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Stochastic Quantization of Topological Field Theory: Generalized Langevin Equation with Memory Kernel

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Abstract

We use the method of stochastic quantization in a topological field theory defined in an Euclidean space, assuming a Langevin equation with a memory kernel. We show that our procedure for the Abelian Chern-Simons theory converges regardless of the nature of the Chern-Simons coefficient.

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1 Introduction

In the Euclidean version of field theory, we are interested in computing the Schwinger functions of a theory. In order to obtain these functions, Parisi and Wu introduced the stochastic quantization [1]. This formalism was introduced as an alternative quantization scheme, different from the usual canonical and the path integral field quantization, based in the Hamiltonian and the Lagrangian respectively. The method starts from a classical equation of motion, but not from Hamiltonian or Lagrangian, and consequently can be used to quantize dynamical systems without canonical formalism and therefore it is useful in situations where the others methods lead to difficult problems.

The main idea of the stochastic quantization is that a d-dimensional quantum system is equivalent to a $(d+1)$-dimensional classical system which undergoes random fluctuations. Some of the most important papers in the subject can be found in Ref. [2]. A brief introduction to the stochastic quantization can be found in the Ref. [3] and Ref. [4]. See also the Ref. [5].

In a previous paper [6], we studied the stochastic quantization of a self-interacting scalar field theory, assuming a non-Markovian process, modifying the Langevin equation by introducing a memory kernel [7] [8] [9]. We have shown that although a system with a stationary, Gaussian, non-Markovian Langevin equation with a memory kernel and a colored noise converges in the asymptotic limit of the Markov parameter τ to the equilibrium, we obtain a non-regularized theory.

In this paper we would like to continue to investigate the virtues of this non-Markovian stochastic quantization method, now employed in the case of a topological field theory. One of the peculiar features within this kind of theory is the appearance of a factor of i in front of the topological action in Euclidean space. Since the topological theory does not depend on the metric of space-time, the path integral measure weighing remains to be e^{iS} even after the Wick rotation. Another feature of a topological action is that it is the integral of a density which is not bounded from below in Euclidean space. So, if one attempts to use a Markovian Langevin equation with a white noise to quantize this theory, one will find serious problems if the factor of i is ignored. This Langevin equation will not tend to any equilibrium in the large τ limit. So, in this sense, the use of a Langevin equation with a complex action [10] becomes essential for stochastically quantizing a topological action [11] [12] [13].

There is, in the literature, an approach to solve the above mentioned convergence problem. Studying the purely topological Chern-Simons theory, Ferrari *et al*, introduced a non-trivial kernel in the Langevin equation [14]. On the other way, Wu *et al* [15] showed that the Langevin equation for a Maxwell-Chern-Simons theory converges to the usual equilibrium result without the need to introduce such kernel. Their method, however, only works in the case where the Chern-Simons coefficient is real.

We show in this paper that, if one uses a non-Markovian Langevin equation with a colored random noise, this convergence problem may be solved in a different way. We will apply this approach to three-dimensional abelian Chern-Simons theory and prove that we obtain convergence towards equilibrium even with an imaginary Chern-Simons coefficient. To simplify the calculations we assume the units to be such that $\hbar = c = 1$.

2 Stochastic quantization of abelian Chern-Simons theory

Let us consider the following action for the three-dimensional Maxwell-Chern-Simons theory, in Euclidean space:

$$S = \int d^3x \left(\frac{1}{4\varepsilon^2} A_\mu(x)(-\Delta\delta_{\mu\nu} + \partial_\mu\partial_\nu)A_\nu(x) - i\frac{\kappa}{8\pi}\epsilon_{\mu\nu\rho}A_\mu(x)\partial_\nu A_\rho(x)\right), \tag{1}$$

where Δ is the three-dimensional Laplace operator. At the end of our calculations we set $\varepsilon \to \infty$ to obtain the results for the purely topological theory, as discussed in Ref. [15]. Notice the factor of i in front of the topological term, as mentioned before. In order to obtain the Schwinger functions o the theory, let us use the stochastic quantization method. Let us introduce a non-Markovian Langevin equation given by

$$\frac{\partial}{\partial\tau}A_\mu(\tau,x) = -\int_0^\tau ds\, M_\Lambda(\tau - s)\frac{\delta\, S}{\delta\, A_\mu(x)}|_{A_\mu(x)=A_\mu(s,x)} + \eta_\mu(\tau,x), \tag{2}$$

