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CENTRO BRASILEIRO
DE PESQUISAS FÍSICAS

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ISOTROPIC SPHERICAL CLUSTERS OF PARTICLES IN GENERAL RELATIVITY

por

A.F. DA F. TEIXEIRA
CENTRO BRASILEIRO DE PESQUISAS FÍSICAS

e

M. M. SOM
INSTITUTO DE FÍSICA DA UFRJ

RIO DE JANEIRO
BRASIL

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A.F. da F. Teixeira
Centro Brasileiro de Pesquisas Físicas
&
M.M. Som
Instituto de Física da UFRJ

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ABSTRACT

We have studied a bounded cluster of many particles moving under their own gravitation. The cluster is spherically symmetric and stationary. All particles have the same rest mass and the same total energy. The T^0_0 component of the energy momentum tensor of the cluster is taken uniform. The distribution of the velocities of all particles that cross any given point inside the cluster is assumed isotropic. Exact solutions of the equations are obtained, and special internal geodesics are discussed. Comparison with interior Schwarzschild solution is given.

1. INTRODUCTION

The gravitational field of a large number of particles moving in a prescribed way under their own gravitation has been stu died by several authors. In an Einstein[1] cluster all particles move in randomly oriented concentric circular orbits, producing a static and spherically symmetric field; some particular Einstein clusters were considered by Florides[2] and by Teixeira and Som[3]. Similar clusters, but with cylindrical symmetry, has been studied by Raychaudhuri and Som[4] and by Teixeira and Som[5]; in these clusters all particles move in concentric circular orbits around the axis of symmetry.

Though mathematically interesting, these spherical and cylindrical clusters present a physically unsatisfactory aspect: since the orbit of each particle crosses the orbits of others, collisions of particles may occur; after such collisions each particle will generally present a non-vanishing radial component in its velocity, a situation which has been discarded in the previous systems.

In the present paper we consider random orientations in the motions of the particles of a spherical cluster. As a result of these random orientations one can reasonably assume that on the average the velocity space of the particles at any given point inside the cluster is isotropic. Due to this isotropy, the off-diagonal components of the energy momentum tensor vanish. An exact solution corresponding to the cluster with a homogeneous T_0^0 is obtained.

2. GENERAL EQUATIONS

In the Einstein equations (Anderson[6])

$$R^\mu_\nu - \delta^\mu_\nu R/2 = -\kappa T^\mu_\nu \quad , \qquad \kappa = 8\pi G/c^4 \tag{1}$$

We use the static spherically symmetric line element

$$ds^2 = e^{\nu}(dx^0)^2 - e^{\lambda}\, dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2 \quad , \qquad (2$$

with ν and λ functions of r alone.

The energy momentum density tensor T^{μ}_{ν} is to correspond to a non-homogeneous cluster of particles in motion; we assume an isotropic distribution for the velocities of all particles that cross any given point inside the cluster. Each particle (i) has mass m and contributes to T^{μ}_{ν} with an individual tensor

$$t^{\mu\nu}_{(i)} = c^2\, m\, u^{\mu}_{(i)} u^{\nu}_{(i)} \quad , \qquad (3)$$

with $u^{\mu}_{(i)}$ representing its four-velocity. We assume the cluster sufficiently dense so as to allow a meaningful space average of the individual $t^{\mu\nu}_{(i)}$ in order to give our $T^{\mu\nu}$.

Due to the random orientations of the paths of the particles $T^{\mu\nu}$ is diagonal; and the imposition of isotropy in the velocity space implies that $T^1_1 = T^2_2 = T^3_3$. So we are led with only two independent components for T^{μ}_{ν}, and write

$$T^{\mu}_{\nu} = c^2\rho\ \mathrm{diag}(1 + 3\beta^2/2 \ ,\ -\beta^2/2 \ ,\ -\beta^2/2 \ ,\ -\beta^2/2) \quad , \qquad (4)$$

with β^2 and the trace $c^2\rho$ functions of r alone.

