f\binom{1}{0}, & i=1,\ldots,N, N>0 \\ \bar{z}^{i-1}\exp\left(-f\right)\binom{0}{1}, & i=1,\ldots,-N, N<0. \end{cases} \tag{18}$$

The next step for obtaining the generating functional (12) is the computation of $\det{}'D^{(N)}$. This can be done if we use the method presented in [10], where a functional differential equation for $\det{}'D^{(N)}$

$$\frac{\delta}{\delta f\left(x\right)}\det{}'D^{(N)} = \det{}'D^{(N)}\left[\frac{a\left(N\right)}{2\pi}\Box f\left(x\right)+2tr\left(\mathbf{P}_0^{(N)}\left(x,x\right)\gamma_5\right)\right],$$

can be solved to give

$$\det{}'D^{(N)} = \exp\left(-\Gamma'\right)\det\left(\left\langle \Phi_{0i}^{(N)\dagger}\Phi_{0j}^{(N)}\right\rangle\right), \tag{19}$$

where

$$\Gamma'\left[A_\mu^{(N)}\right] = \frac{e^2 a\left(N\right)}{4\pi}\int d^2x f\Box f.$$

Finally, if we observe that

$$\mathbf{S}^{(N)}\left(x,y\right) = G^{(N)}\left(x,y\right) - \int d^2z G^{(N)}\left(x,z\right)\mathbf{P}_0^{(N)}\left(z,y\right) - \int d^2z \mathbf{P}_0^{(N)}\left(x,z\right)G^{(N)}\left(z,y\right),$$

where

$$G^{(N)}\left(x,y\right) = \left\{\exp\left(f\left(x\right)-f\left(y\right)\right)\mathbf{P}_+ + \exp\left(-\left(f\left(x\right)-f\left(y\right)\right)\right)\mathbf{P}_-\right\}G_F\left(x,y\right),$$

we conclude, due to the anti-commuting nature of $\left\langle \overline{\eta}'\Phi_{0i}^{(N)}\right\rangle$ and $\left\langle \Phi_{0i}^{(N)\dagger}\eta'\right\rangle$, that we can write

$$\exp\left\langle \overline{\eta}'\mathbf{S}^{(N)}\eta'\right\rangle \prod_{i=1}^{|N|}\left\langle \overline{\eta}'\Phi_{0i}^{(N)}\right\rangle\left\langle \Phi_{0i}^{(N)\dagger}\eta'\right\rangle = \exp\left\langle \overline{\eta}'G^{(N)}\eta'\right\rangle \prod_{i=1}^{|N|}\left\langle \overline{\eta}'\Phi_{0i}^{(N)}\right\rangle\left\langle \Phi_{0i}^{(N)\dagger}\eta'\right\rangle,$$

giving for Z the expression

$$Z\left[J^\mu,\overline{\eta},\eta\right] = \sum_N \int \left[da_\mu\right]\exp\left(-\overline{S} + \left\langle J^\mu A\mu\right\rangle + \left\langle \overline{\eta}'G^{(N)}\eta'\right\rangle\right)\prod_{i=1}^{|N|}\left\langle \overline{\eta}'\Phi_{0i}^{(N)}\right\rangle\left\langle \Phi_{0i}^{(N)\dagger}\eta'\right\rangle, \tag{20}$$

where

$$\overline{S} = \left\langle \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right\rangle + \Gamma\left[a_\mu\right] + \overline{\Gamma}\left[a_\mu, A_\mu^{(N)}\right] + \Gamma'\left[A_\mu^{(N)}\right].$$