where $M_\Lambda(\tau-s)$ is a memory kernel and Λ is a arbitrary parameter. We will have, from Eqs.(1) and (2), in momentum space:

$$\begin{aligned}\frac{\partial}{\partial\tau}A_\mu(\tau,k) &= -\frac{k^2}{\varepsilon^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\int_0^\tau ds\, M_\Lambda(\tau-s)A_\nu(s,k) + \\ &\quad -\frac{\kappa}{4\pi}\epsilon_{\mu\nu\rho}k_\rho\int_0^\tau ds\, M_\Lambda(\tau-s)A_\nu(s,k) + \eta_\mu(\tau,k)\end{aligned} \tag{3}$$

where the stochastic field $\eta_\mu(\tau,k)$ satisfies the modified Einstein relations:

$$\langle\, \eta_\mu(\tau,k)\,\rangle_\eta = 0 \tag{4}$$

and also

$$\langle\, \eta_\mu(\tau,k)\,\eta_\nu(\tau',k')\,\rangle_\eta = 2\delta_{\mu\nu}M_\Lambda(|\tau-\tau'|)\,\delta^d(k+k'). \tag{5}$$

For the initial condition $A_\mu(\tau,k)|_{\tau=0} = 0$, it is easy to see that the solution of the Eq.(3) is given by:

$$A_\mu(\tau,k) = \int_0^\infty d\tau'\, G_{\mu\nu}(k;\tau-\tau')\eta_\nu(\tau',k), \tag{6}$$

where we introduced the retarded Green function $G_{\mu\nu}(k,\tau)$, which satisfies:

$$\begin{aligned}\frac{\partial}{\partial\tau}G_{\mu\nu}(k,\tau) &= -\frac{k^2}{\varepsilon^2}\left(\delta_{\mu\rho} - \frac{k_\mu k_\rho}{k^2}\right)\int_0^\tau ds\, M_\Lambda(\tau-s)\,G_{\rho\nu}(k,s) + \\ &\quad -\frac{\kappa}{4\pi}\epsilon_{\mu\rho\sigma}k_\sigma\int_0^\tau ds\, M_\Lambda(\tau-s)\,G_{\rho\nu}(k,s) + \delta_{\mu\nu}\,\delta(\tau),\end{aligned} \tag{7}$$

for $\tau > 0$ and $G_{\mu\nu}(k,\tau) = 0$ for $\tau < 0$.

To proceed the calculations, let us introduce the Laplace transform of the Eq.(7):

$$\begin{aligned}z\,G_{\mu\nu}(k,z) &= -\frac{k^2}{\varepsilon^2}\left(\delta_{\mu\rho} - \frac{k_\mu k_\rho}{k^2}\right)M_\Lambda(z)\,G_{\rho\nu}(k,z) + \\ &\quad -\frac{\kappa}{4\pi}\epsilon_{\mu\rho\sigma}k_\sigma M_\Lambda(z)\,G_{\rho\nu}(k,z) + \delta_{\mu\nu},\end{aligned} \tag{8}$$

where:

$$M_\Lambda(z) = \int_0^\infty d\tau\, M_\Lambda(\tau)\, e^{-z\tau}. \tag{9}$$

For the result without memory (or, formally, when $M_\Lambda(\tau) \to \delta(\tau)$), we have, from Eq.(7):

$$\begin{aligned} \frac{\partial}{\partial\tau} G_{\mu\nu}(k,\tau) &= -\frac{k^2}{\varepsilon^2}\Big(\delta_{\mu\rho} - \frac{k_\mu k_\rho}{k^2}\Big) G_{\rho\nu}(k,\tau) + \\ & -\frac{\kappa}{4\pi}\epsilon_{\mu\rho\sigma} k_\sigma G_{\rho\nu}(k,\tau) + \delta_{\mu\nu}\delta(\tau), \end{aligned} \tag{10}$$

whose Laplace transform reads:

$$\begin{aligned} z\, G_{\mu\nu}(k,z) &= -\frac{k^2}{\varepsilon^2}\Big(\delta_{\mu\rho} - \frac{k_\mu k_\rho}{k^2}\Big) G_{\rho\nu}(k,z) + \\ & -\frac{\kappa}{4\pi}\epsilon_{\mu\rho\sigma} k_\sigma G_{\rho\nu}(k,z) + \delta_{\mu\nu}. \end{aligned} \tag{11}$$

