NOTAS DE FÍSICA

VOLUME VII

Nº 6

ON THE PHYSICAL INTERPRETATION OF COMPLEX POLES OF THE S-MATRIX - II

by

H. M. Nussenzveig

CENTRO BRASILEIRO DE PESQUISAS FÍSICAS

Av. Wenceslau Braz, 71

RIO DE JANEIRO

1961

Notas de Física - Volume VII - Nº 6

ON THE PHYSICAL INTERPRETATION OF COMPLEX POLES OF THE S-MATRIX - II *

H. M. Nussenzveig

Centro Brasileiro de Pesquisas Físicas - Rio de Janeiro

(Received February 21, 1961)

Summary - The initial-value problem for a Schrödinger particle interacting with a partially transparent sphere (delta-function potential) is solved by an extension of the method described in Part I [1]. The general solution is expanded in terms of the propagators of transient modes. The relation between this expansion and the stationary-state expansion for an impenetrable sphere is discussed. Two special cases are considered: the decay of a wave packet initially confined within the sphere and the scattering of a wave packet by the sphere in the case of a sharp resonance. In the decay problem, the domain of validity of the exponential law and the deviations from this law are investigated. In the resonance scattering problem, the behaviour of the solution in the internal and external regions as a function of the width of the excitation is discussed. The concept of time delay at resonance is analysed.

---

* Submitted for publication in Il Nuovo Cimento.

1.- INTRODUCTION

In the first part of this paper [1] (hereafter referred to as I), the connection between the transient behaviour of a system and the poles of the associated S-matrix was investigated by means of some examples. It was shown in these examples that the general solution of the initial-value problem can be expanded in terms of propagators of transient modes, which are associated with the poles of the S-matrix. This eliminates the difficulties that occur in the usual treatment by the method of complex eigenvalues.

The examples treated in I had some restrictive features in common. They gave rise only to short-lived transient modes, whereas the case of long-lived modes is of greater physical importance. Furthermore, in each example, the S-matrix had only a finite number of poles for given angular momentum. In order to extend the treatment to the interaction of non-relativistic particles with an arbitrary potential of finite range, it is necessary to overcome this limitation, since the S-matrix then has an infinite number of poles for each value of the angular momentum [2].

In the present work, the treatment will be extended to an example which does not suffer from either of the above limitations. We shall consider the initial-value problem for a Schrödinger particle interacting with a partially transparent sphere. The transparency of the sphere can be adjusted to obtain transient modes of arbitrarily long lifetimes. In the limiting case of an impenetrable sphere, the modes go over into the stationary states of a particle in a spherical box with impenetrable walls.

By suitably specializing the initial conditions, one can describe either the decay of a wave packet initially confined within the sphere or the scattering of a wave packet by the sphere. A different treatment of the decay problem for the same model has been given by Petzold [3].

The transient-mode expansion for an infinite number of poles is obtained by means of a Mittag-Leffler expansion. Such an expansion has been employed in electric circuit theory [4], and it has been shown that it can be applied to an arbitrary potential of finite range in stationary scattering theory [5,6]. In this way, the treatment given in I can be extended to the interaction of Schrödinger particles with an arbitrary potential of finite range. It has been found more instructive, however, to consider in detail an explicit example such as the present one.

In section 2, the general solution of the initial-value problem for s-waves and its expansion in transient modes will be derived. The relation between the transient-mode expansion and the stationary-state expansion for an impenetrable sphere will be investigated. It will be shown that the transient-mode expansion behaves like the stationary-state expansion for large times, but it gives a better description of the early stages of propagation within the sphere.

In section 3, the solution will be specialized to the case of decay. The asymptotic form of the decay law and the domain of validity of the exponential law will be discussed.

In section 4, the resonance scattering problem will be considered. The behaviour of the solution in the internal and external regions and its dependence on the width of the excitation will be examined. The relation between the results and the concept of time delay at resonance will be discussed.

2.- GENERAL SOLUTION OF THE INITIAL-VALUE PROBLEM

a) The propagators.

We shall be concerned with the initial-value problem for the s-wave Schrödinger equation (in units $\hbar = m = 1$) associated with the central potential $V(r) = \frac{1}{2}(A/a)\,\delta(r-a)$ $(A>0)$, which describes a penetrable sphere of radius a. In the above units, A is a dimensionless parameter, which measures the opacity of the sphere.

We denote by indices 1 and 2 the interior and the exterior of the sphere, respectively, and introduce corresponding radial functions $\varphi_j(r,t) = r\psi_j(r,t)$ $(j = 1, 2)$. The problem is then equivalent to the solution of the free-particle Schrödinger equations

$$\left(\frac{\partial^2}{\partial r^2} + 2i\,\frac{\partial}{\partial t}\right)\varphi_j(r,t) = 0 \qquad (1)$$

subject to the boundary conditions

$$\varphi_1(0,t) = 0 , \qquad (2)$$

$$\varphi_1(a,t) = \varphi_2(a,t) , \qquad (3)$$

$$\frac{\partial \varphi_2}{\partial r}(a,t) - \frac{\partial \varphi_1}{\partial r}(a,t) = \frac{A}{a}\, \varphi_2(a,t), \tag{4}$$

and the initial conditions

$$\varphi_j(r,0) = f_j(r) \; . \tag{5}$$

We want to express the general solution in terms of propagators:

$$\varphi_j(r,t) = \int_0^a G_{j1}(r,\rho,t)\, f_1(\rho)\, d\rho \; + \int_a^{\infty} G_{j2}(r,\rho,t)\, f_2(\rho)\, d\rho \; . \tag{6}$$

For this purpose, in accordance with the method employed in I, we write down the general solutions of (1)

$$\varphi_j(r,t) = \int_{-\infty}^{+\infty} U(\rho,t)\, \varkappa_j(r+\rho)\, d\rho \; , \tag{7}$$

where

$$U(r,t) = \exp(-i\pi/4)(2\pi t)^{-\frac{1}{2}} \exp(ir^2/2t). \tag{8}$$

The initial conditions give: $\varkappa_1(\rho) = f_1(\rho)$ $(0 \leq \rho \leq a)$; $\varkappa_2(\rho) = f_2(\rho)(\rho \geqslant a)$, and (2) implies $\varkappa_1(-\rho) = -\varkappa_1(\rho)$, so that the unknown functions in (7) are $\varkappa_1(a+\rho)$ and $\varkappa_2(a-\rho)$ for $\rho > 0$. They are determined by conditions (3) and (4), which can be solved by the Laplace transformation method (cf. I).

