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Bethe ansatz solution of the closed anisotropic supersymmetric U model with quantum supersymmetry

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The nested algebraic Bethe ansatz is presented for the anisotropic supersymmetric U model maintaining quantum supersymmetry. The Bethe ansatz equations of the model are obtained on a one-dimensional closed lattice and an expression for the energy is given.

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I. INTRODUCTION

The study of strongly correlated electrons continues to receive a lot of interest due to applications in condensed matter physics.

Some of the well known models are the Hubbard and t-J models [1] and generalizations such as the EKS model [2].

Another example of a model describing strongly correlated electrons is the the supersymmetric U model. This model was first introduced in [3] and was shown to be integrable via the Quantum Inverse Scattering Method (QISM) [4] by demonstrating that the model could be obtained from an R-matrix which is invariant with respect to the Lie superalgebra $gl(2|1)$. The Bethe-ansatz equations for the model were obtained in [5–8]. Subsequently, an anisotropic generalization was presented in [9] which was also shown to be integrable through use of an R-matrix derived from a representation of the quantum superalgebra $U_q(gl(2|1))$ [10].

The anisotropic supersymmetric U model describes a system of correlated electrons and generalizes the Hubbard model in the sense that as well as the presence of the Hubbard on site (Coulomb) interaction there are additional correlated hopping and pair hopping terms. The model acts on the unrestricted 4^k-dimensional electronic Hilbert space $\otimes_{n=1}^{k} C^4$ where k is the lattice length. This means that double occupancy of sites is allowed and distinguishes this model from the anisotropic $t-J$ model [11] which shares the same supersymmetry algebra $U_q(gl(2|1))$. The model contains one free parameter U, (the Hubbard interaction parameter) which arises from the one-parameter family of inequivalent typical four-dimensional irreducible representations of the $U_q[gl(2|1)]$, and another which arises from the deformation parameter q.

Bethe ansatz solutions for the anisotropic model with periodic boundary conditions have been studied [9,12,13], however for this case there is no quantum superalgebra symmetry. In [14–18] some quantum algebra invariant integrable closed chains derived from an R-matrix associated with the Hecke algebra were introduced and investigated. It was subsequently shown [19] that a general prescription for the construction of integrable systems with periodic boundary conditions and quantum algebra invariance existed which could then be applied to higher spin models such as the spin 1 XXZ Heisenberg chain [20].

In the present article we further develop this method by considering the graded case to derive the Hamiltonian of the anisotropic supersymmetric U model with quantum supersymmetry on the closed chain. We will adopt a nested algebraic Bethe ansatz to solve the model and this will be presented in detail in Section 3. Also the energy of the Hamiltonian will be given.

II. QUANTUM ALGEBRA INVARIANT HAMILTONIAN FOR THE SUPERSYMMETRIC U MODEL

The following notation will be adopted. Electrons on a lattice are described by canonical Fermi operators $c_{i,\sigma}$ and $c^\dagger_{i,\sigma}$ satisfying the anti-commutation relations given by $\{c^\dagger_{i,\sigma}, c_{j,\tau}\} = \delta_{ij}\delta_{\sigma\tau}$, where $i, j = 1, 2, .., k$ and $\sigma, \tau = \uparrow, \downarrow$. The operator $c_{i,\sigma}$ ($c^\dagger_{i,\sigma}$) annihilates (creates) an electron of spin σ at site i, which implies that the Fock vacuum $|0>$ satisfies $c_{i,\sigma}|0>= 0$. At a given lattice site i there are four possible electronic states:

$$|0>, \quad |\uparrow>_i = c^\dagger_{i,\uparrow}|0>, \quad |\downarrow>_i = c^\dagger_{i,\downarrow}|0>, \quad |\uparrow\downarrow>_i = c^\dagger_{i,\downarrow}c^\dagger_{i,\uparrow}|0> .$$