Note the similarity between Eqs.(8) and (11). The solution to Eq.(10) is given by [15]

$$\begin{aligned} G_{\mu\nu}(k,\tau) &= \frac{k_\mu k_\nu}{k^2} + \bigg(\Big(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\Big)\cos\Big(\frac{\kappa}{4\pi}k\tau\Big) + \\ & -\epsilon_{\mu\nu\sigma}\frac{k_\sigma}{k}\sin\Big(\frac{\kappa}{4\pi}k\tau\Big)\bigg)\exp\Big(\frac{-k^2}{\varepsilon^2}\tau\Big), \end{aligned} \tag{12}$$

whose Laplace transform is:

$$G_{\mu\nu}(k,z) = \frac{k_\mu k_\nu}{k^2}\frac{1}{z} + \frac{\Big(\Big(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\Big) z - \epsilon_{\mu\nu\sigma} k_\sigma \Big(\frac{\kappa}{4\pi}\Big)\Big)}{\Big(z + \frac{k^2}{\varepsilon^2}\Big)^2 + \Big(\frac{\kappa}{4\pi}\Big)^2 k^2}. \tag{13}$$

Comparing Eqs.(8) and (11), it is trivial to obtain the analog of Eq.(13) with memory:

$$G_{\mu\nu}(k,z) = \frac{k_\mu k_\nu}{k^2}\frac{1}{z} + \frac{\Big(\Big(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\Big) z - \epsilon_{\mu\nu\sigma} k_\sigma \Big(\frac{\kappa'}{4\pi}\Big)\Big)}{\Big(z + \frac{k^2}{\varepsilon'^2}\Big)^2 + \Big(\frac{\kappa'}{4\pi}\Big)^2 k^2}, \tag{14}$$

where:

$$\frac{1}{\varepsilon'^2} \equiv \frac{M_\Lambda(z)}{\varepsilon^2} \tag{15}$$

and

$$\kappa' \equiv \kappa M_\Lambda(z). \tag{16}$$

In the appendix A, we derive in detail the inverse Laplace transform of Eq.(14). It is given by:

$$G_{\mu\nu}(k,\tau) = \bigg(\frac{k_\mu k_\nu}{k^2} + g_{\mu\nu} G_1(k,\tau) + \tilde{g}_{\mu\nu} G_2(k,\tau)\bigg)\theta(\tau), \tag{17}$$

where the quantities $G_i(k,\tau)$, $i=1,2$, $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ are defined in the appendix A. We see that our $G_{\mu\nu}(k,\tau)$ does not approach zero as $\tau\to\infty$. The reason of such behavior is the presence of the longitudinal term $\frac{k_\mu k_\nu}{k^2}$, which is common in the stochastic quantization of all gauge theories without gauge fixing and can be eliminated by a suitable stochastic gauge fixing. In spite of this, the presence of this term will not give any contribution to gauge invariant quantities.

After this discussion, we are able to present the two-point correlation function. We have that $D_{\mu\nu}(k;\tau,\tau')$ is given by

$$\begin{aligned} D_{\mu\nu}(k;\tau,\tau') &\equiv \langle\, A_\mu(\tau,k)\, A_\nu(\tau',k')\,\rangle_\eta = \\ &= \delta^d(k+k')\int_0^\infty ds\int_0^\infty ds'\, G_{\mu\kappa}(k,\tau-s)\, G_{\lambda\nu}(k,\tau'-s')\,\langle\, \eta_\kappa(s,k)\,\eta_\lambda(s',k')\,\rangle_\eta \\ &= 2\delta^d(k+k')\int_0^\infty ds\int_0^\infty ds'\, G_{\mu\lambda}(k,\tau-s)\, G_{\lambda\nu}(k,\tau'-s')\, M_\Lambda(\mid s-s'\mid). \end{aligned} \quad (18)$$

So, inserting Eq.(17) in the above equation and splitting the result in five different contributions yields:

$$D_{\mu\nu}(k;\tau,\tau') = 2\delta^d(k+k')\Big(J_1+J_2+J_3+J_4+J_5\Big) \quad (19)$$

where:

$$J_1 \equiv \int_0^\tau ds\int_0^{\tau'} ds'\,\frac{k_\mu k_\nu}{k^2}M_\Lambda(\mid s-s'\mid), \quad (20)$$

$$J_2 \equiv \int_0^\tau ds\int_0^{\tau'} ds'\, g_{\mu\lambda}g_{\lambda\nu}G_1(k;\tau-s)G_1(k;\tau'-s')M_\Lambda(\mid s-s'\mid), \quad (21)$$