Let $X_1(p)$, $X_2(p)$, $F_1(p)$ and $F_2(p)$ denote the Laplace transforms of $\varkappa_1(a+\rho)$, $\varkappa_2(a-\rho)$, $H(a-\rho)$ $f_1(a-\rho)$ and $f_2(a+\rho)$, respectively, where $H(t)$ is Heaviside's step function. The Laplace transforms of (3) and (4) become

$$[1 - \exp(-2ap)]X_1(p) - X_2(p) = - [F_1(p) - \exp(-2ap)F_1(-p)] + F_2(p),$$

$$p[1 + \exp(-2ap)]X_1(p) + \left(p + \frac{A}{a}\right) X_2(p) = p[F_1(p) - \exp(-2ap)F_1(-p)] + \left(p - \frac{A}{a}\right) F_2(p).$$

Solving these equations for X_1 and X_2, applying the inverse Laplace transformation and substituting the results in (7), we are led to the following expressions for the propagators defined in (6):

$$G_{11}(r,\rho,t) = G_1(x,t) - G_1(y,t), \tag{9}$$

where

$$x = |r - \rho|, \quad y = r + \rho \ , \tag{10}$$

$$G_1(x,t) = U(x,t) + \int_0^\infty [U(2a+x+\xi,t) + U(2a-x+\xi,t)] R_{11}(\xi) d\xi \ ; \tag{11}$$

$$G_{12}(r,\rho,t) = G(x,t) - G(y,t) = G_{21}(\rho,r,t), \tag{12}$$

where

$$G(x,t) = \int_0^\infty U(x+\xi,t)\, R_{12}(\xi)\, d\xi \ ; \tag{13}$$

$$G_{22}(r,\rho,t) = U(x,t) + \int_0^\infty U(y-2a+\xi,t) R_{22}(\xi)\, d\xi \ . \tag{14}$$

The functions $R_{jk}(\xi)$ are given by ($\mathcal{L}^{-1}$ denotes the inverse Laplace transform)

$$R_{11}(\xi) = A\mathcal{L}^{-1}[Q(2ap)] \ , \tag{15}$$

$$R_{12}(\xi) = 2a\mathcal{L}^{-1}[pQ(2ap)], \tag{16}$$

$$R_{22}(\xi) = \mathcal{L}^{-1}\left\{[(A-2ap)\exp(-2ap) - A]\ Q(2ap)\right\} = -\mathcal{L}^{-1}[\exp(-2ap)S(ip)], \tag{17}$$

where

$$Q(z) = [A + z - A\exp(-z)]^{-1} , \quad (18)$$

and

$$S(k) = - Q(-2ika)/Q(2ika) \quad (19)$$

is the S-function (diagonal element of the S-matrix) for s-waves.

b) The poles of the S-matrix.

The poles of the functions of p appearing in (15) to (17) are related by the transformation $p = -ik$ with the poles of S(k) in the k-plane. The latter are roots of the equation

$$A - 2i\beta - A\exp(2i\beta) = 0 , \quad (20)$$

where $\beta = ka$. Methods for locating the roots of complex transcendental equations of this type have been given elsewhere [7]. Here we shall be interested only in some limiting cases.

For each value of A, there exists an infinite number of poles, all of which are simple and are located in the lower half of the k-plane. The pole distribution is symmetrical with respect to the imaginary axis, so that it suffices to consider the lower right quadrant. There is one pole β_n in each strip $(n-1)\pi < \mathrm{Re}\,\beta < n\pi$ $(n = 1, 2, 3, \ldots)$. When A approaches zero (free particles), β_n approaches $(n-\frac{1}{2})\pi - i\infty$. When A increases, β_n moves upwards and away from the imaginary axis, and $\beta_n \rightarrow n\pi$ when $A \longrightarrow \infty$. In this limiting case, therefore, we get the eigenvalues associated with a particle in a spherical box with impenetrable walls, as ought to be expected.

For given A, the asymptotic behaviour of the pole distribution for large n is given by [7]

$$\beta_n \approx (n - \tfrac{1}{4})\pi - \tfrac{1}{2}\, i \log[(2n-\tfrac{1}{2})\pi/A] \qquad (n\pi \gg A). \quad (21)$$

We shall be interested mainly in the case $A \gg 1$, in which the lowest-order poles are very close to their limiting values on the real axis. They are given by

$$\beta_n = n\pi \left[1 - (A+1)^{-1}\right] - i(n\pi/A)^2 + \underline{O}\left[(n\pi/A)^3\right] \quad (n\pi \ll A). \quad (22)$$

The "lifetime" of the transient mode associated with (22) is

$$\tau_n = \tfrac{1}{2}(n\pi)^{-3}A^2a^2. \quad (23)$$

These long-lived transient modes can be interpreted in terms of multiple-reflection interference effects similar to those which occur in the Fabry-Perot interferometer. In particular, we have: $\tau_n = 2a(v_n \Theta_n)^{-1}$, where v_n is the velocity within the sphere and Θ_n is the transmissivity of the potential.

c) The transient-mode expansion.

In order to derive the transient-mode expansion of the propagators, we need the Mittag-Leffler expansion of the expressions within brackets in (15) to (17). The Mittag-Leffler expansions associated with R_{11} and R_{12} are derived in the Appendix. If we define

$$b_o = \tfrac{1}{2}(A/a)(A+1)^{-1};\ b_n = \tfrac{1}{2}(A/a)(A+1-2i\,\beta_n)^{-1}\ (n = \pm 1, \pm 2, \ldots); \quad (24)$$

$$c_n = -2i\,\beta_n b_n/A = -i(\beta_n/a)(A+1-2i\beta_n)^{-1} \quad (n = \pm 1, \pm 2, \ldots), \tag{25}$$

where $\beta_{-n} = -\beta_n^*$, it follows from (15), (16), (A7) and (A8) that

$$R_{11}(\xi) = b_o + \sum b_n \exp(-ik_n\xi), \tag{26}$$

$$R_{12}(\xi) = \tfrac{1}{2}\,\delta(\xi) + \sum c_n \exp(-ik_n\xi), \tag{27}$$

where the terms n and -n must always be taken together in the summations.

Instead of deriving a similar expansion for R_{22}, it is more convenient to express it in terms of R_{12}. It follows from (16) and (17) that

$$R_{22}(\xi) = -\delta(\xi) + R_{12}(\xi) - H(\xi-2a)R_{12}(\xi-2a). \tag{28}$$

Substituting (26) to (28) into (11), (13) and (14), we find

$$G_1(x,t) = U(x,t) + b_o[M(2a+x,0,t) + M(2a-x,0,t)] + + \sum b_n[M(2a+x,k_n,t) + M(2a-x,k_n,t)], \tag{29}$$

$$G(x,t) = \tfrac{1}{2}U(x,t) + \sum c_n M(x,k_n,t), \tag{30}$$

$$G_{22}(r,\rho,t) = U(x,t) - \tfrac{1}{2}[U(y-2a,t) + U(y,t)] + + \sum c_n[M(y-2a,k_n,t) - M(y,k_n,t)], \tag{31}$$

where

$$M(x,k,t) = \int_{-\infty}^{+\infty} H(\xi - x) \exp[ik(x-\xi)] U(\xi,t)\, d\xi \quad (32)$$

is the Schrödinger propagator of a transient mode, which was introduced in I. The transient-mode expansion of the propagators is contained in equations (9), (12) and (29) to (31).