By $n_{i,\sigma} = c^\dagger_{i,\sigma}c_{i,\sigma}$ we denote the number operator for electrons with spin σ on site i, and we write $n_i = n_{i,\uparrow} + n_{i,\downarrow}$. The local Hamiltonian for this model is [9]

$$\begin{aligned} H_{i(i+1)} = & -\sum_\sigma (c^\dagger_{i\sigma}c_{i+1\sigma} + h.c.)\exp\left[-\frac{1}{2}(\zeta - \sigma\gamma)n_{i,-\sigma} - \frac{1}{2}(\zeta + \sigma\gamma)n_{i+1,-\sigma}\right] \\ & + \left[U n_{i\uparrow}n_{i\downarrow} + U n_{i+1\uparrow}n_{i+1\downarrow} + U(c^\dagger_{i\uparrow}c^\dagger_{i\downarrow}c_{i+1\downarrow}c_{i+1\uparrow} + h.c.)\right], \end{aligned} \tag{1}$$

where i labels the sites and

$$U = \epsilon[2e^{-\zeta}\cosh\zeta - \cosh\gamma)]^{\frac{1}{2}}, \qquad \epsilon = \pm 1.$$

This Hamiltonian may be obtained from the R-matrix of a one-parameter family of four-dimensional representations of $U_q[gl(2|1)]$, which is afforded by the module W with highest weight $(0, 0|\alpha)$. The details of this construction may be found in [10].

The Hamiltonian (1) may be modified to ensure quantum superalgebra invariance by adapting the general construction presented in [19].

We can write

$$H = \sum_{i=1}^{k-1} H_{i(i+1)} + H_{1k},$$

where the boundary term is given by

$$H_{1k} = GH_{k,1}G^{-1}$$

with

$$G = R^-_{21}R^-_{31}...R^-_{k1}, \quad k \text{ the lattice length.}$$

Above, R^- is the constant R-matrix obtained as the zero spectral parameter limit of the Yang-Baxter equation solution associated with the model [10]. These operators act in the quantum space and the closed boundary conditions of the model may be explained by the relations

$$GH_{i,i+1} = H_{i+1,i+2}G, \quad i = 1, 2, ..., k-2, \quad GH_{1k} = H_{12}G.$$

The quantum supersymmetry of the Hamiltonian is a result of the intertwining properties of the matrices R.

III. NESTED ALGEBRAIC BETHE ANSATZ

We present the nested algebraic Bethe ansatz for the above Hamiltonian by extending the methods presented in [19,20] to treat quantum group invariant closed higher spin chains to the graded case. We begin with the R-matrix satisfying the Yang-Baxter equation constructed directly from a solution of the twisted representation as given in [13].

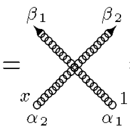
The Yang-Baxter Equation may be written as the operator equation:

$$ {}_{vv}R^{\alpha_1\alpha_2}_{\beta_1\beta_2}(x/y) \;\; {}_{vw}R^{\beta_1 a}_{\gamma_1 b}(x) \;\; {}_{vw}R^{\beta_2 b}_{\gamma_2 c}(y) = {}_{vw}R^{\alpha_2 a}_{\beta_2 b}(y) \;\; {}_{vw}R^{\alpha_1 b}_{\beta_1 c}(x) \;\; {}_{vv}R^{\beta_1\beta_2}_{\gamma_1\gamma_2}(x/y), \tag{2} $$

acting on the spaces $V \otimes V \otimes W$ where V is the vector module and W is the four-dimensional module associated with the one-parameter family of minimal typical representations. Greek indices are used to label the matrix spaces, that is the first two spaces and Roman indices label the quantum space, which is the third space. The quantum space represents the Hilbert space over a site on the one-dimensional lattice. The ${}_{vv}R$-matrix acts in the matrix space and it is between the two matrix spaces that the graded tensor product acts.