$$J_3 \equiv \int_0^\tau ds\int_0^{\tau'} ds'\, \tilde{g}_{\mu\lambda}\tilde{g}_{\lambda\nu}G_2(k;\tau-s)G_2(k;\tau'-s')M_\Lambda(\mid s-s'\mid), \quad (22)$$

$$J_4 \equiv \int_0^\tau ds\int_0^{\tau'} ds'\, g_{\mu\lambda}\tilde{g}_{\lambda\nu}G_1(k;\tau-s)G_2(k;\tau'-s')M_\Lambda(\mid s-s'\mid), \quad (23)$$

and finally

$$J_5 \equiv \int_0^\tau ds\int_0^{\tau'} ds'\, \tilde{g}_{\mu\lambda}g_{\lambda\nu}G_2(k;\tau-s)G_1(k;\tau'-s')M_\Lambda(\mid s-s'\mid). \quad (24)$$

We can solve these equations by ordering the fictitious times s and s', $s>s'$ for instance, and solving the integrals in s (s') in the interval $[0,t]$ ($[0,s]$). We obtain for J_1, in the limit $\tau\to\infty$,

$$J_1 = \frac{1}{2}\frac{k_\mu k_\nu}{k^2}\Big(\tau-\frac{1}{\Lambda^2}\Big). \quad (25)$$

The integrals J_2 and J_3 can be solved by analogy with the scalar case [6]. Making the following replacements:

$$(k^2+m^2)_1 \to \frac{\alpha}{\Lambda^2}(1-\Lambda^4)+\Lambda^2 y_1+\frac{(\alpha^2-y_1^2)}{\Lambda^2}, \quad (26)$$

$$(k^2+m^2)_2 \to \frac{\alpha}{\Lambda^2}(1+\Lambda^4)-\Lambda^2 y_1-\frac{(\alpha^2-y_1^2)}{\Lambda^2}, \quad (27)$$

where the subscript 1 (2) stands for the G_1 (G_2) case (see the appendix A), we will have, in the asymptotic limit $\tau \to \infty$ that

$$\begin{aligned} J_2 &= \left(\frac{\alpha}{\Lambda^2}(1-\Lambda^4)+\Lambda^2 y_1+\frac{(\alpha^2-y_1^2)}{\Lambda^2}\right)^{-1}\left[\frac{\Lambda^2}{(\sigma\gamma)^2}\left(\frac{\Lambda^4}{4}+\frac{(\sigma+\gamma)^2}{4}\right)\left(\frac{\kappa}{4\pi}\right)\epsilon_{\mu\nu\rho}k_\rho + \right. \\ &\left. +(\sigma\gamma)^{-2}\left(\left(\frac{\Lambda^4}{4}+\frac{(\sigma+\gamma)^2}{4}\right)^2-\frac{\Lambda^4}{4}k^2\left(\frac{\kappa}{4\pi}\right)^2\right)\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\right], \end{aligned} \tag{28}$$

and

$$\begin{aligned} J_3 &= \left(\frac{\alpha}{\Lambda^2}(1+\Lambda^4)-\Lambda^2 y_1-\frac{(\alpha^2-y_1^2)}{\Lambda^2}\right)^{-1}\left[-\frac{\Lambda^2}{(\sigma\gamma)^2}\left(\frac{\Lambda^4}{4}+\frac{(\sigma-\gamma)^2}{4}\right)\left(\frac{\kappa}{4\pi}\right)\epsilon_{\mu\nu\rho}k_\rho + \right. \\ &\left. +(\sigma\gamma)^{-2}\left(\left(\frac{\Lambda^4}{4}+\frac{(\sigma-\gamma)^2}{4}\right)^2-\frac{\Lambda^4}{4}k^2\left(\frac{\kappa}{4\pi}\right)^2\right)\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\right]. \end{aligned} \tag{29}$$