It is important to stress here that this formula for the generating functional is independent of the choice of the representative $f\left(x\right)$, that is

$$\frac{\delta Z}{\delta f\left(x\right)} = 0.$$

3 Non Trivial Contributions to Correlation Functions

Being directly proportional to fermionic sources, it is not difficult to see that there are no contributions to bosonic correlation functions and that bosonic-fermionic ones do not give different information (concerning the ambiguities) than that given by the fermionic functions alone. We have non vanishing contributions from non trivial topologies to fermionic correlation functions of the type

$$\left\langle \prod_{i=1}^{k}\psi\left(x_i\right)\prod_{j=1}^{k}\overline{\psi}\left(y_j\right)\right\rangle = \frac{1}{Z\left[0\right]}\frac{\delta}{\delta\overline{\eta}_{\alpha_1}\left(x_1\right)}\cdots\frac{\delta}{\delta\eta_{\beta_k}\left(y_k\right)}\,Z\left[0,\overline{\eta},\eta\right]\Big|_{\overline{\eta}=\eta=0}.$$

It can be easily shown, by induction, that

$$\left\langle \prod_{i=1}^{k} \psi(x_i) \prod_{j=1}^{k} \overline{\psi}(y_j) \right\rangle = \sum_{k=-N}^{N} \int [da_\mu] \exp\left(-\bar{S}\right) \det \begin{vmatrix} \mathbf{\Phi}'^{(N)^\dagger} & \emptyset \\ \mathbf{G}'^{(N)} & \mathbf{\Phi}'^{(N)} \end{vmatrix}, \qquad (21)$$

where $\mathbf{\Phi}'^{(N)^\dagger}$ is a $N \times k$ matrix given by

$$\mathbf{\Phi}'^{(N)^\dagger} = \begin{pmatrix} \Phi_{01}'^{(N)^\dagger}(y_1) \cdots \Phi_{01}'^{(N)^\dagger}(y_k) \\ \vdots \\ \Phi_{0N}'^{(N)^\dagger}(y_1) \cdots \Phi_{0N}'^{(N)^\dagger}(y_k) \end{pmatrix},$$

$\mathbf{\Phi}'^{(N)}$ is $k \times N$,

$$\mathbf{\Phi}'^{(N)} = \begin{pmatrix} \Phi_{01}'^{(N)}(x_1) \cdots \Phi_{0N}'^{(N)}(x_1) \\ \vdots \\ \Phi_{01}'^{(N)}(x_k) \cdots \Phi_{0N}'^{(N)}(y_k) \end{pmatrix},$$

and

$$\mathbf{G}'^{(N)} = \begin{pmatrix} G'^{(N)}(x_1, y_1) \cdots G'^{(N)}(x_1, y_k) \\ \vdots \\ G'^{(N)}(x_k, y_1) \cdots G'^{(N)}(x_k, y_k) \end{pmatrix}$$

and $\emptyset$ (the null matrix) are square matrices $k \times k$ and $N \times N$ respectively, and

$$\begin{aligned} G'^{(N)}(x_i, y_j) &= \exp(-i\rho + \phi\gamma_5) G^{(N)}(x_i, y_j)(i\rho + \phi\gamma_5), \\ \Phi_{0i}'^{(N)}(x_j) &= \exp(-i\rho + \phi\gamma_5) \Phi_{0i}^{(N)}(x_j), \\ \Phi_{0i}'^{(N)^\dagger}(y_j) &= \Phi_{0i}^{(N)^\dagger}(y_j) \exp(i\rho + \phi\gamma_5). \end{aligned}$$

If we define

$$\chi_{ij} = \begin{cases} (z_j)^{i-1} \binom{1}{0}, & N > 0 \\ (\bar{z}_j)^{i-1} \binom{0}{1}, & N < 0 \end{cases}$$

and use the expressions for the zero modes (18) we can show that

$$\begin{aligned} \det\left(\mathbf{\Phi}'^{(\mathrm{N})}\right) &= \exp\left(\sum_{i=1}^{|N|} i\rho(y_i)\right) \exp\left(\pm \sum_{i=1}^{|N|} f(y_i) + \phi(y_i)\right) \det(\chi), \\ \det\left(\mathbf{\Phi}'^{(\mathrm{N})^\dagger}\right) &= \exp\left(-\sum_{i=1}^{|N|} i\rho(x_i)\right) \exp\left(\pm \sum_{i=1}^{|N|} f(x_i) + \phi(x_i)\right) \det\left(\chi^\dagger\right) \end{aligned}$$

