ISSN 0029-3865

CBPF-NF-007/88

# POTTS FERROMAGNET: TRANSFORMATIONS AND CRITICAL EXPONENTS IN PLANAR HIERARCHICAL LATTICES

by

Paulo R. Hauser* and Evaldo M.F. Curado

Centro Brasileiro de Pesquisas Físicas - CBPF/CNPq
Rua Dr. Xavier Sigaud, 150
22290 - Rio de Janeiro, RJ - Brasil

*Departamento de Física
Universidade Federal da Santa Catarina
88000 - Florianópolis, SC - Brasil

ABSTRACT

We prove that the duality transformation for a Potts ferromagnet on two-rooted planar hierarchical lattices (HL) preserves the thermal eigenvalue. This leads to a relation between the correlation length critical exponents $\nu$ of a HL and its corresponding dual lattice. Using hyperscaling we show that their specific heat critical exponents $\alpha$ coincide. For a smaller class of HL-namely of diamond and tress types - we prove that another transformations also preserves $\nu$ and $\alpha$.

Key-words: Critical exponents; Hierarchical lattices; Duality.

Phase transitions of the q-state Potts model on hierarchical lattices (HL) have been largely studied with real space renormalization group methods because exact calculations can be performed on such lattices[1-5]. More recently Hu[6] and da Silva and Tsallis[7] have obtained some intriguing results studying critical properties of Ising and Potts ferromagnets on HL's. Hu[6] exhibits two different HL's (see below, figures la e lc) that present the same thermal eigenvalue $\lambda$. da Silva and Tsallis[7] have shown that generalized diamond and tress HL's (see examples in our fig. 1) have the same correlation length critical exponents $\nu$. We will show that these results are consequence of two HL transformations rather than singular cases. The first one is the duality[2], a property of *any* planar HL. The second is related to a smaller class of HL's, namely the generalized diamond and tress ones[7,8]; it transforms a diamond HL into a tress HL and conversely.

In order to show the properties induced by these transformations we will consider a q-state Potts ferromagnet on a HL. The Hamiltonian is given by

$$\mathcal{H} = - qJ \sum_{<ij>} \delta_{\sigma_i \sigma_j} \quad , \qquad (\sigma_i = 0,1,\ldots,q-1) \tag{1}$$

where the sum is over nearest neighbors sites and $\delta$ is the kronecker delta. The $\sigma$-variables are on the sites of the HL and the coupling constants are associated with the bonds. We will use a very convenient variable, the thermal transmissivity

$$t \equiv [1 - \exp(-qJ/k_B T)]/[1 + (q-1)\exp(-qJ)/k_B T)]^{(9)} \quad ,$$

associated with each bond of the HL. Its dual variable $\tau$ is defined by the relation[9]

$$\tau \equiv \frac{1-t}{1+(q-1)t} \quad . \tag{2}$$

The recursive relation of a two-rooted graph corresponding to the HL-basic cell with length b and aggregation number A[3] is given by $t' = G(t)$ where $G(t)$ is a ratio of two polynomials of $t$[10,11]. The thermal eigenvalue of this HL is given by

$$\lambda \equiv \left.\frac{\partial G}{\partial t}\right|_{t^*} \quad , \tag{3}$$

where $t^*$ satisfies $t^* = G(t^*)$.

Considering that, for HL's whose basic cells are two-rooted planar graphs, the function G associated with a HL is related with the function $\tilde{G}$ associated with the dual HL by the equation[10]

$$G(t) = \frac{1-\tilde{G}(\tau)}{1+(q-1)\tilde{G}(\tau)} \quad , \tag{4}$$

we are able to prove the following property.

*Property* 1: The Potts thermal eigenvalues of a two-rooted planar HL and of its dual lattice coincide.

The proof is straightforward. We must take the derivative of eq. (4) with respect to t, then use the chain rule in the right-hand-side (having in mind that $\tau$ is related with t by eq. (2)) and evaluate the derivatives at the fixed point $t^*$. Considering that $\tau^* = \tau(t^*)$ and $\tilde{G}(\tau^*)=\tau^*$ we verify that

$$\left.\frac{\partial G}{\partial \tilde{G}}\right|_{\tilde{G}=\tau^*} = \left[\left.\frac{d\tau}{dt}\right|_{t^*}\right]^{-1} .$$

This leads to the equality between the thermal eigenvalues of the HL's associated with G and $\tilde{G}$(dual)

$$\lambda = \tilde{\lambda} \quad , \tag{5}$$

where $\lambda$ is given by eq. (3) and $\tilde{\lambda} = \left.\frac{\partial \tilde{G}}{\partial \tau}\right|_{\tau^*}$.