The remaining integrals J_4 and J_5 can be solved without any further complications [16]. Again, in the asymptotic limit $\tau \to \infty$, we obtain:

$$\begin{aligned} J_4+J_5 &= \\ &\frac{f(\Lambda,\sigma,\gamma)}{g(\Lambda,\sigma,\gamma)}\left[-\frac{\Lambda^2}{(2\sigma\gamma)}\left(\frac{\kappa}{4\pi}\right)\epsilon_{\mu\nu\rho}k_\rho + \right. \\ &\left. +(\sigma\gamma)^{-2}\left(\left(\frac{\Lambda^4}{4}+\frac{(\sigma+\gamma)^2}{4}\right)\left(\frac{\Lambda^4}{4}+\frac{(\sigma-\gamma)^2}{4}\right)+\frac{\Lambda^4}{4}k^2\left(\frac{\kappa}{4\pi}\right)^2\right)\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\right], \end{aligned} \tag{30}$$

where:

$$\begin{aligned} f(\Lambda,\sigma,\gamma) &\equiv 153\Lambda^{14}+\Lambda^{10}\Big[18(\sigma+\gamma)^2+17(\sigma-\gamma)^2+9\sigma\gamma\Big]+\Lambda^6\Big[(\sigma+\gamma)^4+(\sigma-\gamma)^4+ \\ &+\frac{\sigma\gamma}{2}\Big((\sigma+\gamma)^2+(\sigma-\gamma)^2\Big)-\frac{9}{2}\sigma\gamma\Big((\sigma+\gamma)^2-\sigma\gamma(1-2\sigma\gamma)\Big)\Big]+ \\ &+\Lambda^2\sigma\gamma\Big[\frac{(\sigma+\gamma)^2}{2}-\sigma\gamma(\sigma+\gamma)^2-\frac{1}{2}(\sigma+\gamma)^2(\sigma-\gamma)^2\Big], \end{aligned} \tag{31}$$

and

$$g(\Lambda,\sigma,\gamma)\equiv\Big(9\Lambda^4+(\sigma-\gamma)^2\Big)\Big(9\Lambda^4+(\sigma+\gamma)^2\Big)(\Lambda^4+\sigma^2)(\Lambda^4+\gamma^2). \tag{32}$$

As mentioned before, the linearly divergent longitudinal term, found in Eq.(25), can be eliminated by a stochastic gauge fixing. Now, taking the limit $\varepsilon \to \infty$, it is easy to see that the contribution $J_2 + J_3$ vanishes identically. Then, finally, we obtain, for the purely topological two-point correlation function:

$$\begin{aligned} D_{\mu\nu}(k;\tau,\tau') &= 2\delta^d(k+k')\left[\frac{1}{2}\frac{k_\mu k_\nu}{k^2}\left(\tau-\frac{1}{\Lambda^2}\right)+\frac{f'(\Lambda,\sigma,\gamma)}{g'(\Lambda,\sigma,\gamma)}\left(-\frac{\Lambda^2}{2Q(y_1')}\left(\frac{\kappa}{4\pi}\right)\epsilon_{\mu\nu\rho}k_\rho + \right.\right. \\ &\left.\left. +\left(\frac{\beta'}{Q^2(y_1')}-1\right)\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\right)\right], \end{aligned} \tag{33}$$

where $\beta' = \beta|_{\varepsilon\to\infty} = \frac{\Lambda^4}{4}k^2(\frac{\kappa}{4\pi})$, $y_1' = y_1|_{\varepsilon\to\infty}$ and:

$$\begin{aligned} f'(\Lambda,\sigma,\gamma) &= f|_{\varepsilon\to\infty} = 120\Lambda^{14} + \frac{19}{2}\Lambda^{10}Q(y_1') + 9\Lambda^6 Q^2(y_1') + \\ & -\frac{\Lambda^6}{2}Q(y_1') + \frac{9}{2}\Lambda^4 Q(y_1') + \Lambda^2 Q^2(y_1') - 18Q^2(y_1') + \frac{9}{2}Q(y_1'), \end{aligned} \quad (34)$$

$$g'(\Lambda,\sigma,\gamma) = g|_{\varepsilon\to\infty} = 64\Lambda^8 Q^2(y_1') + 32\Lambda^4 Q^3(y_1') + 4Q^4(y_1'), \quad (35)$$

$$Q(y_1') \equiv \Big(\Lambda^4 y_1' - (y_1')^2\Big)^{1/2}. \quad (36)$$

We see that in our last expression for the propagator remained a term proportional to the Maxwell transversal propagator. This is a anomalous situation, since the Maxwell contribution is absent in the usual purely topological Chern-Simons theory. The origin of this anomalous situation is the use of a non-Markovian Langevin equation. To circumvent this problem and recover the usual result, we have to make the following choice:

$$\beta' = Q^2(y_1'), \quad (37)$$

which lead us to:

$$y_1' = \frac{\Lambda}{2} \pm \frac{(\Lambda^8 - 4\beta')^{1/2}}{2}. \quad (38)$$