and

$$\begin{aligned} \det\left(\mathbf{G}'^{(\mathrm{N})}\right) = & \exp\left(\sum_{i=|N|+1}^{k} i(\rho(y_i) - \rho(x_i))\right) \times \\ & \exp\left(\pm \sum_{i=|N|+1}^{k} f(y_i) - f(x_i) + \phi(y_i) - \phi(x_i)\right) \det(\mathbf{G_F}), \end{aligned}$$

where, according to the positiveness or not of N,

$$\begin{aligned}
\det(\chi) &= \prod_{\substack{i,j=1\\ i>j}}^{|N|} |z_i - z_j| \otimes \begin{pmatrix}1\\0\end{pmatrix}, \qquad N>0\\
\det\left(\chi^\dagger\right) &= \prod_{\substack{i,j=1\\ i>j}}^{|N|} |\bar{z}_i - \bar{z}_j| \otimes \begin{pmatrix}1 & 0\end{pmatrix}, \qquad N>0
\end{aligned}$$

or

$$\begin{aligned}
\det(\chi) &= \prod_{\substack{i,j=1\\ i>j}}^{|N|} |\bar{z}_i - \bar{z}_j| \otimes \begin{pmatrix}0\\1\end{pmatrix}, \qquad N<0\\
\det\left(\chi^\dagger\right) &= \prod_{\substack{i,j=1\\ i>j}}^{|N|} |z_i - z_j| \otimes \begin{pmatrix}0 & 1\end{pmatrix}, \qquad N<0.
\end{aligned}$$

Collecting these results, we arrive at

$$\left\langle \prod_{i=1}^{k} \psi(x_i) \prod_{j=1}^{k} \overline{\psi}(y_j) \right\rangle = \sum_{k=-N}^{N} \int [da_\mu] \exp\left(-\bar{S}_{\text{sources}}\right) \det \begin{vmatrix} \chi^\dagger & \emptyset \\ \mathbf{G}_F & \chi \end{vmatrix}, \tag{22}$$

where

$$\bar{S}_{\text{sources}} = \bar{S} - \left\langle i\left(j_\rho + j'_\rho\right)\rho\right\rangle \mp \left\langle (j+j')(f+\phi)\right\rangle$$

and j_ρ, j'_ρ, j and j' are defined by

$$\begin{aligned}
j_\rho &= \sum_{i=1}^{|N|} \delta(y_i - z) - \delta(x_i - z),\\
j'_\rho &= \sum_{i=|N|+1}^{k} \delta(y_i - z) - \delta(x_i - z),\\
j &= \sum_{i=1}^{|N|} \delta(y_i - z) + \delta(x_i - z),\\
j' &= \sum_{i=|N|+1}^{k} \delta(y_i - z) - \delta(x_i - z),
\end{aligned}$$

with $\langle\rangle$ representing integration over z.

There is still a last integration over the scalar fields ρ and ϕ in terms of which the gauge field is written. So we write the effective action $\overline{S}$ in terms of these fields

$$\overline{S} = \frac{1}{2e^2}\left\langle (\phi+f)\,\Box\left(\Box - \frac{e^2 a(N)}{\pi}\right)(\phi+f)\right\rangle + \frac{(1-a(N))}{2\pi}\langle \rho\Box\rho\rangle,$$

and do the following change of variables using the sources j_ρ, j'_ρ, j and j':

$$\sigma = \rho - \frac{1}{\lambda}\left\langle \Delta_F\left(j_\rho + j'_\rho\right)\right\rangle \tag{23}$$

and

$$\varphi = \phi + f \mp e^2 \left\langle \Delta\left(m; x-y\right)\left(j+j'\right)\right\rangle \tag{24}$$

where

$$\Delta_F\left(x-y\right) \equiv \Box^{-1}\left(x-y\right) = -\frac{1}{2\pi}\ln\left|x-y\right|$$

