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An Exact Approach to the Oscillator Radiation Process in an Arbitrarily Large Cavity

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Abstract

Starting from a solution of the problem of a mechanical oscillator coupled to a scalar field inside a reflecting sphere of radius R, we study the behaviour of the system in free space as the limit of an arbitrarily large radius in the confined solution. From a mathematical point of view we show that this way of facing the problem is not equivalent to consider the system *a priori* embedded in infinite space. In particular, the matrix elements of the transformation turning the system to principal axis, do not tend to distributions in the limit of an arbitrarily large sphere as it should be the case if the two procedures were mathematically equivalent. Also, we introduce "dressed" coordinates which allow an exact description of the oscillator radiation process for any value of the coupling, strong or weak. In the case of weak coupling, we recover from our exact expressions the well known decay formulas from perturbation theory.

1 Introduction

Since a long time ago the experimental and theoretical investigations on the polarization of atoms by optical pumping and the possibility of detecting changes in their polarization states has allowed the observation of resonant effects associated to the coupling of these atoms with strong radiofrequency fields [1]. As remarked in [2], the theoretical understanding of these effects using perturbative methods requires the calculation of very high-order terms in perturbation theory, what makes the standard Feynman diagrams technique practically unreliable in those cases. The trials of treating non-perturbativelly such kind of systems consisting of an atom coupled to the electromagnetic field, have lead to the idea of "dressed atom", introduced in refs [3] and [4]. This approach consists in quantizing the electromagnetic field and analyzing the whole system consisting of the atom coupled to the electromagnetic field. Along the years since then, this concept has been extensively used to investigate several situations involving the interaction of atoms and electromagnetic fields. For instance, atoms embedded in a strong radiofrequency field background in refs. [5] and [6], atoms in intense resonant laser beans in ref. [7] or the study of photon correlations and quantum jumps. In this last situation, as showed in refs. [8], [9] and [10], the statistical properties of the random sequence of outcoming pulses can be analyzed by a broadband photodetector and the dressed atom approach provides a convenient theoretical framework to perform this analysis.

Besides the idea of dressed atom in itself, another aspect that desserves attention is the non-linear character of the problem involved in realistic situations, which implies, as noted above, in very hard mathematical problems to be dealt with. An way to circunvect these mathematical difficulties, is to assume that under certain conditions the coupled atom-electromagnetic field system may be approximated by the system composed of an

harmonic oscillator coupled *linearly* to the field trough some effective coupling constant g.

In this sense, in a slightly different context, recently a significative number of works has been spared to the study of cavity QED, in particular to the theoretical investigation of higher-generation Schrodinger cat-states in high-Q cavities, as has been done for instance in [11]. Linear approximations of this type have been applied along the last years in quantum optics to study decoherence, by assuming a linear coupling between a cavity harmonic mode and a thermal bath of oscillators at zero temperature, as it has been done in [12] and [13]. To investigate decoherence of higher generation Schrodinger cat-states the cavity field reduced matrix for these states could be calculated either by evaluating the normal-ordering characteristic function, or by solving the evolution equation for the field-resevoir state using the normal mode expansion, generalizing the analysis of [12] and [13].

In this paper we adopt a general physicist's point of view, we do not intend to describe the specific features of a particular physical situation, instead we analyse a simplified linear version of the atom-field system and we try to extract the more detailed information we can from this model. We take a linear simplified model in order to try to have a clearer understanding of what we believe is one of the essential points, namely, the need of non-perturbative analytical treatments to coupled systems, which is the basic problem underlying the idea of dressed atom. Of course, such an approach to a realistic non-linear system is an extremelly hard task and here we make what we think is a good agreement between physical reality and mathematical reliability, with the hope that in future work our approach could be transposed to more realistic situations.