The ${}_{vv}R$-matrix acts on $V \otimes V$ and has the form [21], [22]

$$ {}_{vv}R^{\beta_1\beta_2}_{\alpha_1\alpha_2}(x) = $$



$$ = \begin{pmatrix} A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & E & 0 & C & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 & 0 & 0 & C & 0 & 0 \\ 0 & xC & 0 & E & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & E & 0 & C & 0 \\ 0 & 0 & xC & 0 & 0 & 0 & E & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & xC & 0 & E & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} $$

where A, E and C depend on the spectral parameter as follows $A(x) = \frac{1-xq^2}{x-q^2}$, $E(x) = \frac{(1-x)q}{x-q^2}$ and $C(x) = \frac{1-q^2}{x-q^2}$. The ${}_{vv}R$-matrix satisfies the Yang-Baxter equation

$$ R_{12}(x/y)R_{13}(x)R_{23}(y) = R_{23}(y)R_{13}(x)R_{12}(x/y). $$

By construction, these R-matrices also satisfy the generalized Cherednik reflection property [23]

$$ R^{\alpha\beta}_{\alpha'\beta'}(x){R^{-1}}^{\alpha'\beta'}_{\gamma\delta}(1/y) = R^{\alpha\beta}_{\alpha'\beta'}(y){R^{-1}}^{\alpha'\beta'}_{\gamma\delta}(1/x), \tag{3} $$

and crossing unitarity [24]

$$ {R^{st_1}}^{\alpha\beta}_{\alpha'\beta'}(x\zeta)K^{\alpha'}_{\alpha''}{(R^{-1})^{st_1}}^{\alpha''\beta'}_{\gamma'\delta}(x){K^{-1}}^{\gamma'}_{\gamma} = \delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta}, \tag{4} $$

where st_1 denotes matrix supertransposition in the first space and K is the crossing matrix given below.

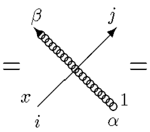
It will be necessary to rewrite the ${}_{vv}R$-matrix in terms of constant matrices ${}_{vv}R^{+}$ and ${}_{vv}R^{-}$ that is,

$$ {}_{vv}R(x) = \left(\frac{-1}{x-q^2}\right)\left(x \;\; {}_{vv}R^{+} - \;\; {}_{vv}R^{-}\right), $$

where ${}_{vv}R^{+}$ (${}_{vv}R^{-}$) corresponds to the leading term in the limit as $x \to \infty$ ($x \to 0$).

The ${}_{vw}R$-matrix was constructed in [13] in the $V \otimes W$ space and has the following form

$$ {}_{vw}R^{\beta j}_{\alpha i}(x) = $$



$$ = \begin{pmatrix} J & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & J & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Y & 0 & 0 & Q' & 0 & 0 & -S' & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 & 0 & 0 & 0 & -T' & 0 & 0 \\ 0 & 0 & 0 & 0 & J & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Q & 0 & 0 & Y & 0 & 0 & -P' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & J & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & L & 0 & 0 & -T' & 0 \\ 0 & 0 & S & 0 & 0 & P & 0 & 0 & M & 0 & 0 & 0 \\ 0 & 0 & 0 & -T & 0 & 0 & 0 & 0 & 0 & -N & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -T & 0 & 0 & -N & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} $$

where the dependence of these elements on the spectral parameter is given by

$$ J(x) = \frac{(x - q^{-\alpha-2})}{(xq^{-\alpha-2}-1)}, \quad Y(x) = J(x)(D+B) + \frac{1}{[\alpha+2]}, $$

$$ L(x) = \frac{1}{[\alpha+2]}([\alpha+1]J(x)+1), \quad M(x) = F^2 DJ(x) + \frac{[\alpha]}{[\alpha+2]}, \quad N(x) = \frac{1}{[\alpha+2]}(J(x)+[\alpha+1]), $$

$$ Q(x) = (qB - Dq^{-1})J(x) - \frac{q^{-1}}{[\alpha+2]}, \quad Q'(x) = (q^{-1}B - qD)J(x) - \frac{q}{[\alpha+2]}, $$