The physical interpretation of M(x,k,t) was discussed in I. It was also shown there that, if A and B denote the regions of the complex plane above and below the second bisessar, respectively, and if (33)

$$w = (2t)^{-\frac{1}{2}}(x-kt), \quad (33)$$

the following asymptotic expansions are valid:

$$M(x,k,t) = M_A(x,k,t) = it(x-kt)^{-1} U(x,t) [1 - \tfrac{1}{2}iw^{-2} + \ldots + + (-\tfrac{1}{2}i)^n (2n-1)!!w^{-2n} + R_n(w)] \quad \text{if } w \in A, \quad (34)$$

$$M(x,k,t) = M_B(x,k,t) = \exp[i(kx-Et)] + M_A(x,k,t) \quad \text{if } w \in B, \quad (35)$$

where

$$(2n-1)!! = 1.3.5 \ldots (2n-1), \quad E = \tfrac{1}{2} k^2, \quad \text{and}$$

$$|R_n(w)| \leqslant \pi^{\frac{1}{2}} 2^{-n-1}(2n+1)!!|w|^{-2n-1} \quad (n = 0, 1, 2, \ldots). \quad (36)$$

d) The limiting case of an impenetrable sphere.

Before applying the general solution to the special cases of decay and of resonance scattering, it is instructive to consider the limiting case of an impenetrable sphere. In this limit ($A \to \infty$), (15) becomes

$$R_{11}(\xi) = \mathcal{L}^{-1}\left\{\left[1 - \exp(-2ap)\right]^{-1}\right\} = \tfrac{1}{2}\,\delta(\xi) + \tfrac{1}{2}a^{-1}\left[1 + \Sigma \exp(-ik_n\xi)\right], \tag{37}$$

where $k_n = n\pi/a$. Substituting this in (11), we get

$$G_1(x,t) = U(x,t) + \tfrac{1}{2}\left[U(2a+x,t) + U(2a-x,t)\right] + \tfrac{1}{2}\,a^{-1}\left[M(2a+x,0,t) + M(2a-x,0,t)\right] + \tfrac{1}{2}\,a^{-1}\,\Sigma\left[M(2a+x,k_n,t) + M(2a-x,k_n,t)\right]. \tag{38}$$

It is also possible to expand G_{11} in terms of the stationary states of the particle within the impenetrable sphere:

$$G_{11}(r,\rho,t) = 2a^{-1} \sum_{n=1}^{\infty} \sin(k_n r)\,\sin(k_n \rho)\,\exp(-\tfrac{1}{2}ik_n^{\,2}t).$$

This can be rewritten in the form (9), with *

$$G_1(x,t) = \tfrac{1}{2}\,a^{-1} + a^{-1} \sum_{n=1}^{\infty} \cos(k_n x)\,\exp(-\tfrac{1}{2}ik_n^{\,2}t) = \theta\left(\frac{x}{2a}\,\middle|\,-\frac{\pi t}{2a^2}\right), \tag{39}$$

where $\theta\,(x|\tau)$ is the Jacobi theta function [8]. If we apply Jacobi's transformation formula [8]

---

* The constant term in (39) is arbitrary; the choice was dictated by reasons of convenience.

$$\theta(x|\tau) = (\tau/i)^{-\frac{1}{2}} \exp(-\pi i x^2/\tau)\ \theta\left(\frac{x}{\tau}\,\middle|\,-\frac{1}{\tau}\right) , \qquad (40)$$

equation (39) becomes

$$G_1(x,t) = U(x,t) + \sum_{n=1}^{\infty} \left[U(2na-x,t) + U(2na+x,t)\right] \quad (41)$$

Equations (38), (39) and (41) give three different representations of the same propagator. In the theory of heat conduction, the transformation (40) is employed to transform a series which converges rapidly for large τ (large times) into a series which converges rapidly for small τ (small times) [8]. A similar result is valid here. The characteristic time interval is $T = 4a^2/\pi$, the period of the ground state.

For $t \gg T$, the contributions from large values of n in the stationary-state expansion (39) oscillate very rapidly, and tend to cancel one another by destructive interference. Thus, the main contribution arises from the lowest values of n. On the other hand, for $t \ll T$, a large number of terms will contribute, so that the convergence becomes very slow.

The expansion (41), when replaced in (9), corresponds to the result which is found by applying the method of images: there is an infinite series of images, corresponding to the successive reflections at $r = 0$ and $r = a$. This series is rapidly convergent for $t \ll T$, on account of the rapid oscillation of the terms with large n. For very short times, the propagator does not differ very much from the free-particle propagator, as ought to be expected. However, for $t \gg T$, the series (41) does not converge well.

It can easily be shown, with the help of (34) and (35), that the transient-mode expansion (38) behaves like the stationary--state expansion for $t \gg T$, whereas, for $t \ll T$, it is dominated by the free-particle propagator. Thus, the transient-mode expansion combines the advantages of the stationary-state expansion with those of the expansion by the method of images: it converges well both for small and for large times. Furthermore, unlike the others, it can still be applied when A has a finite value.

Since each term in the transient-mode expansion approaches the corresponding term in the stationary-state expansion for large t, it is clear that the expansion coefficient will give the probability amplitude associated with the level in question. Thus, in the limiting case of an impenetrable sphere, the transient-mode expansion is closely related with the stationary-state expansion.

3.- THE DECAY PROBLEM

a) Behaviour of the propagators.

In order to describe the decay of a particle which is initially confined within the sphere, it suffices to specialize the initial conditions (5), by requiring that $f_2(\rho) = 0$. The behaviour of the solution in regions 1 and 2 is then determined by the propagators G_{11} and G_{21}, respectively.

Let us consider the behaviour of G_{11} as a function of time. We shall be interested only in times much larger than the "period" associated with the lowest transient mode, i.e. $t \gg a^2$. Under these conditions, $|w_n| \gg 1$ for all the functions $M(2a \pm x, k_n, t)$

in (29), where w_n is the parameter defined in (33). We can therefore employ the expansions (34) and (35). Denoting by Σ_A the sum over all poles located above the second bisector, we get

$$G_1(x,t) = U(x,t) + \Sigma_A\, b_n \exp\left[i(k_n x - E_n t)\right] + \Big\{ b_o M(2a+x,0,t) + $$
$$+ itU(2a+x,t)\,\Sigma b_n\left[(2a+x-k_n t)^{-1} - it(2a+x-k_n t)^{-3} - 3t^2(2a+x-k_n t)^{-5}\right] +$$
$$+ (x \leftrightarrow -x)\Big\} + a^{-1}\underline{O}\left[(a^2/t)^3\right], \qquad (42)$$

where the last term in the expression within curly brackets denotes antisymmetrization of that expression with respect to x.