So, if we choose:

$$y_1 = \frac{\Lambda}{2} \pm \frac{(\Lambda^8 - 4\beta')^{1/2}}{2} + \frac{C}{\varepsilon^n}, \quad (39)$$

where C is a real constant and n is an arbitrarily large integer number. Inserting this latter equation in Eq.(A.23), we will get a cubic equation in C. From the usual Galois theory of radical solutions for polynomials [17] [18], we can always choose a real root from the three possible ones. So, in other words, we can always choose a real constant such that the two-point correlation function converges to a "purely topological" term, with some minor differences from the usual one. We notice as well that our approach still works when κ is purely imaginary (which is mathematically analogous to writing $A_\mu = A'_\mu + iA'_\mu$, where A'_μ is real, and taking the real part of the Langevin equation (3) in coordinate space).

3 Conclusions

In this paper we discussed the stochastic quantization for Maxwell Chern-Simons theory using a non-Markovian Langevin equation and examined the field theory that appears in the asymptotic limit of this non-Markovian process.

This paper is the second one of a program where it is investigated the possibility that the Parisi-Wu quantization method can be extended assuming a Langevin equation with a memory kernel with the modified Einstein relations. To make sure that this modification can be used, one must first check that the system evolves to the equilibrium in the asymptotic limit. Second we have to show that converges to the correct equilibrium distribution. We proved that although the system evolves to equilibrium, in the propagator remained a term proportional to the Maxwell transversal propagator. This is a anomalous situation, since the Maxwell contribution is absent in the usual purely topological Chern-Simons theory. To circumvent this problem and recover the usual result, we have imposed a constraint in the parameters of our theory.

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A Appendix

In this appendix, we derive the retarded Green function for the diffusion problem $G_{\mu\nu}(k,\tau)$. Expanding the denominator in Eq.(A.1), given by

$$G_{\mu\nu}(k,z) = \frac{k_\mu k_\nu}{k^2}\frac{1}{z} + \frac{\left(\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)z - \epsilon_{\mu\nu\sigma}k_\sigma\left(\frac{\kappa'}{4\pi}\right)\right)}{\left(z + \frac{k^2}{\varepsilon'^2}\right)^2 + \left(\frac{\kappa'}{4\pi}\right)^2 k^2}, \tag{A.1}$$

we have:

$$G_{\mu\nu}(k,z) = \frac{k_\mu k_\nu}{k^2}\frac{1}{z} + \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)I_1(z) - \epsilon_{\mu\nu\sigma}k_\sigma\left(\frac{\kappa}{4\pi}\right)I_2(z), \tag{A.2}$$

where:

$$I_1(z) \equiv \frac{z}{P(z)}, \tag{A.3}$$

$$I_2(z) \equiv \frac{M_\Lambda(z)}{P(z)}, \tag{A.4}$$

and:

$$P(z) \equiv z^2 + 2\frac{k^2}{\varepsilon^2}M_\Lambda(z)z + \frac{k^4}{\varepsilon^4}M_\Lambda^2(z) + \left(\frac{\kappa}{4\pi}\right)^2 k^2 M_\Lambda^2(z). \tag{A.5}$$

Using the following exponential representation for the memory kernel $M_\Lambda(\tau)$:

$$M_\Lambda^n(\tau) = \frac{1}{2n!}\Lambda^2(\Lambda^2 \mid \tau \mid)^n \exp(-\Lambda^2 \mid \tau \mid), \tag{A.6}$$

where Λ is a parameter, we will have, for the case $n = 0$:

$$I_1(z) = \frac{z^3 + 2\Lambda^2 z^2 + \Lambda^4 z}{\Omega(z)}, \tag{A.7}$$

$$I_2(z) = \frac{\frac{\Lambda^2}{2}z + \frac{\Lambda^4}{2}}{\Omega(z)}, \tag{A.8}$$

and:

$$\Omega(z) \equiv z^4 + 2\Lambda^2 z^3 + (\Lambda^4 + \alpha)z^2 + \alpha\Lambda^2 z + \beta, \tag{A.9}$$

where:

$$\alpha \equiv \frac{k^2\Lambda^2}{\varepsilon^2}, \tag{A.10}$$

and

$$\beta \equiv \left(\frac{k^4}{\varepsilon^4} + \left(\frac{\kappa}{4\pi}\right)^2 k^2\right)\frac{\Lambda^4}{4}. \tag{A.11}$$