$$ S(x) = \frac{\sqrt{[\alpha]}}{[\alpha+2]}q^{-(\alpha+3)/2} - q^{(\alpha+1)/2}FDJ(x), \quad S'(x) = -\frac{\sqrt{[\alpha]}}{[\alpha+2]}q^{(\alpha+3)/2} + q^{-(\alpha+1)/2}FDJ(x), $$

$$ T(x) = \frac{\sqrt{[\alpha+1]}}{[\alpha+2]}(q^{(\alpha+2)/2}J(x) - q^{-(\alpha+2)/2}), \quad T'(x) = \frac{\sqrt{[\alpha+1]}}{[\alpha+2]}(q^{(\alpha+2)/2} - q^{-(\alpha+2)/2}J(x)), $$

$$ P(x) = q^{(\alpha+3)/2}FDJ(x) - \frac{\sqrt{[\alpha]}}{[\alpha+2]}q^{-(\alpha+1)/2}, \quad P'(x) = -q^{-(\alpha+3)/2}FDJ(x) + \frac{\sqrt{[\alpha]}}{[\alpha+2]}q^{(\alpha+1)/2}, $$

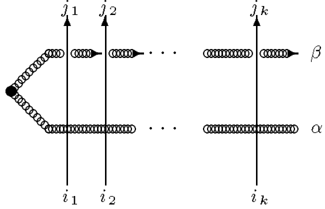
with constants

$$ D = \frac{[\alpha]}{[\alpha+2](q+q^{-1})}, \quad F = \frac{(q+q^{-1})}{\sqrt{[\alpha]}}, \quad B = 1/(q+q^{-1}), \quad \text{and} \quad [\alpha] = \frac{q^{\alpha}-q^{-\alpha}}{q-q^{-1}}. $$

The ${}_{vw}R$-matrix as well as satisfying the Yang Baxter relation (2), also satisfies the generalized Cherednick reflection property (3) and crossing unitarity (4).

We now introduce an auxiliary doubled monodromy matrix

$$ {}_{vw}U(x)^{\beta\{j\}}_{\alpha\{i\}} = $$



$$= {}_{vw}R_{+}{}^{\beta_2 j_1}_{\alpha' j_1'}\ {}_{vw}R_{+}{}^{\beta_3 j_2}_{\beta_2 j_2'} \dots {}_{vw}R_{+}{}^{\beta j_k}_{\beta_k j_k'}\ {}_{vw}R^{\alpha' j_1'}_{\alpha_2 i_1}(1/x)\ {}_{vw}R^{\alpha_2 j_2'}_{\alpha_3 i_2}(1/x) \dots {}_{vw}R^{\alpha_k j_k'}_{\alpha i_l}(1/x),$$

acting on $V \otimes W^{\otimes k}$. Above ${}_{vw}R_{+}$ represents the leading term in the matrix ${}_{vw}R^{-1}(x)$ for the limit as $x \to \infty$.

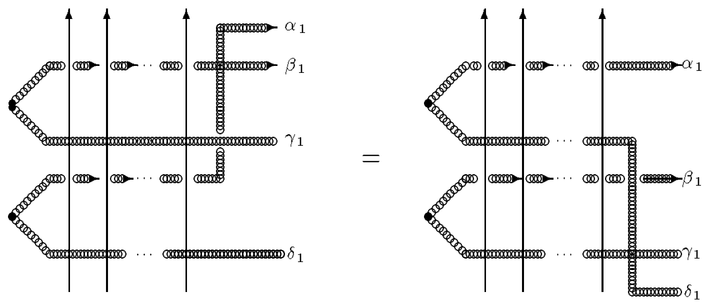
Represent the doubled monodromy matrix in the following way:

$$ {}_{vw}U^{\gamma}_{\alpha}(x) = \begin{pmatrix} {}_{vw}U^1_1(x) & {}_{vw}U^1_2(x) & {}_{vw}U^1_3(x) \\ {}_{vw}U^2_1(x) & {}_{vw}U^2_2(x) & {}_{vw}U^3_2(x) \\ {}_{vw}U^3_1(x) & {}_{vw}U^3_2(x) & {}_{vw}U^3_3(x) \end{pmatrix}. \tag{5}$$