and

$$\Delta\left(m; x-y\right) \equiv \left[\Box\left(\Box - m^2\right)\right]^{-1}\left(x-y\right) = -\frac{1}{2\pi m^2}\left\{K_0\left[m\left|x-y\right|\right] + \ln\left|x-y\right|\right\}$$

and we have defined $\lambda \equiv \left(1-a\left(N\right)\right)/2\pi$ and $m^2 = \left(e^2 a\left(N\right)\right)/\pi$. Now we have for $\overline{S}$ plus the sources the expression

$$\begin{aligned} -\overline{S}_{sources} &= \frac{1}{2e^2}\left\langle \varphi\Box\left(\Box - \frac{e^2 a\left(N\right)}{\pi}\right)\varphi\right\rangle + \frac{e^2}{2}\left\langle\left(j+j'\right)\Delta\left(m\right)\left(j+j'\right)\right\rangle + \\ & \frac{\left(1-a\left(N\right)\right)}{2\pi}\left\langle\sigma\Box\sigma\right\rangle - \frac{1}{2\lambda}\left\langle\left(j_\rho + j'_\rho\right)\Delta_F\left(j_\rho + j'_\rho\right)\right\rangle. \end{aligned}$$

As we have already said, the scalar fields ρ and ϕ are such that a_μ does not carry a topological charge in the limit $|x| \to \infty$. So it is desirable that the new fields σ and φ behave like the old ones, going to zero at infinity. If this would not be the case, it would be equivalent to perform transformations that change the topological sector, which would lead us to compute jacobians over noncompact spaces, what is very difficult to obtain [18][19]. So, altough keeping in mind the general case, we will restrict ourselves to transformations which do not change the topological sector.

In the case of the σ field we have

$$\begin{aligned} \lim_{|x|\to\infty}\sigma\left(x\right) &= \lim_{|x|\to\infty}\rho\left(x\right) - \frac{1}{\lambda}\lim_{|x|\to\infty}\left\langle\Delta_F\left(x-z\right)\left(j_\rho + j'_\rho\right)\right\rangle \\ &= \frac{1}{2\pi\lambda}\lim_{|x|\to\infty}\left\{\sum_{i=1}^{k}\left(\ln\left|x-x_i\right| - \ln\left|x-y_i\right|\right)\right\} \\ &= 0, \end{aligned}$$

once $\lim_{|x|\to\infty}\rho\left(x\right) = 0$, in agreement with the conditions imposed.

For the field φ, we have

$$\begin{aligned} \lim_{|x|\to\infty}\varphi\left(x\right) &= \lim_{|x|\to\infty} f\left(x\right) + \lim_{|x|\to\infty}\phi\left(x\right) \mp \lim_{|x|\to\infty} e^2\left\langle\Delta\left(m; x-z\right)\left(j+j'\right)\right\rangle \\ &= -N\ln|x| \pm \frac{e^2}{2\pi m^2}\lim_{|x|\to\infty}\left\langle\left(K_0\left[m\left|x-z\right|\right] + \ln\left|x-z\right|\right)\left(j+j'\right)\right\rangle \\ &= -N\ln|x| \pm \frac{1}{2a\left(N\right)}2\left|N\right|\ln|x| \\ &= -\left(N \mp \frac{|N|}{a\left(N\right)}\right)\ln|x|, \end{aligned}$$

once K_0 is well behaved in the limit considered and $\lim_{|x|\to\infty} \phi(x) = 0$. Here, the $\mp$ sign corresponds to sectors with topological charge N and $-N$, respectivelly. The assymptotic behavior of φ is then

$$\lim_{|x|\to\infty} \varphi(x) = \begin{cases} -\left(N - \frac{N}{a(N)}\right) \ln|x|, & N > 0, \\ -\left(N - \frac{N}{a(N)}\right) \ln|x|, & N < 0. \end{cases}$$

which is singular unless we have

$$a(N) = 1, \qquad \forall \;\; N \neq 0.$$

4 Correlation functions in sectors with trivial topology

In this case, bosonization gives us

$$Z[J^\mu, \overline{\eta}, \eta] = \int [dA_\mu] \Im(A_\mu) \exp\left(-\left\langle \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right\rangle + \langle J^\mu A_\mu \rangle + \left\langle \overline{\eta}' \left(i\gamma^\mu \partial_\mu\right)^{-1} \eta' \right\rangle \right), \quad (25)$$