In order to get the inverse Laplace transform of Eq.(A.2), we must seek for the solutions of the quartic equation $\Omega(z) = 0$. As it is well known, a general quartic equation is a fourth-order polynomial equation of the form:

$$z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0 = 0. \tag{A.12}$$

Using the familiar algebraic technique developed by Ferrari and Cardano [19], it is easy to show that the roots of Eq.(A.12) are given by:

$$z_1 = -\frac{1}{4}a_3 + \frac{1}{2}R + \frac{1}{2}D, \tag{A.13}$$

$$z_2 = -\frac{1}{4}a_3 + \frac{1}{2}R - \frac{1}{2}D, \tag{A.14}$$

$$z_3 = -\frac{1}{4}a_3 - \frac{1}{2}R + \frac{1}{2}E, \tag{A.15}$$

$$z_4 = -\frac{1}{4}a_3 - \frac{1}{2}R - \frac{1}{2}E, \tag{A.16}$$

where:

$$R \equiv \left(\frac{1}{4}a_3^2 - a_2 + y_1\right)^{1/2}, \tag{A.17}$$

$$D \equiv \begin{cases} \left(F(R) + G\right)^{1/2} & \text{for } R \neq 0 \\ \left(F(0) + H\right)^{1/2} & \text{for } R = 0, \end{cases} \tag{A.18}$$

$$E \equiv \begin{cases} \left(F(R) - G\right)^{1/2} & \text{for } R \neq 0 \\ \left(F(0) - H\right)^{1/2} & \text{for } R = 0, \end{cases} \tag{A.19}$$

$$F(R) \equiv \frac{3}{4}a_3^2 - R^2 - 2a_2, \tag{A.20}$$

$$H \equiv 2\left(y_1^2 - 4a_0\right)^{1/2}, \tag{A.21}$$

$$G \equiv \frac{1}{4}(4a_3a_2 - 8a_1 - a_3^3)R^{-1}, \tag{A.22}$$

and y_1 is a real root of the following cubic equation:

$$y^3 - a_2 y^2 + (a_1 a_3 - 4a_0)y + (4a_2 a_0 - a_1^2 - a_3^2 a_0) = 0. \tag{A.23}$$

Therefore, the inverse Laplace transform of $I_1(z)$ and $I_2(z)$ reads:

$$\begin{aligned} I_1(\tau) \quad = \quad & \frac{z_1^3 + 2\Lambda^2 z_1^2 + \Lambda^4 z_1}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)} e^{z_1 \tau} + \\ & + \frac{z_2^3 + 2\Lambda^2 z_2^2 + \Lambda^4 z_2}{(z_2 - z_1)(z_2 - z_3)(z_2 - z_4)} e^{z_2 \tau} + \end{aligned}$$

$$+\frac{z_3^3+2\Lambda^2 z_3^2+\Lambda^4 z_3}{(z_3-z_1)(z_3-z_2)(z_3-z_4)}e^{z_3\tau}+$$
$$+\frac{z_4^3+2\Lambda^2 z_4^2+\Lambda^4 z_4}{(z_4-z_1)(z_4-z_2)(z_4-z_3)}e^{z_4\tau}, \tag{A.24}$$

and

$$\begin{aligned} I_2(\tau) &= \frac{\frac{\Lambda^2}{2}z_1+\frac{\Lambda^4}{2}}{(z_1-z_2)(z_1-z_3)(z_1-z_4)}e^{z_1\tau}+ \\ &+\frac{\frac{\Lambda^2}{2}z_2+\frac{\Lambda^4}{2}}{(z_2-z_1)(z_2-z_3)(z_2-z_4)}e^{z_2\tau}+ \\ &+\frac{\frac{\Lambda^2}{2}z_3+\frac{\Lambda^4}{2}}{(z_3-z_1)(z_3-z_2)(z_3-z_4)}e^{z_3\tau}+ \\ &+\frac{\frac{\Lambda^2}{2}z_4+\frac{\Lambda^4}{2}}{(z_4-z_1)(z_4-z_2)(z_4-z_3)}e^{z_4\tau}. \end{aligned} \tag{A.25}$$

Now, let us study a simple convergence criterium in order that $G_{\mu\nu}(k,\tau)\to 0$ as the Markov paramter goes to infinity, i.e., $\tau\to\infty$. In this situation, the system converges to an equilibrium. Comparing the polynomial $\Omega(z)$ with expression Eq.(A.12), it is trivial to make the following identifications: $a_0=\beta$, $a_1=\alpha\Lambda^2$, $a_2=\alpha+\Lambda^4$ and, finally, $a_3=2\Lambda^2$.