From (1), (2) and (4) we obtain the independent equations

$$(r^{-2} - r^{-1}\lambda_1)e^{-\lambda} - r^{-2} = -\kappa c^2 \rho (1 + 3\beta^2/2) \quad , \qquad (5)$$

$$(r^{-2} + r^{-1}\nu_1)e^{-\lambda} - r^{-2} = \kappa c^2\, \rho\, \beta^2/2 \quad , \qquad (6)$$

$$(\rho\, \beta^2)_1 + \rho(1 + 2\beta^2)\nu_1 = 0 \quad , \qquad (7)$$

where a subscript 1 means d/dr. Since these three equations cowith four functions $(\nu,\lambda,\rho,\beta^2)$, one constraint is necessary in order to get explicit solutions: we choose T_0^0 = const, that is ,

$$\kappa c^2 \rho(1 + 3\beta^2/2) = 6\omega^2/c^2 \quad , \quad \omega^2 \text{ const} \quad ; \tag{8}$$

the choice of the constant in this way will become clear later.

3. SOLUTION OF EQUATIONS

From (5) and (8) we obtain the regular solution at the origin

$$\lambda = - 2 \log f \quad , \tag{9}$$

where

$$f(r) = (1 - 2\omega^2 r^2/c^2)^{1/2} \quad . \tag{10}$$

From (7) and (8) we obtain the first integral

$$\rho \beta^2 = Ae^{-\nu/2} - 12 \omega^2/\kappa c^4 \quad , \quad A\text{=const.} \tag{11}$$

From (6) one then easily obtains, using (9) and (11)

$$e^{\nu/2} = B f + A\kappa c^4/8\omega^2 \quad , \quad B\text{=const} . \tag{12}$$

In order to fix the constants A and B we impose that any particle on the boundary r = a of the spherical cluster be (momentarily) at rest: from (4) one finds then that this amounts to impose that

$$\beta^2 (a) = 0 \quad ; \tag{13}$$

and we must have on the boundary r = a

$$\nu(a) = - \lambda(a) \quad , \tag{14}$$

since the line element (2) must be continuous with the external

Schwarzschild line element, for which $\nu = -\lambda$. These two conditions (13) and (14) then give for our cluster ($r \leq a$)

$$\rho(r) = (3\omega^2/2\pi G)(3f_a - 2f)(3f_a - f)^{-1} \quad , \qquad (15)$$

$$\beta^2(r) = (f - f_a)(3f_a - 2f)^{-1} \quad , \qquad (16)$$

$$ds^2 = \frac{1}{4}(3f_a - f)^2 (dx^0)^2 - f^{-2}\, dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \quad , \qquad (17)$$

where $f_a = f(a)$; and outside the cluster ($r > a$) we have

$$ds^2 = (1 - 2\omega^2 a^3/c^2 r)(dx^0)^2 - (1 - 2\omega^2 a^3/c^2 r)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 . \qquad (18)$$

Incidentally we verify that also the radial derivative of g_{00} is continuous at $r = a$.

In order to avoid negative values for $\rho(r)$ one finds from (15) that $3f_a - 2f \geq 0$; since the maximum value of $f(r)$ is $f(0) = 1$, one must have $f_a \geq 2/3$, or

$$\omega^2 a^2 \leq 5\, c^2/18 \quad , \qquad (19)$$

which coincides with the requirement $\beta^2(r) \geq 0$. Then all metric coefficients in (17) and (18) only assume regular and non-zero values.

The energy differential (Tolman[7])

$$dE = (-g)^{1/2}(2T_0^0 - T)\, dx^1\, dx^2\, dx^3 \quad , \quad g = \det g_{\mu\nu} \qquad (20)$$

integrated inside a spherical shell of radius r_1 and r_2 or our cluster gives the surprisingly simple result

$$E(r_1, r_2) = \omega^2 c^2 (r_2^3 - r_1^3)/G \quad ; \qquad (21)$$

for a given value of the radiis $\underline{a}$ of the cluster, the maximum possible energy of the cluster is then, from (19),

$$E_{max} = (5\ c^4/18\ G)a \ . \qquad (22)$$

4. INTERNAL GEODESICS

To study the motion of the particles belonging to the cluster we consider the geodesic equation

$$d\,u^\mu/ds + \{^{\ \mu}_{\nu\lambda}\}\, u^\nu\, u^\lambda = 0 \quad , \qquad (23)$$

which gives for an arbitrary static spherically symmetric metric(2)