It may be shown that this monodromy matrix satisfies the following Yang-Baxter relation

$$ {}_{vv}R^{\alpha_1\beta_1}_{\alpha_2\beta_2}(y/x)\ \ {}_{vw}U^{\beta_2 a}_{\gamma_2 b}(x)\ \ {}_{vv}R_{+}{}^{\gamma_2\alpha_2}_{\gamma_1\delta_2}\ \ {}_{vw}U^{\delta_2 b}_{\delta_1 c}(y) = {}_{vw}U^{\alpha_1 a}_{\alpha_2 b}(y)\ \ {}_{vv}R_{+}{}^{\alpha_2\beta_1}_{\delta_2\beta_2}\ \ {}_{vw}U^{\beta_2 b}_{\gamma_2 c}(x)\ \ {}_{vv}R^{\gamma_2\delta_2}_{\gamma_1\delta_1}(y/x), \tag{6}$$

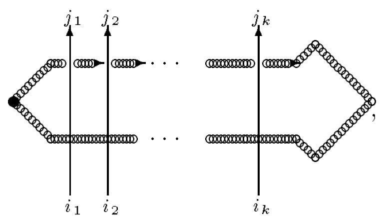
depicted graphically below.



The occurance of the constant matrices ${}_{vv}R_{+}$ will greatly simplify the calculations of the algebraic Bethe ansatz.

In order to perform the nested algebraic Bethe ansatz (NABA) we define an auxiliary transfer matrix as the (super) Markov trace of the monodromy matrix, that is,

$$ {}_{vw}\tau^{\{j\}}_{\{i\}}(y) = \sum_i {}_vK^{\alpha}_{\alpha}\ \ {}_{vw}U^{\alpha\{j\}}_{\alpha\{i\}}(y) = \tag{7}$$



where

$$ {}_vK = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q^2 & 0 \\ 0 & 0 & -q^2 \end{pmatrix}. \tag{8}$$

Therefore the ${}_{vw}\tau(y)$ form a one-parameter family of commuting operators and it may be shown that they commute with the transfer matrix, ${}_{ww}\tau(y)$ of the Hamiltonian (1). This means that they have a common set of eigenvectors. We find that

$$ {}_{vw}\tau(y) = {}_{vw}U^1_1(y) + q^2 \ {}_{vw}U^2_2(y) - q^2 \ {}_{vw}U^3_3(y). \tag{9} $$

Take the lowest weight state as a reference state (pseudo-vacuum) in W, which we denote as $|0>_i$.

Then $|0>=\otimes_{i=1}^k|0>_i$ and we find the action of the doubled monodromy matrix on this reference state to be given by

$$ {}_{vw}U(x)_k|0>= \begin{pmatrix} I(x)^k & 0 & 0 \\ 0 & I(x)^k & 0 \\ {}_{vw}U^3_1(x) & {}_{vw}U^3_2(x) & 1 \end{pmatrix} |0>, \tag{10} $$

where

$$ I(x) = \frac{(1-xq^{\alpha})}{(1-xq^{-\alpha-2})}. $$

We construct a set of eigenstates of the transfer matrix using the technique of the NABA.