where $\overline{\eta}'$ and η' are the transformed fermionic sources (13), and

$$\Im(A_\mu) = \frac{\det D}{\det i\gamma^\mu \partial_\mu},$$

can be written as an effective term to be added to the action

$$\overline{S} = \left\langle \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right\rangle - \left\langle \frac{e^2}{2\pi} A_\mu \left\{ a\delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box} \right\} A_\nu \right\rangle.$$

To compute the photon self-energy we consider the fermionic external sources to be absent $\overline{\eta} = \eta = 0$, and write

$$\begin{aligned} \langle J_\mu A^\mu \rangle &= \left\langle J_\mu \left(\partial^\mu \rho + \tilde{\partial}^\mu \phi \right) \right\rangle \\ &= -\left\langle \left(\partial_\mu J^\mu\right) \rho + \left(\tilde{\partial}_\mu J^\mu \right) \phi \right\rangle \\ &= -\left\langle J_L \rho + J_T \phi \right\rangle, \end{aligned}$$

where J_L and J_T are the sources associated to the longitudinal and transverse terms, respectivelly. Now the action goes into

$$\overline{S} - \langle J_\mu A^\mu \rangle = \int d^2x \left\{ \frac{1}{2e^2} \phi \Box \left(\Box - m^2 \right) \phi - \frac{\lambda}{2} \rho \Box \rho + J_L \rho + J_T \phi \right\}. \quad (26)$$

We perform a change of variables on the scalar fields ϕ and ρ to obtain the following gaussian expression for $Z[J^\mu]$:

$$Z[J_\mu] = \exp\left\langle \frac{e^2}{2} J_T \left[\Box \left(\Box - m^2 \right) \right]^{-1} J_T \right\rangle \exp\left\langle -\frac{1}{2\lambda} J_L \Box^{-1} J_L \right\rangle.$$

In momentum space, this is simply

$$Z\left[J_{\mu}\right]=\exp\left\{\int\frac{d^{2}k}{\left(2\pi\right)^{2}}\tilde{J}_{\mu}\left(-k\right)\left[\frac{e^{2}}{2}\frac{k^{2}\delta^{\mu\nu}-k^{\mu}k^{\nu}}{k^{2}\left(k^{2}+m^{2}\right)}-\frac{1}{2\lambda}\frac{k_{\mu}k_{\nu}}{k^{2}}\right]\tilde{J}_{\nu}\left(k\right)\right\},$$

which gives to the photon self-energy the expression

$$G^{\mu\nu}\left(x-y\right)=\int\frac{d^{2}p}{\left(2\pi\right)^{2}}\exp\left(-ip\left(x-y\right)\right)\left[\frac{e^{2}}{2}\frac{p^{2}\delta^{\mu\nu}-p^{\mu}p^{\nu}}{p^{2}\left(p^{2}+m^{2}\right)}-\frac{1}{2\lambda}\frac{p_{\mu}p_{\nu}}{p^{2}}\right].$$

We see that the use of the point-splitting regularization in the computation of the fermionic determinant introduced the Jackiw parameter explicitly at the pole of the propagator. This implies that the ambiguity will be present in any physical quantity we compute, which depends on this pole.

One can easily see, through the expression above or the next

$$G^{\mu\nu}\left(x-y\right)=-e^{2}\left(\Box\delta^{\mu\nu}-\partial^{\mu}\partial^{\nu}\right)\left[2\pi K_{0}\left(m\left|x-y\right|\right)\right]+\frac{1}{\lambda}\partial^{\mu}\partial^{\nu}\left[-\frac{1}{2\pi}\ln\left(m\left|x-y\right|\right)\right]$$

that $G^{\mu\nu}$ is free of ultraviolet and infrared divergences.