$$du^0/ds + \nu_1\, u^0\, u^1 = 0 \quad , \qquad (24)$$

$$du^1/ds + e^{-\lambda}\left[\tfrac{1}{2}\nu_1 e^\nu (u^0)^2 + \tfrac{1}{2}\lambda_1 e^\lambda (u^1)^2 - r\,\sin^2\theta (u^3)^2 - r(u^2)^2\right] = 0 \qquad (25).$$

$$du^2/ds + 2u^1u^2/r - \sin\theta\,\cos\theta\,(u^3)^2 = 0 \ , \qquad (26)$$

$$du^3/ds + 2u^1u^3/r + 2\cot\theta\, u^2u^3 = 0 \quad ; \qquad (27)$$

these four equations (24) to (27) are not independent, however , since they obey the identity $u^\mu u_\mu = 1$, that is

$$e^\nu(u^0)^2 - e^\lambda(u^1)^2 - r^2(u^2)^2 - r^2\,\sin^2\theta\,(u^3)^2 = 1 \ . \qquad (28)$$

The first integral of (24) to (28) is then

$$u^0 = \alpha^2\, e^{-\nu} \quad , \qquad (29)$$

$$(u^1)^2 = (\alpha^4\, e^{-\nu} - 1 - \gamma^2/r^2)e^{-\lambda} \quad , \qquad (30)$$

$$(u^2)^2 = (\gamma^2 - \delta^2/\sin^2\theta)/r^4 \quad , \qquad (31)$$

$$u^3 = \delta/(r^2\,\sin^2\theta) \quad . \qquad (32)$$

where α^2, γ^2 and δ are the three constants of integration, related somehow to the three components of a given initial velocity these results are valid for arbitrary $\nu(r)$ and $\lambda(r)$.

We now impose that all particles of the cluster have the same energy; since all of them have also the same rest mass, it follows from (29) that α^2 is a constant common to all particles of the cluster. To obtain the value of this constant we remember the imposition (13) of the null velocity of any particle of the cluster on the boundary; then from (29) and (28) we obtain

$$\alpha^2 = f_a \quad . \tag{33}$$

In view of the difficulty in integrating (29) to (32) we consider here only the case of non-relativistic clusters, for which $\omega^2 a^2 << c^2$. Due to the spherical symmetry we only consider the motion on the plane of the equator; then with $\theta = \pi/2$ and $u^2 = 0$ in (31) we obtain $\gamma^2 = \delta^2$, and the equations (30) and (32) give, with $x^0 = ct$,

$$(dr/dt)^2 = \omega^2(a^2 - r^2) - c^2\delta^2/r^2 \ , \tag{34}$$

$$d\phi/dt = c\delta/r^2 \ . \tag{35}$$

The solutions of these equations are ellipses centered in the origin, with semi-axes given by

$$2\, r_{max}^2 = a^2 + (a^4 - 4c^2\delta^2/\omega^2)^{1/2} \ , \tag{36}$$

$$2\, r_{min}^2 = a^2 - (a^4 - 4c^2\delta^2/\omega^2)^{1/2} \ .$$

The velocity of the particle is given by

$$v^2 = \omega^2(a^2 - r^2) \ , \tag{37}$$

and can be seen not to depend on δ; also the period τ of a revolution does not depend on δ in this limit $\omega^2 a^2 << c^2$, and is

$$\tau = 2\pi/\omega \quad . \tag{38}$$

A radial geodesic corresponds to $\delta = 0$; a particle with radial motion starts from the boundary of the cluster ($r=a$) and moves harmonically through the center of the cluster; the singularity at the origin in (34) and (35) is only apparent, since all particles of the cluster which cross the origin have $\delta = 0$. With increasing values of δ^2 the orbit does not reach the boundary $r=a$ anymore, since $r_{max} < a$; nor it crosses the origin, since $r_{min} \neq 0$. The maximum value of δ^2 is $a^4\omega^2/4c^2$, and corresponds to a circle with radius $a/\sqrt{2}$; the velocity of the particle on this circle is a constant $v = \omega a/\sqrt{2}$, as can be seen from (37).

5. DISCUSSIONS

We have constructed a spherically symmetric cluster of particles of same rest mass, all with same total energy. In such cluster all particles must be always in motion, except the particles that perform radial motion; these are momentarily at rest when they reach the boundary of the sphere.

