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NON-ABELIAN GAUGE FIELDS IN THE POINCARÉ GAUGE

by

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ABSTRACT

The canonical structure of non-Abelian gauge fields is analyzed in the (non-covariant) Poincaré gauge. General aspects of the gauge conditions and quantization prescriptions are discussed.

Key-words: Gauge theory; Constrained systems; Quantization.

The purpose of this note is to consider the canonical structure and quantization of a general non-Abelian gauge field $A_\mu(x) = A^a{}_\mu(x)T_a$, where $\{T_a\}$ is a hermitian representation of the (semi-simple, compact) gauge group, under the Poincaré gauge conditions defined by

$$x_i A_i^a(x) = 0 \quad , \quad A_0^a(x) = - x_i \int_0^1 d\lambda \, \Pi_i^a(t,\lambda\vec{x}) \qquad (1)$$

(We will restrict our considerations to regular gauge fields.)

The above equations are the non-Abelian generalization of the gauge conditions discussed in [1] and widely used in quantum optics in connection with the interaction of electromagnetic radiation with matter in the long-wavelength limit. (In the old literature the Abelian version of conditions (1) were known as "multipolar gauge". See [2], for instance.)

The first of the conditions (1) can be viewed as the non-covariant version of the Fock-Schwinger gauge condition [3] which, in its more general (Lorentz covariant) form, is defined by $(x^\mu-\xi^\mu)A_\mu(x) = 0$, where ξ is a constant vector. The Fock-Schwinger gauge condition has recently acquired some popularity [4] due to its many remarkable virtues, the most important of which being the fact that it provides an expression for the gauge fields in terms of the field strength tensor,

$$A_\mu(x) = \int_0^1 d\lambda \; \lambda x^\nu \; F_{\nu\mu}(\lambda x) \quad . \qquad (2)$$

This "inversion formula" holds true for both Abelian and non--Abelian fields.

Brittin et al. [1] has called the attention to the fact that the inversion formula (2) is in fact ensured by the Poincaré lemma, so that the way of reasoning can be reversed since it is clear from eq. (2) that the gauge field automatically satisfies the Fock-Schwinger condition, $x^\mu A_\mu(x) = 0$. Now, in ref. [1] it has been shown that the Poincaré lemma also provides a non-covariant way of expressing the Abelian fields $A_i(x)$ and $A_0(x)$ in terms of the components of the field strength tensor as follows:

$$A_i(x) = \int_0^1 d\lambda \ \lambda x_k F_{ki}(t,\lambda\vec{x}) \quad ,$$
$$A_0(x) = \int_0^1 d\lambda \ x_i F_{i0}(t,\lambda\vec{x}) \tag{3}$$

The main differences between the above expressions and the corresponding components of the gauge field obtained from eq. (2) are clear. From the first of eqs. (3) it follows that $x_i A_i(x) = 0$ which is the non-covariant form of the Fock--Schwinger gauge conditions. The set of gauge conditions (3) has been called by Brittin et al. the "Poincaré gauge conditions".

The non-Abelian version of eqs. (3) can be obtained by the same procedure used in ref. [1], i.e., by applying the Poincaré lemma. Reversing the argument we can impose $x_i A_i^a(x)=0$ as a gauge condition, and then obtain the second of eqs. (1) as the consistency condition (time preservation) of the first one, as we will show shortly. Now, conditions (1) can be shown to be attainable and complete, thus constituting a good set of gauge conditions.

In what follows we will not go into details of the formal properties of the Poincaré gauge. A detailed discussion will be presented somewhere [5].

We shall now proceed with the canonical formalism for the Yang-Mills field. Due to the presence of two first class constraints in the theory,

$$\Omega_a^{(1)} = \Pi_a^0 \approx 0 \quad ,$$
$$\Omega_a^{(2)} = \partial_i \Pi_{ai} - g C_{abc} A_i^b \Pi_i^c \approx 0 \quad , \tag{4}$$

we need to impose two (non-covariant) gauge conditions and we choose one of them to be

$$\Omega_a^{(4)} = x_i A_i^a(x) \approx 0 \quad , \quad (i = 1,2,3) \quad . \tag{5}$$

Once the condition (5) is imposed we cannot arbitrarily choose a condition on $A_0^a(x)$ for such a condition is dictated by the time preservation of eq. (5). Indeed using the canonical Hamiltonian

$$H_c = \int d^3x\, \mathcal{H}_c = \int d^3x \left[\frac{1}{2}\Pi_i^a \Pi_i^a + \Pi_i^a \partial_i A_0^a + \frac{1}{4} F_{ij}^a F_{ij}^a - g C_{abc} A_0^b A_i^c \Pi_i^a \right] \tag{6}$$

it follows from $\dot{\Omega}_a^{(4)} = \{\Omega_a^{(4)}, H_c\} \approx 0$ that

$$x_i \partial_i A_0^a \approx x_i \Pi_i^a \tag{7}$$

whose solution provides us with the second gauge condition we are looking for:

$$\Omega_a^{(2)} = A_0^a(x) + x_i \int_0^1 d\lambda\, \Pi_i^a(\lambda \vec{x}) \approx 0 \quad . \tag{8}$$