In the case of the fermionic self-energy, we can consider the bosonic external source to be absent instead, $J^{\mu}=0$. We define

$$\begin{aligned}S^{\pm}_{\alpha\beta}\left(x-y\right)_{J^{\mu}=0} \equiv& \left(\left(i\gamma^{\mu}\partial_{\mu}\right)^{-1}\mathbf{P}_{\pm}\right)_{\alpha\beta}\\ &\int dA_{\mu}\exp\left(-\overline{S}\right)\exp\left(i\left(\rho\left(x\right)-\rho\left(y\right)\right)\mp\left(\phi\left(x\right)-\phi\left(y\right)\right)\right),\end{aligned}$$

which allow us to write

$$S_{\alpha\beta}\left(x-y\right)=S^{+}_{\alpha\beta}\left(x-y\right)+S^{-}_{\alpha\beta}\left(x-y\right).$$

Similar calculations as before give

$$S^{\pm}_{\alpha\beta}\left(x-y\right)_{J^{\mu}=0}=\left(\left(i\gamma^{\mu}\partial_{\mu}\right)^{-1}\mathbf{P}_{\pm}\right)_{\alpha\beta}\exp\left\langle\frac{e^{2}}{2}j\left[\Box\left(\Box-m^{2}\right)\right]^{-1}j+\frac{1}{2\lambda}j\Box^{-1}j\right\rangle.$$

Remembering that $\mathbf{P}_{+}+\mathbf{P}_{-}=\mathbf{1}$, we find

$$S_{\alpha\beta}\left(x-y\right)=\exp\left\langle\frac{e^{2}}{2}j\left[\Box\left(\Box-m^{2}\right)\right]^{-1}j+\frac{1}{2\lambda}j\Box^{-1}j\right\rangle S^{F}_{\alpha\beta}\left(x-y\right),$$

where $S^{F}_{\alpha\beta}\left(x-y\right)$ is the two point function of free fermions.

About the ultraviolet behavior we see that, being $S^{F}_{\alpha\beta}$ finite, the expression

$$\begin{aligned}\left\langle\frac{e^{2}}{2}j\left[\Box\left(\Box-m^{2}\right)\right]^{-1}j\right\rangle &= e^{2}\int\frac{d^{2}k}{\left(2\pi\right)^{2}}\frac{1-\exp\left(ik\left(x-y\right)\right)}{k^{2}\left(k^{2}+m^{2}\right)}\\ &= e^{2}\left(2\pi K_{0}\left(0\right)\right)-e^{2}\left(2\pi K_{0}\left(m\left|x-y\right|\right)\right),\end{aligned} \tag{27}$$

is free of singularities, while

$$\begin{aligned}\left\langle \frac{1}{2\lambda} j \Box^{-1} j \right\rangle &= -\frac{1}{\lambda} \int \frac{d^2k}{(2\pi)^2} \frac{1-\exp\left(ik\left(x-y\right)\right)}{k^2} \\ &= \lim_{\alpha\to 0} -\frac{1}{\lambda} \int d^2k \frac{\exp\left(ik\alpha\right) - \exp\left(ik\left(x-y\right)\right)}{k^2} \\ &= \lim_{\alpha\to 0} -\frac{1}{2\pi\lambda} \left[\ln\left(m\alpha\right) - \ln\left(m\left|x-y\right|\right)\right] \\ &= \lim_{\alpha\to 0} -\frac{1}{2\pi\lambda} \ln\left(\frac{\alpha}{\left|x-y\right|}\right) \\ &= \begin{cases} -\infty, & \text{if } \lambda > 0 \\ +\infty, & \text{if } \lambda < 0 \end{cases}\end{aligned} \tag{28}$$

give us

$$S_{\alpha\beta}\left(x-y\right) = \begin{cases} 0, & \text{if } \lambda > 0 \\ \infty, & \text{if } \lambda < 0 \end{cases},$$

which, in the case $\lambda < 0$, has a divergence that depends explicitly on the ambiguity.