The density $\rho(r)$ increases from the center to the boundary of the sphere, where

$$\rho(a) = 3\omega^2/4\pi G \quad ; \tag{39}$$

this means that if all particles were momentarily stopped we would see a larger condensation in the outermost shells. In non-relativistic clusters this density ρ is constant, with the value

$$\rho = 3\omega^2/4\pi G \quad , \qquad \omega^2 a^2 \ll c^2 \quad , \tag{40}$$

We can evaluate the quantity $v(r)$ that is analogous to the classical three-velocity,

$$(v/c)^2 = (g_{0i}\, g_{0j} - g_{00}\, g_{ij})u^i u^j/(g_{0\mu}u^\mu)^2 \quad ; \qquad (41)$$

from (17) and (29) to (33) we obtain for our cluster

$$(v/c)^2 = \frac{1}{4}\,(5-f/f_a)(f/f_a-1) \qquad ; \qquad (42)$$

this quantity increases from the value zero on the boundary $(f=f_a)$ to a maximum value less than unity at the center of the cluster $(f=1;\ 0<f_a<1)$. In non-relativistic clusters we have

$$v^2 = \omega^2(a^2 - r^2) \quad , \quad \omega^2 a^2 << c^2 \quad . \qquad (43)$$

The quantity $\beta(r)$ is also a measure of the velocity field of the particles. This quantity also decreases from a maximum value at the origin to the value zero on the boundary; in the non-relativistic limit $(\omega^2 a^2 << c^2)$ the two quantities $\beta^2(r)$ and $v^2(r)/c^2$ coincide.

We verified that while $\rho(r)$ increases from the center to the boundary the quantity $\beta^2(r)$ decreases in a rate such that the combination $\rho(1+3\beta^2/2)$ is held constant. Also the combination $\rho(1+3\beta^2)e^{(\nu+\lambda)/2}$ which appears in the energy differential (20) is a constant; this constancy explains the simplicity of the expression (21) for the energy content of a shell.

Our cluster has been specified by only two constants, the radius a and the parameter ω. An easy interpretation for ω is obtained in non-relativistic clusters; we found (38) that in such clusters all particles perform elliptic paths with same period $\tau = 2\pi/\omega$.

The clusters of maximum energy ($\omega^2 a^2 = 5c^2/18$, $f_a = 2/3$) deserve a special consideration: their rest mass density $\rho(r)$ (15) increases from the value zero at the origin to the non-relativistic value $3\omega^2/4\pi G$ on the boundary. The quantity $\beta^2(r)$ (16) is zero on the boundary and diverges on the origin; however,this divergence does not affect the bounded values of the components of the energy momentum tensor (4). Incidentally we verify that T^μ_ν assumes the form

$$T^\mu_\nu(0) = (2\omega^2/\kappa c^2)\text{diag}(3,-1,-1,-1) \ , \quad f_a = 2/3 \qquad (44)$$

on the center of the cluster, a form which is characteristic of a null fluid. The quantity v(r) analogous to the classical three velocity (42) of the particles gives on the center of the cluster an acceptable value, near 2c/3. The amount of energy (22) present in these clusters in formidable; if we take for a the radius usualy attributed for elementary particles (1 fm) we get for $c^{-2}E_{max}$ a mass of the order of that of earth (10^3 Tg).

In our cluster all particles have the same mass; and all particles that cross a given point have the same modulus of velocity at that point. So when two particles of the cluster collide elastically, the emerging particles have also that same modulus of velocity. And since the velocities of the elastically colliding particles were randomly oriented, the velocities of the emerging particles have also random orientation, only constrained by momentum conservation, assuming absence of gravitational radiation and of spin effects. This attributes to the system an equilibrium configuration.

One of the interesting features of the solution is that the cluster of particles with randomly oriented velocities can also be described in terms of a fluid system. In a fluid description T^0_0 is still an energy density of the system, while the negative of $T^1_1 = T^2_2 = T^3_3$ is now the pressure of the fluid. In our particular system, where T^0_0 = const, we find that the cluster produces the same gravitational field as that of a static sphere of incompressible fluid. Our cluster thus represents a new physical interpretation of the well known interior Schwarzschild solution of Einstein's field equations.

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