The sum of each of the series that appear within the curly brackets can be explicitly computed by employing the Mittag--Leffler expansions given in the Appendix. For instance, according to (24) and (A8),

$$\Sigma\, b_n(\xi - k_n t)^{-1} = -\tfrac{1}{2}\, A\left\{ \left[a(A+1)\xi\right]^{-1} + 2it^{-1}Q(-2ia\xi/t)\right\},$$

and the sums of the remaining series follow by repeated differentiation with respect to ξ. The results can then be expanded in powers of $a\xi/t$. Making similar expansions for the other terms of (42), and substituting the results in (9), we finally get, for $t \gg a^2$,

$$G_{11}(r,\rho,t) = 2a^{-1}\,\Sigma_A\left[1 - (A+1-2i\beta_n)^{-1}\right]\sin(k_n\rho)\,\sin(k_n r)\exp(-iE_n t) -$$
$$- (2/\pi)^{\frac{1}{2}} \exp(i\pi/4)(A+1)^{-2}\,\rho r t^{-3/2} + 2(2/\pi)^{\frac{1}{2}} \exp(-i\pi/4)\, A(A+1)^{-3}\Big\{\Big[A - 2 -$$
$$- \tfrac{1}{3}(1+\tfrac{1}{2}i)(2A-1)A(A+1)^{-1}\Big]a^2 + \tfrac{1}{8}A^{-1}(A+1)(r^2+\rho^2)\Big\}\rho r t^{-5/2} + a^{-1}\underline{O}\left[(a^2/t)^3\right].$$

(43)

Each term of the series appearing in (43) decays exponentially (the lifetime of the nth term is given by (23)). For $t \to \infty$, G_{11} is dominated by the term in $t^{-3/2}$. The same term has been found in reference 3. The asymptotic decay law is therefore in general * an inverse third power law. This agrees with the discussion given in I.

Equation (43) enables us to obtain the decay law corresponding to a given excitation, for all times $t \gg a^2$. A specific example will be given below.

The propagator G_{21} may be treated in a similar way. Each transient mode $k_n = k_n' - i\kappa_n$ ($k_n \in A$) in (30) gives rise to an exponential wave train with a diffuse wave front at $r \approx (k_n' - \kappa_n)t$, ahead of which the amplitude decreases rapidly (cf. the general discussion in I). Behind the wavefront associated with the lowest mode, all the wave trains overlap. For points far behind this wave front ($r \ll k_1' t$) and times $t \gg a^2$, we find

$$G_{21}(r,\rho,t) = -2i\,\Sigma_A\, c_n \sin(k_n\rho)\exp\left[i(k_n r - E_n t)\right] + (A+1)^{-1}\Big\{1 + \\ + \tfrac{1}{3}A(A+1)^{-1}\left[3A(A+1)^{-1} - 2\right](a^2/t)\Big\}\left[U(r-\rho,t) - U(r+\rho,t)\right] - \\ - iA(A+1)^{-2}(a/t)\left[(r-\rho)U(r-\rho,t) - (r+\rho)U(r+\rho,t)\right] + (aA)^{-1}\underline{O}\left[(a^2/t)^2\right]. \quad (44)$$

In the derivation of this result, we have made use of (30), (34), (35) and the Mittag-Leffler expansion (A7).

---

* An exception occurs if $f_1(\rho)$ is chosen orthogonal to ρ over the interval $(0,a)$.

The terms of the series that appear in (43) and (44) correspond to the "complex energy wave functions" which are employed in the method of complex eigenvalues. Each term $\sin(k_n r)$ in (43) corresponds to a term $T_n \exp(ik_n r)$ in (44), where T_n is the "transmission coefficient", given by

$$T_n = - \beta_n \exp(-2i\beta_n)/A . \qquad (45)$$

For $t \gg r^2$, (44) becomes

$$G_{21}(r,\rho,t) = -2i \, \Sigma_A \, c_n \sin(k_n \rho) \exp\left[i(k_n r - E_n t)\right] -$$

$$- (2/\pi)^{\frac{1}{2}} \exp(i\pi/4)(A+1)^{-1}\left[r - A(A+1)^{-1}a\right] \rho \, t^{-3/2} + \ldots \qquad (46)$$

and the last term predominates for $t \rightarrow \infty$. This term has also been given in reference 3.

b) The decay law.

In order to investigate the domain of validity of the exponential decay law under the most favourable conditions, let us consider the decay of a long-lived mode, e.g. the lowest mode for $A \gg 1$. To concentrate the excitation as much as possible on this mode, we shall choose the initial (unnormalized) wave function * $f_1(\rho) = \sin(k_1 \rho)$, which corresponds to a "complex reso-

---

* The choice $f_1(\rho) = \sin(\pi\rho/a)$ would lead to amplitudes of the order A^{-1} (instead of A^{-2}) for the higher modes.

nance".

Employing (43) and (22), this leads to

$$\varphi_1(r,t) = \left[1 - i\pi A^{-2} + \underline{O}(A^{-3})\right]\sin(k_1 r)\exp(-iE_1 t) +$$

$$+ \pi^{-3/2}A^{-1}\left[2^{\frac{1}{2}}\exp(i\pi/4)A^{-1} + \tfrac{4}{3}i(a^2/t)\right]\left[1 + \underline{O}(A^{-1})\right]ra^2t^{-3/2} +$$

$$+ \Delta\varphi_1(r,t) \quad (\text{for } t \gg a^2), \qquad (47)$$

where

$$\Delta\varphi_1(r,t) = \sum_{n=2}^{\infty}\left[1-(A+1-2i\beta_n)^{-1}\right]\left[(\beta_n-\beta_1)^{-1}\sin(\beta_n-\beta_1) -\right.$$

$$\left.- (\beta_n+\beta_1)^{-1}\sin(\beta_n+\beta_1)\right]\sin(k_n r)\exp(-iE_n t) + \underline{O}\left[(a^2/t)^3\right]. \qquad (48)$$

It follows from (21) and (22) that only modes with $n\pi \ll A$ give an appreciable contribution to (48), so that we may write

$$\Delta\varphi_1(r,t) \approx 2i\pi A^{-2}\sum_{n=2}^{\bar{n}}(-)^n(n/n+1)\sin(n\pi r/a)\exp\left[-\tfrac{1}{2}i(n\pi/a)^2t - \tfrac{1}{2}n^3t/\tau_1\right]$$

$$+ \underline{O}\left[(a^2/t)^3\right], \qquad (49)$$

where $\bar{n}$ is such that the contribution from $n \gtrsim \bar{n}$ is negligible ($\bar{n}\pi \ll A$) and τ_1 is the lifetime of the mode $n = 1$, which is given by (23).