The creation operators are ${}_{vw}U^3_1(x)$, ${}_{vw}U^3_2(x)$ due to the choice of reference state. Thus we use the following for the ansatz for the eigenstates of ${}_{vw}\tau(y)$:

$$ \Psi = {}_{vw}U^3_{a_1}(x_1) \ {}_{vw}U^3_{a_2}(x_2) ... {}_{vw}U^3_{a_r}(x_r) \ \ \Psi^{(1)}_{\{a\}}|0>, \tag{11} $$

where indices a_i have values 1 or 2. We seek a solution of the eigenvalue equation

$$ {}_{vw}\tau(y)\Psi = {}_{vw}\Lambda(y)\Psi. \tag{12} $$

The action of these states is determined by the monodromy matrix and the relations (6). The relations necessary for the construction of the NABA are

$$ \begin{aligned} {}_{vw}U^3_3(y) \ {}_{vw}U^3_\alpha(x) = &-\frac{1}{q\ E(y/x)} \ {}_{vw}U^3_\alpha(x) \ {}_{vw}U^3_3(y) - \frac{y\ C(y/x)}{x\ q\ E(y/x)} \ {}_{vw}U^3_\alpha(y) \ {}_{vw}U^3_3(x) \\ &- \left(\frac{q-q^{-1}}{q}\right) \sum_\beta {}_{vw}U^3_\alpha(y) \ {}_{vw}U^\beta_\alpha(x), \end{aligned} \tag{13} $$

$$ {}_{vw}U^\gamma_\beta(y) \ {}_{vw}U^3_\alpha(x) = \frac{r_{+\delta'\gamma'}^{\ \gamma\alpha'} \ r_{\beta\alpha}^{\beta'\gamma'}(x/y)}{q\ E(x/y)} \ {}_{vw}U^3_{\alpha'}(x) \ {}_{vw}U^{\delta'}_{\beta'}(y) - \frac{x \ \ r_{+\delta'\beta}^{\ \gamma\alpha'} \ \ C(x/y)}{y\ q\ E(x/y)} \ {}_{vw}U^3_{\alpha'}(y) \ {}_{vw}U^{\delta'}_\alpha(x), \tag{14} $$

with the indices taking values of 1 and 2. It can be seen that this R-matrix $r(y)$ also fulfills a Yang-Baxter equation and can be identified with the R-matrix of the quantum spin $\frac{1}{2}$ Heisenberg (XXZ) model.

The action of the ansatz (11) on the diagonal elements of the monodromy matrix (5) is given by

$$ {}_{vw}U^3_3(y)\Psi = \frac{(-1)^r}{q^r} \prod_{i=1}^r \frac{1}{E(y/x_i)} \Psi + \text{ u.t.} $$

$$ [{}_{vw}U^1_1(y) + q^2 \ {}_{vw}U^2_2(y)]\Psi = \frac{I(y)^k}{q^r} \prod_{j=1}^r \frac{1}{E(x_j/y)} \prod_{l=1}^r {}_{vw}U^3_{b_l}(x_l)|0> q\tau^{(1)}(y)^{b_1 \ldots b_r}\Psi^{(1)} + \text{ u.t.} $$

where

$$ {}_{vw}\tau^{(1)}(y) = q^{-1} \ {}_{vw}U^{(1)}{}^1_1(y) + q \ {}_{vw}U^{(1)}{}^2_2(y). \tag{15} $$

In order that the eigenvalue problem (12) is satisfied, it is necessary to solve a new eigenvalue problem for the nesting as follows:

$$ {}_{vw}\tau^{(1)}(y)\Psi^{(1)} = \Lambda^{(1)}(y, \{y_j\})\Psi^{(1)}, $$

where

$$ \Psi^{(1)} = {}_{vw}U^{(1)}{}^2_1(y_1)\ \ {}_{vw}U^{(1)}{}^2_1(y_2)...{}_{vw}U^{(1)}{}^2_1(y_m)|0>^{(1)} . \tag{16} $$

The second level reference state is given by $|0>^{(1)} = \otimes_{i=1}^r |2>_i$.