The set of constraints (4), (5) and (8) is second class. The matrix

$$\mathbb{C} = (C^{ij}_{ab}(y,z)) \equiv (\{\Omega^{(i)}_a(y),\Omega^{(j)}_b(z)\}), \quad i,j=1,2,3,4,$$

is non-singular. For the inverse matrix $\mathbb{C}^{-1} = (\not{C}^{ij}_{ab}(x,z))$ we find that the only non-vanishing elements are

$$\not{C}^{ab}_{12}(x,y) = \delta_{ab}\int_0^1 d\alpha \int_0^1 d\lambda\alpha\lambda^2 x_i x_i \overset{(3)}{\delta}(\alpha\lambda\vec{x}-\vec{y}) \ , \quad (9a)$$

$$\not{C}^{ab}_{13}(x,y) = -\delta_{ab} \overset{(3)}{\delta}(\vec{x}-\vec{y}) \ , \quad (9b)$$

$$\not{C}^{ab}_{14}(x,y) = C_{abc}\int_0^1 d\lambda \int_0^1 d\alpha\ \alpha^2 \Pi^c_k(\alpha\vec{y})x_k \overset{(3)}{\delta}(\alpha\vec{y}-\lambda\vec{x}) \ , \quad (9c)$$

$$\not{C}^{ab}_{24}(x,y) = \delta_{ab}\int_0^1 d\lambda\ \lambda^2 \overset{(3)}{\delta}(\vec{x}-\lambda\vec{y}) \quad (9d)$$

Using the above expressions we obtain the following fundamental Dirac brackets:

$$\{A^a_i(x),A^b_0(z)\}^* = -(\delta^{ab}\frac{\partial}{\partial x^i} - C_{adc}A^d_i(x))C^{cd}_{12}(x,z) + \delta^{ab}\int_0^1 d\lambda\ z_i \overset{(3)}{\delta}(\vec{x}-\lambda\vec{z}) \ , \quad (10a)$$

$$\{A^a_i(x),\Pi^b_k(z)\}^* = \delta^{ab}\delta_{ik}\overset{(3)}{\delta}(\vec{x}-\vec{z}) + z_k(\delta^{ab}\frac{\partial}{\partial x^i} - C_{acb}A^c_i(x))\int_0^1 d\lambda\lambda^2 \overset{(3)}{\delta}(\vec{x}-\lambda\vec{z}) \ , \quad (10b)$$

$$\{A^a_0(x),\Pi^b_k(z)\}^* = -C_{abc}\int_0^1 d\lambda \int_0^1 d\alpha\ \alpha^2 \cdot \left[\Pi^c_i(\lambda\vec{x})x_i z_k \overset{(3)}{\delta}(\lambda\vec{x}-\alpha\vec{z}) + \Pi^c_k(\vec{z})x_i x_i \lambda \overset{(3)}{\delta}(\lambda\alpha\vec{x}-\vec{z})\right]. \quad (10c)$$

All other brackets are null. It is easy to verify that the above Dirac brackets are compatible with the constraints equations (4), (5), (8). The equations of motion for the dynamical variables read

$$\dot{A}^a_k(z) = \Pi^a_k(z) + \frac{\partial A^a_0}{\partial z^k} - g\, C_{abc} A^b_0(z) A^c_k(z) \quad , \tag{11}$$

$$\dot{\Pi}^a_k(z) = D_i F^a_{ki}(z) + g\, C_{abc} \Pi^b_k A^0_c(z) \quad . \tag{12}$$

Equation (11) is clearly compatible with equation (7).

The canonical quantization procedure can now be implemented as the fundamental Dirac brackets relations obtained above provide a basis for the construction of the equal time commutation relations through the usual transition scheme

$$\{C(x), B(z)\}^* \longrightarrow \frac{1}{i\hbar} [\hat{C}(x), \hat{B}(z)] \quad ,$$

with the second class constraints (4), (5), (8) converted into strong operators identities acting on the physical states of the system. It is worth noting that the right hand sides of the fundamental Dirac brackets (10) do not pose any operators ordering problems.

Let us now consider the path integral quantization. It is done by expressing the generating functional for the evolution operator as

$$Z = N \int [d\Pi^a_\mu][dA^\mu_a] \det(M_F) \prod_a \delta(\Omega^{(1)}_a)\delta(\Omega^{(2)}_a)\delta(\Omega^{(3)}_a)\delta(\Omega^{(4)}_a)$$

$$\times \exp\left[\frac{i}{\hbar} \int d^4x (\Pi^\mu{}_a \dot{A}^a{}_\mu - \mathcal{H}_c)\right] \tag{13}$$

where $\mathcal{H}_c$ is given by equation (6). After some length manipulations using standard techniques we obtain the desired result,

$$Z = N \int [dA^{\mu}{}_{a}]\,\delta(x_i A_i^a)\ \exp[\frac{i}{\hbar}\, S[A]] \tag{14}$$

where S[A] is the action functional expressed only in terms of the fields $A_{\mu}^{a}(x)$ and its derivatives.

Comments and Conclusions

We have considered the canonical framework for a general gauge field under the Poincaré gauge conditions. The gauge conditions are attainable and complete and thus constitute a good set of gauge conditions. Besides introducing some technical complications the gauge conditions break the translational invariance of the theory. As a consequence, on one hand, the Green's functions lack translational invariance but, on the other hand, they are endowed with a very simple structure. Also, the gauge conditions renders a theory which is free from Fadeev-Popov ghosts, a remarkable property which is shared by the class of axial gauges.

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