According to (47) and (49), there are only very small corrections to the exponential law for $a^2 \ll t \lesssim \tau_1$. For $t \gtrsim \tau_1$, the main correction arises from the term in $t^{-3/2}$, so that

$$\varphi_1(r,t) \approx \sin([redacted]/a)\exp\left[-\tfrac{1}{2}i(\pi/a)^2t - \tfrac{1}{2}t/\tau_1\right] -$$

$$- 2^{\frac{1}{2}}\pi^{-3/2}\exp(i\pi/4)A^{-2}a^{2}rt^{-3/2} \quad (t \gtrsim \tau_1). \qquad (50)$$

The probability that the particle is still confined within the sphere after a time t gives the decay law:

$$P(t) = \left[\int_0^a |f_1(r)|^2 dr\right]^{-1} \int_0^a |\varphi_1(r,t)|^2 \, dr. \quad (51)$$

According to (50), we get

$$P(t) \cong \exp(-t/\tau_1) - (2/\pi)^{5/2}A^{-2}\cos(\tfrac{1}{2}\pi^2 a^{-2}t + \tfrac{\pi}{4})\exp(-\tfrac{1}{2}t/\tau_1)(a^2/t)^{3/2}$$

$$+ \tfrac{4}{3}\pi^{-3}A^{-4}(a^2/t)^3 \qquad (t \gtrsim \tau_1). \quad (52)$$

The deviations from the exponential law will become important when the first term of (52) becomes of the same order of magnitude as the last one, i.e. $\exp(t/\tau_1) \approx (t/\tau_1)^3 A^{10}$, which gives

$$t/\tau_1 \approx 10 \log A + 3 \log\log A. \qquad (53)$$

Thus, the t^{-3} law predominates only after the probability that the particle still has not decayed becomes smaller than A^{-10}. This is extremely small for all values of A which correspond to even moderately long-lived modes. In this case, therefore, it seems to be extremely difficult to detect deviations from the exponential law, in the present model. A similar conclusion has been reached by Petzold [3].

According to Krylov and Fock [9], the decay law is entirely determined by the energy spectrum of the initial state. This follows from their definition of the decay law as the probability of finding the system in the initial state at time t, namely,

$|(\Psi(\underset{\sim}{r},t), \Psi(\underset{\sim}{r},0))|^2$, where $\Psi(\underset{\sim}{r},t)$ is the normalized wave function. It must be pointed out, however, that this definition, although it would lead to essentially the same results as (51) in the case of (50), becomes inadequate in other cases. For instance, it would imply a definite "decay law" even for a wave packet confined within an impenetrable sphere. The definition (51) does not suffer from this disadvantage. In the case of non-zero angular momentum, however, due to the presence of the centrifugal barrier, the definition of the decay law requires further consideration.

4.- THE RESONANCE SCATTERING PROBLEM

a) Solution in the internal region.

The general solution derived in section 2 can also be applied to describe the scattering of an arbitrary initial wave packet by the sphere. In this case, $f_1(\rho) = 0$ in (6), so that the behaviour of the solution in regions 1 and 2 is determined by the propagators G_{12} and G_{22}, respectively.

We shall consider only the case of a sharp resonance. In order to investigate the effect of the excitation conditions, we shall take an initial wave packet depending on two variable parameters, which correspond to its mean momentum and its width in momentum space. A convenient choice for this purpose is *

* For the sake of simplicity, the initial instant is taken to coincide with the time at which the incident wave front impinges on the surface. It would amount to the same to take the initial position of the wave front at any reasonable distance from the sphere (for not too large distances, the spread of the wave packet on its way to the surface can be neglected).

$$f_2(\rho) = \exp\left[-ik_o(\rho-a)\right], \qquad (54)$$

where $k_o = k_o' - i\kappa_o$ is a complex parameter with negative imaginary part ($\kappa_o > 0$).

For definiteness, we shall associate the resonance with the lowest transient mode. Thus, it will be assumed throughout that $\left|(k_o-k_1)/k_1\right| \ll 1$ and $A \gg 1$. For $k_o = k_1$, we have a "complex resonance".

Substituting (54) in (6), and taking into account (12) and (30), we find

$$\varphi_1(r,t) = F(r,t) - F(-r,t) , \qquad (55)$$

where

$$F(r,t) = \tfrac{1}{2}\int_a^\infty U(\rho-r,t)\exp\left[-ik_o(\rho-a)\right]d\rho +$$

$$+ \Sigma\, c_n \int_a^\infty M(\rho-r,k_n,t)\exp\left[-ik_o(\rho-a)\right]d\rho . \qquad (56)$$

It follows from (32) that

$$\int_a^\infty U(\rho-r,t)\exp\left[-ik_o(\rho-a)\right]d\rho = M(a-r,k_o,t), \qquad (57)$$

$$\int_a^\infty M(\rho-r,k_n,t)\exp\left[-ik_o(\rho-a)\right]d\rho = \qquad (58)$$

$$= i(k_o-k_n)^{-1}\left[M(a-r,k_o,t) - M(a-r,k_n,t)\right] \quad \text{if} \quad k_o \neq k_n ,$$

$$= itU(a-r,t) - (a-r-k_nt)\, M(a-r,k_n,t) \quad \text{if} \quad k_o = k_n .$$

The last term of (58) appears only at the complex resonance ($k_o = k_1$). Excluding this case, for the moment, (56) becomes

$$F(r,t) = -2i\beta_o Q(-2i\beta_o)M(a-r,k_o,t) - i\sum c_n(k_o-k_n)^{-1} M(a-r,k_n,t), \quad (59)$$

where $\beta_o = k_o a$, and (A7) has been employed to compute the coefficient of the first term.

Let us investigate the behaviour of (55) for times much larger than the "period" of the lowest mode ($t \gg a^2$). Under these circumstances, the asymptotic expansions (34) and (35) can be employed in (59). The resulting series can be summed with the help of the Appendix, and the results can be expanded in powers of a^2/t (cf. the similar treatment of G_{11} in section 3(a)). Grouping together the first term of (59) and the resonance term ($n = 1$) in the series, and expanding them in powers of k_o-k_1, we finally get

$$\varphi_1(r,t) = -2i\beta_1\exp(i\,\beta_1)(A+1-2i\beta_1)^{-1}\Big\{\sin(k_1r)g(k_o,t) +$$

$$+ \beta_1^{-1}\left[k_1r\cos(k_1r) + \sin(k_1r)\right]\exp(-iE_ot)\Big\} + 2i\,\Sigma_A'\,\beta_n\exp(i\beta_n)\cdot$$

$$\cdot\,(\beta_o-\beta_n)^{-1}\,(A+1-2i\beta_n)^{-1}\sin(k_nr)\exp(-iE_nt) + (2/\pi)^{\frac{1}{2}}\exp(i\pi/4)\cdot$$

$$\beta_1^{-2}(1+i\beta_1)(A+1)^{-1}\,a^2rt^{-3/2} + \ldots \quad (t\gg a^2), \qquad (60)$$

where the accent in the summation sign indicates the exclusion of $n = 1$, and

$$g(k_o,t) = (\beta_o-\beta_1)^{-1}\left[\exp(-iE_ot) - \exp(-iE_1t)\right] \quad (k_o\neq k_1). \qquad (61)$$

At the complex resonance, according to (58), (60) is still valid, with

$$g(k_1,t) = \lim_{k_o\to k_1} g(k_o,t) = -i(k_1t/a)\exp(-iE_1t)\;. \quad (62)$$

According to (60), the amplitudes of the non-resonant modes are at most of the order of the corresponding transmission coefficients (45), and decrease as they get further away from the resonant mode, on account of the factor $(\beta_o-\beta_n)^{-1}$. The effect of the resonance is contained in the "amplitude gain factor" $|g(k_o,t)|$, which measures the increase in the amplitude of the mode $n = 1$ due to the resonance.