For convenience, let us assume that R, defined by Eq.(A.17), does not vanish. To proceed with the calculations, let us introduce the following real quantities σ and γ defined respectively by

$$\sigma\equiv\left(a_2-\frac{1}{4}a_3^2-y_1\right)^{1/2}=(\alpha-y_1)^{1/2} \tag{A.26}$$

and

$$\gamma\equiv\left(a_2+y_1-\frac{1}{2}a_3^2\right)^{1/2}=(\alpha+y_1-\Lambda^4)^{1/2}, \tag{A.27}$$

where we used the identifications $a_2=\alpha+\Lambda^4$ and $a_3=2\Lambda^2$. Then, we shall have:

$$R=i\sigma, \tag{A.28}$$

and

$$E=i\gamma. \tag{A.29}$$

So, with the above identifications, it is easy to see to prove that G, defined by Eq.(A.22), vanishes identically. Therefore, we will have, from Eq.(A.18) and Eq.(A.19), that $D=E$. We also see that:

$$\sigma^2+\gamma^2=2\alpha-\Lambda^4>0, \tag{A.30}$$

which implies:

$$k^2>\frac{\varepsilon^2\Lambda^2}{2}, \tag{A.31}$$

where we used Eq.(A.10), which is a convergence criterium similar to the massless scalar field case [6].

Thus, from Eqs.(A.13) - (A.16), Eq.(A.28) and Eq.(A.29), we obtain the following solutions to $\Omega(z) = 0$:

$$z_1 = -\frac{\Lambda^2}{2} + \frac{1}{2}i\sigma + \frac{1}{2}i\gamma, \tag{A.32}$$

$$z_2 = -\frac{\Lambda^2}{2} + \frac{1}{2}i\sigma - \frac{1}{2}i\gamma, \tag{A.33}$$

$$z_3 = -\frac{\Lambda^2}{2} - \frac{1}{2}i\sigma + \frac{1}{2}i\gamma, \tag{A.34}$$

$$z_4 = -\frac{\Lambda^2}{2} - \frac{1}{2}i\sigma - \frac{1}{2}i\gamma. \tag{A.35}$$

So, from these last results, we will have, finally, for $G_{\mu\nu}(k,\tau)$:

$$G_{\mu\nu}(k,\tau) = \left(\frac{k_\mu k_\nu}{k^2} + g_{\mu\nu}G_1(k,\tau) + \tilde{g}_{\mu\nu}G_2(k,\tau)\right)\theta(\tau), \tag{A.36}$$

where:

$$G_1(k,\tau) \equiv \left(\frac{\Lambda^2}{(\sigma+\gamma)}\sin\left(\frac{(\sigma+\gamma)}{2}\tau\right) + \cos\left(\frac{(\sigma+\gamma)}{2}\tau\right)\right)e^{-\frac{\Lambda^2}{2}\tau}, \tag{A.37}$$

$$G_2(k,\tau) \equiv \left(\frac{\Lambda^2}{(\sigma-\gamma)}\sin\left(\frac{(\sigma-\gamma)}{2}\tau\right) + \cos\left(\frac{(\sigma-\gamma)}{2}\tau\right)\right)e^{-\frac{\Lambda^2}{2}\tau}, \tag{A.38}$$

and $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ appearing in Eq.(A.36) are defined by:

$$g_{\mu\nu} \equiv \Pi_{\mu\nu} - h_{\mu\nu}, \tag{A.39}$$

$$\tilde{g}_{\mu\nu} \equiv h_{\mu\nu} - \tilde{\Pi}_{\mu\nu}, \tag{A.40}$$

with:

$$h_{\mu\nu} \equiv -\frac{\Lambda^2}{2\gamma\sigma}\epsilon_{\mu\nu\rho}k_\rho\left(\frac{\kappa}{4\pi}\right), \tag{A.41}$$

$$\Pi_{\mu\nu} \equiv \frac{1}{\gamma\sigma}\left(\frac{\Lambda^4}{4} + \frac{(\sigma+\gamma)^2}{4}\right)\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right), \tag{A.42}$$

and

$$\tilde{\Pi}_{\mu\nu} \equiv -\frac{1}{\gamma\sigma}\left(\frac{\Lambda^4}{4} + \frac{(\sigma-\gamma)^2}{4}\right)\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right). \tag{A.43}$$

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