*Corollary* 1: Being b the basic cell minimum length of a HL and $\tilde{b}$ the corresponding length of its dual lattice, then eq. (5) can be written as

$$b^{1/\nu} = \tilde{b}^{1/\tilde{\nu}} \tag{6}$$

where $\nu$ and $\tilde{\nu}$ are the correlation lengths critical exponents of a HL and its dual lattice respectively. The definition of intrinsic dimension[3], namely $D \equiv \ell og A/\ell og b$, and the fact that duality transformation preserves the aggregation number A, permit us to rewrite eq. (6) as

$$D\nu = \tilde{D}\tilde{\nu} . \tag{7}$$

*Corollary* 2: Using the hyperscaling relation for a HL[4], $D\nu = 2 - \alpha$, we have

$$\alpha = \tilde{\alpha} \ , \tag{8}$$

thus showing that the specif heat critical exponents of a HL and of its dual lattice are the *same*. We remark that the relations between critical exponents given by eqs. (7) and (8) are valid for *all* planar HL. Furthermore, if $b = \tilde{b}$ then also $\nu = \tilde{\nu}$.

The second property is related to a smaller class of planar HL. This class is partitioned in two subclasses namely the diamond-like and tress-like HL's. The basic cell of a diamond HL is constituted by N branches in parallel, each one with b bonds in series and the tress basic cell is constituted by b clusters in series, each one with N bonds in parallel. For example in figure 1 the basic cells (1a) and (1c) generate diamond HL's with $b = 2, N = 3$ and $b = 3, N = 2$ respectively, and (1b) and (1d) generate tress HL's with $b = 3, N = 2$ and $b = 2, N = 3$ respectively. The expression for $G_D(G_T)$ of the diamond (tress) HL, for any b N, is given by:

$$G_D(t,b,N) = \frac{1 - \left[\frac{1-t^b}{1+(q-1)t^b}\right]^N}{1+(q-1)\left[\frac{1-t^b}{1+(q-1)t^b}\right]^N} \tag{9}$$

$$G_T(t,b,N) = \left[\frac{1 - \left[\frac{1-t}{1+(q-1)t}\right]^N}{1+(q-1)\left[\frac{1-t}{1+(q-1)t}\right]^N}\right]^b \ . \tag{10}$$

It is easy to verify by equations (9) and (10) that there is a relation between $G_D$ and $G_T$ given by

$$G_T(\omega, b, N) = [G_D(t, b, N)]^b \tag{11}$$

where $\omega = t^b$. Thus, the diamond and tress HL can be connected by two transformations, the diamond-tress ($T_{DT}$) given by

$$T_{DT}: G_D(t, b, N) \longrightarrow [G_D(t, b, N)]^b = G_T(t^b, b, N) \tag{12}$$

and its inverse (tress-diamond $T_{TD}$)

$$T_{TD}: G_T(t, b, N) \longrightarrow [G_T(t, b, N)]^{1/b} = G_D(t^{1/b}, b, N) \quad , \tag{13}$$

where the equalities in (12) and (13) follow from eq. (11). Clearly, $T_{DT}$ and $T_{TD}$ are related to the diamond-tress transformations proposed by Ottavi and Albinet[8].

*Property* 2: A diamond-like and a tress-like HL with the same b and N share the same Potts correlation length critical exponent $\nu$.

Also this proof is straightforward. We take the derivative of eq. 11 with respect to t and evaluate this derivative at the critical point $t^*$. Having in mind that $\omega^* = \omega(t^*) = t^{*b}$ this leads to $\lambda_T = \lambda_D$. As b is the same for diamond and tress HL connected by $T_{DT}$ this implies that $\nu_T = \nu_D$.

*Corollary*: As the diamond and tress HL connected by $T_{DT}$ (or $T_{TD}$) have the same b this implies that their intrinsic dimensionalities are the same. By using hyperscaling it follows that their specific heat critical exponents are the same; $\alpha_T = \alpha_D$.

It is worthwhile to note that if $b \neq N$ ($D \neq 2$) then tress HL is not the dual of a diamond HL. Also, if $b = N$ ($D = 2$) then $T_{DT}$ transformation turns out to be the duality transformation.

For diamond and tress HL's the conjugation of the transformations defined by equations (4) and (11), namely $\tilde{T}$ e $T_{DT}$, connects four HL's as illustrated in figure 1.

In conclusion, the HL's connected by $\tilde{T}$ share the same $\alpha$ and their correlation length critical exponents are related by $D\nu = \tilde{D}\tilde{\nu}$ ; the HL's connected by $T_{DT}$ have both $\alpha$ and $\nu$ equal.

*Acknowledgements*: We would like to thank C. Tsallis for fruitful discussions and useful suggestions.

CAPTIONS

Fig. 1: Diamond ((a) and (c)) and tress ((b) and (d)) HL basic cells connected by the transformations $\tilde{T}$, $T_{dt}$ and $T_{td}$ (o and ● denote the roots and internal sites respectively).