At the complex resonance, (62) gives

$$|g(k_1,t)| \approx \tfrac{1}{2}(A/\pi)^2(t/\tau_1)\exp(-\tfrac{1}{2}t/\tau_1) \; . \qquad (63)$$

Thus, the amplitude increases linearly with the time, to begin with, and attains its maximum value $|g|_{max} = e^{-1}(A/\pi)^2$ after a rise time $t = 2\tau_1$; thereafter, it decreases, with a decay time which is also of the order of $2\tau_1$.

Outside of the complex resonance, we have to consider the effect of the displacement of the center of the exciting line (54) and the effect of its width variation. These two effects can be considered separately.

If only the center of the exciting line is shifted, its width remaining the same, we find, as ought to be expected, that this gives rise to "beats" with the difference frequency, and the maximum gain decreases in proportion with the distance from exact resonance.

It is more interesting to consider the effect of the width variation. Let the center of the exciting line be kept at its resonance value, while the width is changed. Then, (61) gives

$$|g(k_o,t)| \approx (A/\pi)^2 \left(1 - \frac{\tau_1}{\tau_o}\right)^{-1} \left[\exp(-\tfrac{1}{2}t/\tau_o) - \exp(-\tfrac{1}{2}t/\tau_1)\right], \quad (64)$$

where τ_o is the "lifetime" associated with the exciting line.

It follows from (64) that, in the case of excitation by a narrow line ($\tau_o \gg \tau_1$), the rise time and the maximum gain are of the same order as at the complex resonance, whereas the decay time is of the order of $2\tau_o$. On the other hand, for excitation by a broad line ($\tau_o \ll \tau_1$), the decay time is of the same order as at the complex resonance, but both the rise time and the maximum gain are reduced by a factor of the order of τ_o/τ_1.

The above results can be summed up as follows: <u>the rise (decay) time is the shorter (longer) of τ_o and τ_1 ; the maximum amplitude gain is of the order of $(A/\pi)^2$ times the fraction of the width of the excitation that falls upon the width of the resonant mode</u>.

b) Solution in the external region.

It follows from (54), (6), (31), (56), (57) and (59) that

$$\varphi_2(r,t) = \varphi_{2H}(r,t) + \varphi_{2R}(r,t) , \quad (65)$$

where

$$\varphi_{2H}(r,t) = M(a-r,k_o,t) - M(r-a,k_o,t), \quad (66)$$

$$\varphi_{2R}(r,t) = F(2a-r,t) - F(-r,t) = -2i\beta_o Q(-2i\beta_o)[M(r-a,k_o,t) -$$

$$- M(r+a,k_o,t)] - i\sum c_n(k_o-k_n)^{-1}[M(r-a,k_n,t) - M(r+a,k_n,t)] . \quad (67)$$

This corresponds to a decomposition of φ_2 into a "hard sphere term" φ_{2H}, which is identical to the solution for an impenetrable sphere, and a "resonance term" φ_{2R}.

We shall consider only the case of an excitation centered at the resonance momentum but of variable width, i.e. $k_o = k_1' - i\kappa_o$, where $k_1 = k_1' - i\kappa_1$. We want to find the shape of the scattered wave at a time t larger than the lifetime $\tau_1 = \frac{1}{2}(k_1' \kappa_1)^{-1}$, but still much smaller than the "spreading time" $t_s = \mathrm{Min}(\kappa_1^{-2}, \kappa_o^{-2})$, so that the effect of the spreading of the wave packet will be small. According to (23), this means that *

$$A^2 \lesssim t/a^2 \ll A^4. \tag{68}$$

The wave front associated with the scattered wave is located at $r-a \approx k_1' t$. Around this wave front, according to the general discussion given in I, there is a domain of width $(2t)^{\frac{1}{2}}$, where "diffraction in time" effects play an important role. We shall consider only the behaviour of the wave function far behind this region, so that

$$\zeta = k_1' t - r \gg t^{\frac{1}{2}} . \tag{69}$$

This allows us to employ the asymptotic expansions (34) and (35), restricting (34) to its first term. Thus, (66)

* Notice that $t \ll \kappa_o^{-2}$ also implies a restriction on the width of the incident wave packet, namely, $\kappa_o a \ll (\kappa_1 a)^{\frac{1}{2}} \approx \pi/A$. We restrict our consideration to wave packets satisfying this condition.

becomes

$$\varphi_{2H}(r,t) = -2i\ \sin[k_o(r-a)]\exp(-iE_ot) - 2it(r-a)[(r-a)^2-k_o^2t^2]^{-1}U(r-a,t) + \dots, \qquad (70)$$

and (67), with the help of (A7) and expansions in powers of k_o-k_1, gives

$$\varphi_{2R}(r,t) = f(r,t) + 2i\pi A^{-1}\exp\left[i(k_or-E_ot)\right] + 2i\Sigma_A'\beta_n\sin\beta_n(\beta_o-\beta_n)^{-1}\cdot$$

$$\cdot(A+1-2i\beta_n)^{-1}\exp\left[i(k_nr-E_nt)\right] + 2a(r-a)(r-a-k_ot)^{-1}Q\left[-2ia(r-a)/t\right]\cdot$$

$$\cdot U(r-a,t) - 2a(r+a)(r+a-k_ot)^{-1}\ Q\left[-2ia(r+a)/t\right]\ U(r+a,t) + \dots, \qquad (71)$$

where the accent in the summation sign indicates the exclusion of $n = 1$, and

$$f(r,t) = 2\kappa_1(\kappa_o-\kappa_1)^{-1}\left[\exp(-\kappa_o\zeta) - \exp(-\kappa_1\zeta)\right]\exp\left[i(k_1'r-\tfrac{1}{2}k_1'^2t)\right] .(72)$$

According to (45), $\sin\beta_n = T_n\exp(i\ \beta_n)$, so that the amplitude of each non-resonant mode in (71) differs from the corresponding amplitude in (60) only by the transmission factor, and is of order A^{-2} for $n\pi << A$. Under the assumptions (68) and (69), it can be shown that the dominant term in (71) is $f(r,t)$ (the remaining terms give only small corrections).

Since $f(r,t)$ is the transmitted wave corresponding to the first term of (60), its spatial behaviour is just the transmitted counterpart of the time behaviour within the sphere. The amplitude of the resonance term at a distance ζ behind the wave front corresponds to the amplitude of the resonant mode within the sphere at a time $t = \zeta/k_1'$.

The total outgoing wave, under the above conditions, is given by

$$\varphi_{2,out}(r,t) \approx \left\{ 2\kappa_1(\kappa_o-\kappa_1)^{-1}\left[\exp(-\kappa_o\zeta) - \exp(-\kappa_1\zeta)\right] + \exp(-\kappa_o\zeta) \right\}$$

$$\cdot \exp\left[i(k_1'r-\tfrac{1}{2}k_1'^2t)\right] , \qquad (73)$$

where the last term in the curly brackets represents the contribution from hard sphere scattering.