We represent the nested monodromy matrix as

$$ {}_{vw}U_m^{(1)}(y) = \begin{pmatrix} {}_{vw}U^{(1)}{}^1_1(y) & {}_{vw}U^{(1)}{}^1_2(y) \\ {}_{vw}U^{(1)}{}^2_1(y) & {}_{vw}U^{(1)}{}^2_2(y) \end{pmatrix} . \tag{17} $$

The action of the nested monodromy matrix ${}_{vw}U_m^{(1)}(y)$ on the second level reference state is

$$ \begin{aligned} {}_{vw}U^{(1)}{}^1_1(y)|0>^{(1)} &= q^r \prod_{j=1}^r E(x_j/y)|0>^{(1)}, \\ {}_{vw}U^{(1)}{}^2_2(y)|0>^{(1)} &= \prod_{j=1}^r A(x_j/y)|0>^{(1)} . \end{aligned} \tag{18} $$

The action of ${}_{vw}\tau^{(1)}(y)$ on the ansatz (16) is computed similar to the first level case from the relations (6). We obtain

$$ \begin{aligned} {}_{vw}U^{(1)}{}^1_1(y)\Psi^{(1)} &= q^{r-m} \prod_{i=1}^m \frac{A(y_i/y)}{E(y_i/y)} \prod_{l=1}^r E(x_l/y)\Psi^{(1)} + \text{ u.t.} \\ {}_{vw}U^{(1)}{}^2_2(y)\Psi^{(1)} &= q^{m} \prod_{i=1}^m \frac{A(y/y_i)}{E(y/y_i)} \prod_{l=1}^r A(x_l/y)\Psi^{(1)} + \text{ u.t.} \end{aligned} \tag{19} $$

The eigenvalues for the auxilliary transfer matrix, ${}_{vw}\tau(y)$ are found to be

$$ {}_{vw}\Lambda(y) = q^{-m} I(y)^k \prod_{i=1}^m \frac{A(y_i/y)}{E(y_i/y)} + q^{2+m-r} I(y)^k \prod_{i=1}^r \frac{A(x_i/y)}{E(x_i/y)} \prod_{j=1}^m \frac{A(y/y_j)}{E(y/y_j)} - (-1)^r q^{2-r} \prod_{i=1}^r \frac{1}{E(y/x_i)}, \tag{20} $$

provided the "unwanted terms" cancel. The cancellation of these terms lead to the Bethe ansatz equations obtained by eliminating the poles of the eigenvalues (20)

$$ \begin{aligned} \prod_{i\neq n}^m \frac{qy_i - q^{-1}y_n}{q^{-1}y_i - qy_n} &= q^{2(1+m)-r} \prod_{j=1}^r \frac{q^{-1}y_n - qx_j}{y_n - x_j} \qquad n = 1, ..., m, \\ q^m I(x_l)^k &= \prod_{j=1}^m \frac{y_j - x_l}{q^{-1}y_j - x_l q} \qquad l = 1, ..., r. \end{aligned} \tag{21} $$

Associated with these solutions, the energies of the Hamiltonian are given by

$$ E = \sum_j \frac{-(q^{\alpha+1} - q^{-\alpha-1})^2}{(q^{\alpha/2}x_j^{-1/2} - q^{-\alpha/2}x_j^{1/2})(q^{-\alpha/2-1}x_j^{-1/2} - q^{\alpha/2+1}x_j^{1/2})} + k\left(q^{\alpha+1} + q^{-\alpha-1}\right) . $$

This expression reduces to the normal periodic case [5] in the rational limit as $q \to 1$.

IV. CONCLUSION

In this work, we have constructed a quantum algebra invariant supersymmetric U model on a closed lattice and derived the Bethe ansatz equations. Notice that in the Bethe ansatz equations (21) the presence of "q" terms in comparison with the corresponding equations for the usual periodic boundary conditions [13]. In fact, this feature also appeared in other models [15,16,18,25] and seems to be a peculiarity of quantum-group-invariant closed spin chains. In the limit for $q \to 1$, the usual Bethe ansatz equations for the periodic chain [6] are recovered.

An appealing direction for further study of the present closed supersymmetric U model with $U_q[gl(2|1)]$ invariance would be to investigate its thermodynamic properties. In particular the partition function in the finite size scaling limit which can be used to derive the operator content [15] of the related statistical models.

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