In obtaining this result we have used both infrared (m) and ultraviolet (α) regulators. The infrared divergence cancels but the ultraviolet one remains. It is important to stress the fact that for $\lambda = 0$, we do not have any divergences.

For $\lambda < 0$, we can perform a wave function renormalization

$$\begin{aligned}\psi &= Z_\psi^{\frac{1}{2}} \psi', \\ \overline{\psi} &= Z_{\overline{\psi}}^{\frac{1}{2}} \overline{\psi}',\end{aligned} \tag{29}$$

which gives

$$S^R_{\alpha\beta}\left(x-y\right) = Z_\psi^{\frac{1}{2}} Z_{\overline{\psi}}^{\frac{1}{2}} S_{\alpha\beta}\left(x-y\right).$$

By choosing $Z_\psi^{\frac{1}{2}} = Z_{\overline{\psi}}^{\frac{1}{2}} = Z^{\frac{1}{2}}$, we will need just one renormalization condition to fix $Z^{\frac{1}{2}}$, for example,

$$Z^{\frac{1}{2}} = \exp\left\{-\frac{1}{2\pi\lambda} \ln\alpha + \beta\right\},$$

where β is an additional ambiguity over the finite part of $Z^{\frac{1}{2}}$ to be fixed by the requirement of external renormalization conditions. This will lead to a finite result at the end.

To proceed, we compute the mixed four point function

$$G^{\mu\nu}_{\alpha\beta}\left(x,y,z,w\right) = \frac{\delta^4}{\delta J_\mu\left(x\right)\delta J_\nu\left(y\right)} \left\{T^+_{\alpha\beta}\left[J_\mu; x-y\right] + T^-_{\alpha\beta}\left[J_\mu; x-y\right]\right\},$$

where

$$T^\pm_{\alpha\beta}\left[J_\mu; x-y\right] = \left(\left(i\gamma^\mu\partial_\mu\right)^{-1}\left(x-y\right)\mathbf{P}_\pm\right)_{\alpha\beta} \int \left[dA_\mu\right] \exp\left(-S^\pm\right),$$

and

$$S^{\pm} = \overline{S} - \frac{e^2}{2}\left\langle j\left(\left[\Box\left(\Box - m^2\right)\right]^{-1} + \frac{1}{e^2\lambda}\Box^{-1}\right)j\right\rangle \mp \frac{e^2}{2}\left\langle J_T\left[\Box\left(\Box - m^2\right)\right]^{-1} j\right\rangle$$
$$\mp\frac{e^2}{2}\left\langle j\left[\Box\left(\Box - m^2\right)\right]^{-1} J_T\right\rangle - \frac{i}{2\lambda}\left\langle J_L\Box^{-1}j\right\rangle - \frac{i}{2\lambda}\left\langle j\Box^{-1}J_L\right\rangle.$$

We still have to compute functional derivatives with respect to J_μ of the above expression, which is in the form $\exp\left(-\langle JKJ\rangle - \langle JL\rangle\right)$. When taking the limit $J = 0$, we find

$$\frac{\delta}{\delta J_x \delta J_y}\exp\left(-\langle JKJ\rangle - \langle JL\rangle\right)_{J_x=J_y=0} = -2K_{xy} + L_x L_y,$$

where K_{xy} stands for the bosonic two point function, already calculated and

$$L^{\mu}_{\pm}(z) = \int d^2z\left(\pm\frac{e^2}{2}\epsilon^{\mu\nu}\partial^z_\nu\left[\Box\left(\Box - m^2\right)\right]^{-1}(z,z')\,j(z') + \frac{i}{2\lambda}\partial^{\mu}_z\Box^{-1}(z,z')\,j(z')\right).$$

Putting all together, we can write

$$G^{\mu\nu}_{\alpha\beta}(x,y,z,w) = S_{\alpha\beta}(z-w)\,G^{\mu\nu}(x-y) + S^{+}_{\alpha\beta}(z-w)\,L^{\mu}_{+}(x)\,L^{\nu}_{+}(y)$$
$$+S^{-}_{\alpha\beta}(z-w)\,L^{\mu}_{-}(x)\,L^{\nu}_{-}(y)$$

where $L^{\mu}_{\pm}(x) \equiv L^{\mu}_{\pm}(x,z,w)$ and $L^{\mu}_{\pm}(y) \equiv L^{\mu}_{\pm}(y,z,w)$.