This result has a simple interpretation in terms of the expansion in stationary scattering states,

$$\varphi_2 = \varphi_{2,in} + \varphi_{2,out} = -\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \exp\left[-ik(r-a)-\tfrac{1}{2}ik^2t\right] \frac{dk}{(k-k_o)} +$$

$$+\frac{1}{2\pi i}\int_{-\infty}^{+\infty} S_a(k) \exp\left[ik(r-a)-\tfrac{1}{2}ik^2t\right] \frac{dk}{(k-k_o)} , \qquad (74)$$

where $S_a(k) = \exp(2ika)S(k)$, and $S(k)$ is given by (19). It can readily be shown that (73) corresponds to the result which is obtained by taking the <u>one-level approximation</u> [10]

$$S_a(k) = (k-k_1^*)(k-k_1)^{-1} . \qquad (75)$$

Thus, under the above conditions, only the immediate neighbourhood of the resonant level gives an appreciable contribution to the integral.

The hard-sphere term in (73) always interferes destructively with the resonance term. The resulting "absorption dip" in the surface-reflected wave represents the part of the incident wave packet which penetrates within the scatterer to

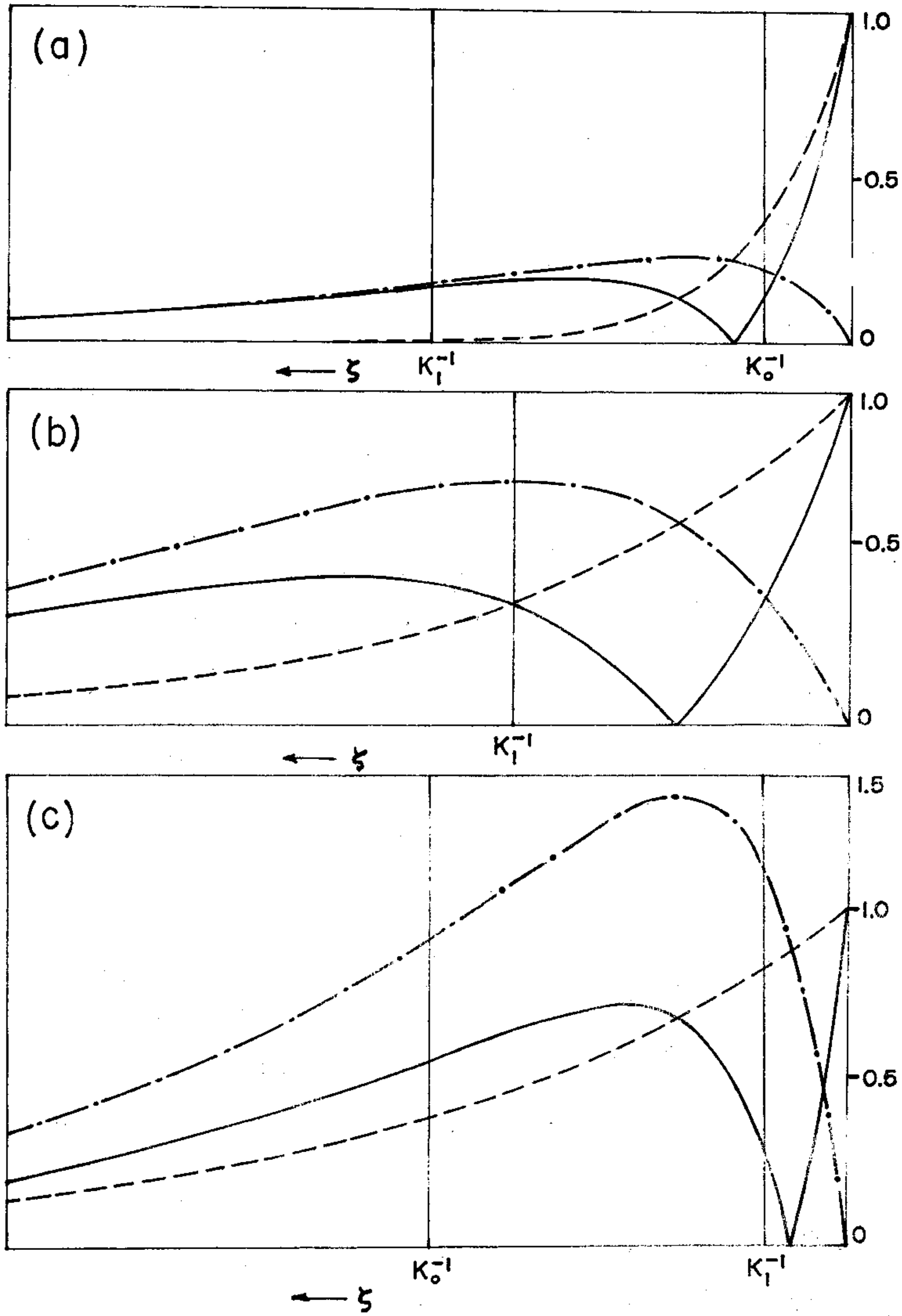

Fig. 1 - Profiles of the hard sphere term (- - - -), the resonance term (-.-.-.) and the total outgoing wave (——) as a function of the distance ξ behind the wave front. (a) $\kappa_0 = 5\,\kappa_1$ (broad line); (b) $\kappa_0 = \kappa_1$ (complex resonance); (c) $\kappa_0 = \kappa_1/5$ (narrow line).

build up the resonant mode.

The absolute values of the factor within curly brackets in (73) and its component terms are plotted as a function of ζ in Fig. 1, in the following cases: (a) broad line ($\kappa_0 \gg \kappa_1$); (b) complex resonance ($\kappa_0 = \kappa_1$); (c) narrow line ($\kappa_0 \ll \kappa_1$). The amplitude of the resonance term also represents the behaviour of the resonant mode within the sphere as a function of time.

The curves in Fig. 1 represent probability amplitudes. The corresponding probability distributions can be obtained by squaring them. In all cases, the total outgoing wave consists of a peak due to direct reflection at the surface, followed by a tail which represents the effect of resonance scattering.

c) The time delay.

According to Eisenbud [11], the energy derivative of the scattering phase shift represents the time delay suffered by the incident wave packet in the scattering process. The retardation suffered by the center of the outgoing wave packet is given by $\Delta = 2 \, d\eta/dk$, where $\eta(k)$ is the phase shift.

According to Wigner [12], this result leads to a simple physical interpretation of the energy dependence of η. At energies for which the incident particle hardly enters the scatterer the "retardation" will be close to $-2a$ ($\underline{a}$ being the radius of the scatterer), whereas it will assume large positive values close to resonances, where the incident particle is captured and retained for some time by the scatterer.