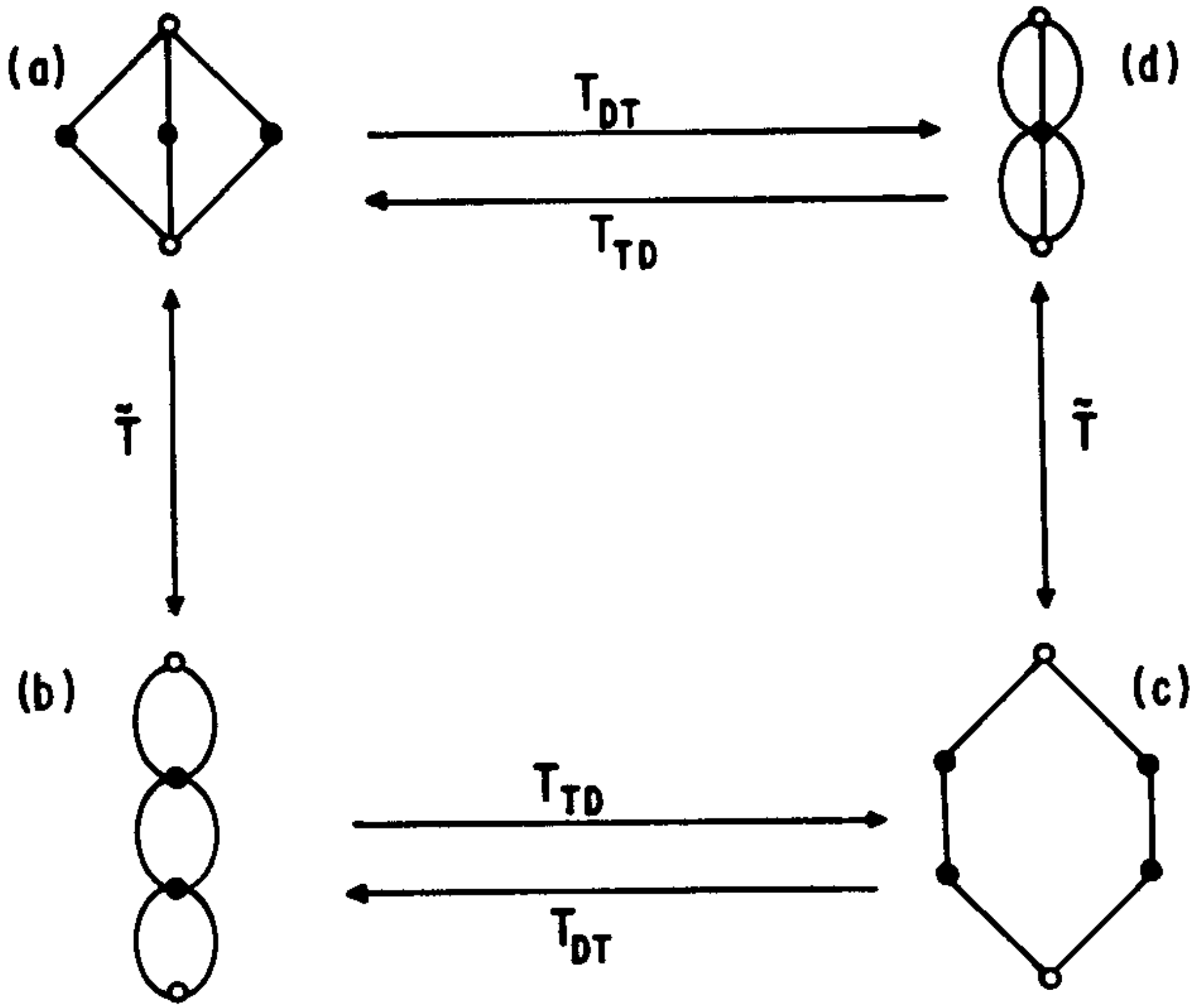


FIG. 1

## REFERENCES

1) Griffiths RB and Kaufman M, Phys. Rev. B 26 (1982) 502; 30 (1984) 241

2) Melrose JR, J. Phys. A 16 (1983) L407

3) Melrose JR, J. Phys. A 16 (1983) 3077

4) Tsallis C, J. Phys. C 18 (1985) 6581

5) Hauser PR and Saxena VK, Phys. Rev. B 34 (1986) 8193

6) Hu B, preprint

7) da Silva LR and Tsallis C, J. Phys. A 20 (1987) 6013

8) Ottavi H and Albinet G, J. Phys. A 20 (1987) 2961

9) Tsallis C and Levy SVF, Phys. Rev. Lett. 47 (1981) 950

10) Tsallis C; To appear in Phys. Reports

11) Costa UMS, Roditi I and Curado EMF, J. Phys. A 20 (1987) 6001.