We see that $S^{\pm}_{\alpha\beta}$ has the same ultraviolet behaviour as $S_{\alpha\beta}$. $L^{\mu}_{\pm}$ is given by

$$L^{\mu}_{\pm}(x,z,w) = \mp\pi e^2\varepsilon_{\mu\nu}\partial^{\nu}\left[K_0(m|x-z|) + K_0(m|x-w|)\right] -$$
$$\frac{i}{4\pi\lambda}\partial_\mu\left[\ln(m|x-z|) + \ln(m|x-w|)\right],$$

which is free of singularities. This shows that the four point function will be finite and non vanishing if the fermionic two point function is. This analysis can be extended to correlation functions with arbitrary number of legs and the same conclusion will be reached, i.e., the only divergence to be regulated is that of the two point fermionic function.

5 Conclusion and remarks

The ambiguity in the Jackiw parameter can now be restricted, in the case of trivial topology. We have found a renormalization for the fermionic self-energy. This means that the theory is finite and has non vanishing fermionic correlation functions for

$$\lambda < 0 \Longrightarrow \frac{a-1}{\pi} < 0,$$

or

$$a < 1.$$

For $a > 1$, every correlation function involving fermions will vanish, thus giving an inconsistent theory in the sense that we begin considering fermionic operators and find, at the end, that these operators are identically null.

On the other side, we can restrict even more the values of a if we do not admit a tachyon in the spectrum. This extra consideration puts the ambiguity in the interval

$$0 \leq a \leq 1.$$

We can interpret this range if we compare our results with those obtained after the computation of the conservation of the gauge and axial currents in the Schwinger model given by

$$\langle \partial_\mu J_5^\mu \rangle = -\frac{e}{4\pi}\epsilon^{\mu\nu}F_{\mu\nu}\left(1+a_s\right) = -\frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}a$$

and

$$\langle \partial_\mu J^\mu \rangle = \frac{e}{2\pi}\partial_\mu A^\mu \left(1-a_s\right) = \frac{e}{\pi}\partial_\mu A^\mu \left(1-a\right)$$

where, $a_s = (1+a)/2$ is the original parameter introduced by Jackiw and Johnson [7]. We see that the a parameter interpolates continuously beetwen regularization schemes that preserve chiral ($a=0$) and gauge ($a=1$) invariances respectively.

At the same time, the divergence found for all values of $a \neq 1$ means that perhaps the ambiguity is only apparent. As is well known, whenever we have to renormalize a theory, we are forced to fix our renormalization counterterms through the use of renormalization conditions. These conditions usually introduce an arbitrary parameter μ in the correlation functions, but for a physical quantity R one can always prove that [9]

$$\frac{d}{d\mu}R = 0.$$

This is equally valid for the physical masses of the model and simply means that once we have fixed the experimental values of the parameters which enter into the lagrangean, it does not matter the way one chooses to renormalize the theory. In this way, we intend to investigate a possible dependence of a on renormalization group parameters, through a careful study of the renormalization conditions, in a nonperturbative setting. The main problem that we have to face is how to express these conditions directly in configuration space, instead of momentum space, where bosonization is rather involved. Progress in this direction will be reported elsewhere.

Finally, we would like to remark that, in nontrivial topology sectors, the question seems to be even more difficult to answer. As we have seen, there is an infinite amount of ambiguity in the theory, due to arbitrary choices of $a(N)$ for each N. A simple criterium to choose $a(N)$ seems to be the one which does not allow changes in the topological sector. It gives a value for $a(N)$ which coincides with the one obtained through the requirement of gauge invariance. The connection between gauge invariance and preservation of topology is not completely clear and perhaps can only be clarified if one could compute the correlation functions without these criteria. It is our aim to explore also this direction in the near future.

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