In the case of a sharp resonance, where the one-level approximation (75) can be applied, the retardation is given by $\Delta = 2/\kappa_1$, where κ_1 is the width of the level in wave-number units. This corresponds to a time delay of twice the life-time associated with the level. If κ_0 is the width of the incident wave packet, it is assumed in this case that $\kappa_0 \ll \kappa_1$, so that the variation of $d\eta/dk$ over the width κ_0 can be neglected.

According to (73) and Fig. 1 (c), the results which have been found in the present example in the case of a narrow line ($\kappa_0 \ll \kappa_1$) are not in agreement with Eisenbud's expression. The "retardation" of the outgoing wave front is $\approx -2a$, and the shape of the outgoing wave packet differs from that of the incident wave, so that a description in terms of a retardation of the center of the wave packet is not appropriate.

The reason for this discrepancy is that the momentum distribution of the incident wave packet (cf. (74)) does not fulfil one of the conditions which are required for the validity of Eisenbud's expression. This condition is that the momentum distribution should go to zero sufficiently rapidly outside of its width κ_0. An example is provided by a Gaussian wave packet of width κ_0 *. In this case, the strong surface reflection which was found above disappears, because the internal region is excited adiabatically. The initial excitation within the scatterer does not vanish, but it can be made arbitrarily small by taking the

---

* The author wishes to thank Professor E. P. Wigner for drawing his attention to this point.

initial position of the center of the wave packet sufficiently far from the scatterer. The outgoing wave packet is also of Gaussian shape, and the retardation of its center is given by Eisenbud's expression.

It should be emphasized, however, that the retardation, under these conditions, is a very small effect. A retardation $\Delta = 2/\kappa_1$ produces a decrease in the probability density at the unretarded position of the center of the Gaussian wave packet by an amount $1 - \exp(-\kappa_0^2 \Delta^2) \approx 4(\kappa_0/\kappa_1)^2$, which is very small for $\kappa_0 \ll \kappa_1$.

In order to render the retardation effect more conspicuous, one can try to associate it with a sharper signal. However, this necessarily involves a violation of the requirements for the validity of Eisenbud's expression, as we have seen in the case of an initial wave packet having a sharp front. If one takes an incident wave packet of width $\kappa_0 \gg \kappa_1$, so that the corresponding uncertainty in position is much smaller than the retardation in question, Eisenbud's expression cannot be applied, because $d\eta/dk$ varies greatly over the width κ_0. As shown in Fig. 1 (a), the outgoing wave packet then has a tail of small amplitude $(\sim \kappa_1/\kappa_0)$, which decays with the lifetime of the level, i.e. much more slowly than the incident wave. This can also be called a time delay, but it is again a small effect.

ACKNOWLEDGMENT

The author is indebted to Professor S. A. Wouthuysen for suggesting the example treated in this paper. He wishes to thank Professor N. G. Van Kampen and Professor G. Beck for valuable discussions, and Professor L. Van Hove for the hospitality of the Institute for Theoretical Physics, Utrecht, where part of this work was done. A fellowship of the National Research Council of Brazil is gratefully acknowledged.

* * *

Appendix - The Mittag-Leffler expansions.

We want to find Mittag-Leffler expansions of $Q(z)$ and of $zQ(z)$, where $Q(z)$ is defined by (18). Let z_n $(n = 1, 2, \ldots)$ be the poles of $zQ(z)$ in the lower left quadrant of the z-plane. The poles in the upper left quadrant are $z_{-n} = z_n^*$. The residue at z_n is

$$r_n = z_n(A+1+z_n)^{-1}, \qquad (A1)$$

and, according to (21),

$$z_n \approx -\log[(2n-\tfrac{1}{2})\pi/A] - i(2n-\tfrac{1}{2})\pi \quad (n\pi \gg A). \qquad (A2)$$

The Mittag-Leffler expansion of $zQ(z)$ can be obtained by applying Cauchy's method [13]. For this purpose, consider a sequence of squares C_n with corners at the points $(2n+\frac{1}{2})(\pm 1 \pm i)\pi$. It is easily shown that $zQ(z)$ is bounded on this system of contours taken as a whole.

It follows that

$$zQ(z) = \frac{1}{A+1} + \Sigma\, r_n \left(\frac{1}{z-z_n} + \frac{1}{z_n} \right) , \qquad (A3)$$

where the terms n and -n must be taken together in the summation. According to (A1) and (A2), each of the series within brackets is absolutely convergent, so that we may write

$$\frac{1}{A+1} + \Sigma \frac{r_n}{z_n} = C = zQ(z) - \Sigma \frac{r_n}{z-z_n} . \qquad (A4)$$

To determine the constant C, add to (A4) the same equation with z replaced by -z. This gives

$$C = \tfrac{1}{2}[zQ(z) - zQ(-z)] - \Sigma \frac{r_n z_n}{(z^2-z_n^2)} . \qquad (A5)$$

According to (A1) and (A2), the series in (A5) converges uniformly on the real axis as a whole (which is not true for the series in (A4)). Therefore, it does not contribute in the limit $z \to \pm\infty$, and we find

$$C = \tfrac{1}{2} \lim_{z \to \pm\infty} [zQ(z) - zQ(-z)] = \tfrac{1}{2} , \qquad (A6)$$

so that the Mittag-Leffler expansion finally becomes

$$zQ(z) = \tfrac{1}{2} + \Sigma \frac{r_n}{z-z_n} . \qquad (A7)$$

Dividing both members of (A3) by z, we obtain the Mittag-Leffler expansion of $Q(z)$:

$$Q(z) = \frac{1}{(A+1)z} + \Sigma \frac{r_n}{z_n(z-z_n)} . \qquad (A8)$$

REFERENCES

1. G. BECK and H. M. NUSSENZVEIG: Nuovo Cimento, 16, 416 (1960).
2. T. REGGE: Nuovo Cimento, 8, 671 (1958).
3. J. PETZOLD: Zeits. Phys., 155, 422 (1959).
4. B. GROSS: Suppl. Nuovo Cimento, 3, 235 (1956).
5. J. HUMBLET: Mém. in-8° Soc. Roy. Sci. Liège, 12, nº 4 (1952).
6. V. I. SERDOBOLSKII: Sov. Phys. Journ. Exp. Theor. Phys., 36, 1354 (1959).
7. H. M. NUSSENZVEIG: Nuclear Physics, 11, 499 (1959).
8. A. SOMMERFELD: Partial Differential Equations in Physics (New York, 1949), p. 72.
9. N. S. KRYLOV and V. A. FOCK: Žurn. Ėksp. Teor. Fiz., 17, 93 (1947). Cf. also L. A. KHALFIN: Sov. Phys. Journ. Exp. Theor. Phys., 6, 1053 (1958), and J. PETZOLD: Zeits. Phys., 157, 122 (1959).
10. A. M. LANE and R. G. THOMAS: Rev. Mod. Phys., 30, 257, 321 (1958).
11. L. EISENBUD: Princeton dissertation, unpublished (1948).
12. E. P. WIGNER: Phys. Rev., 98, 145 (1955).
13. E. C. TITCHMARSH: The Theory of Functions, 2nd ed. (Oxford, 1939), p. 110.

* * *