We consider a non relativistic system composed of a harmonic oscillator coupled linearly to a scalar field in ordinary Euclidean 3-dimensional space. We start from an analysis

of the same system confined in a reflecting sphere of radius R, and we assume that the free space solution to the radiating oscillator should be obtained taking a radius arbitrarily large in the R-dependent quantities. The limit of an arbitrarily large radius in the mathematics of the confined system is taken as a good description of the ordinary situation of the radiating oscillator in free space. We will see that this is not equivalent to the alternative continuous formulation in terms of distributions, which is the case when we consider *a priori* the system in unlimited space. The limiting procedure adopted here allows to avoid the inherent ambiguities present in the continuous formulation. From a physical point of view we give a non-perturbative treatment to the oscillator radiation introducing some coordinates that allow to divide the coupled system into two parts, the "dressed" oscillator and the field, what makes unecessary to work directly with the concepts of "bare" oscillator, field and interaction to study the radiation process. These are the main reasons why we study a simplified linear system instead of a more realistic model, to make evident some subtleties of the mathematics involved in the limiting process of taking a cavity arbitrarily large, and also to exhibit an exact solution valid for weak as well as for strong coupling. These aspects would be masked in the perturbative approach used to study non-linear couplings.

We start considering a harmonic oscillator $q_0(t)$ of frequency ω_0 coupled linearly to a scalar field $\phi(\mathbf{r},t)$, the whole system being confined in a sphere of radius R centered at the oscillator position. The equations of motion are,

$$\ddot{\mathrm{q}}_0(t) + \omega_0^2 q_0(t) = 2\pi\sqrt{gc}\int_0^R d^3\mathbf{r}\,\phi(\mathbf{r},t)\delta(\mathbf{r}) \tag{1}$$

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi(\mathbf{r},t) = 2\pi\sqrt{gc}\,q_0(t)\delta(\mathbf{r}) \tag{2}$$

which, using a basis of spherical Bessel functions defined in the domain $< |\mathbf{r}| < R$, may

be written as a set of equations coupling the oscillator to the harmonic field modes,

$$\ddot{q}_0(t) + \omega_0^2 q_0(t) = \eta \sum_{i=1}^{\infty} \omega_i q_i(t) \tag{3}$$

$$\ddot{q}_i(t) + \omega_i^2 q_i(t) = \eta \omega_i q_0(t). \tag{4}$$

In the above equations, g is a coupling constant, $\eta = \sqrt{2g\Delta\omega}$ and $\Delta\omega = \pi c/R$ is the interval between two neighbouring field frequencies, $\omega_{i+1} - \omega_i = \Delta\omega = \pi c/R$.

2 The transformation to principal axis and the eigenfrequencies spectrum

2.1 - Coupled harmonic Oscillators

Let us consider for a moment the problem of a harmonic oscillator q_0 coupled to N other oscillators. In the limit $N \to \infty$ we recover our original situation of the coupling oscillator-field after redefinition of divergent quantities, in a manner analogous as renormalization is done in field theories. In terms of the cutoff N the coupled equations (3) and (4) are simply rewritten taking the upper limit N instead of ∞ for the summation in the right hand side of Eq.(3) and the system of $N+1$ coupled oscillators q_0 $\{q_i\}$ corresponds to the Hamiltonian,

$$H = \frac{1}{2}\left[p_0^2 + \omega_0^2 q_0^2 + \sum_{k=1}^{N} p_k^2 + \omega_k^2 q_k^2 - 2\eta\omega_k q_0 q_k\right]. \tag{5}$$

The Hamiltonian (5) can be turned to principal axis by means of a point tranformation,

$$q_\mu = t_\mu^r Q_r\,, \quad p_\mu = t_\mu^r P_r, \tag{6}$$

performed by an orthonormal matrix $T = (t_\mu^r)$, $\mu = (0,k)$, $k = 1, 2, \ldots N$, $r = 0, \ldots N$. The subscript 0 and k refer respectively to the oscillator and the harmonic modes of the

field and r refers to the normal modes. The transformed Hamiltonian in principal axis is

$$H = \frac{1}{2}\sum_{r=0}^{N}(P_r^2 + \Omega_r^2 Q_r^2), \tag{7}$$

where the Ω_r's are the normal frequencies corresponding to the possible collective oscillation modes of the coupled system.

Using the coordinate transformation $q_\mu = t_\mu^r Q_r$ in the equations of motion and explicitly making use of the normalization condition $\sum_{\mu=0}^{N}(t_\mu^r)^2 = 1$, we get,

$$t_k^r = \frac{\eta\omega_k}{\omega_k^2 - \Omega_r^2} t_0^r, \tag{8}$$

$$t_0^r = \left[1 + \sum_{k=1}^{N} \frac{\eta^2\omega_k^2}{(\omega_k^2 - \Omega_r^2)^2}\right]^{-\frac{1}{2}} \tag{9}$$

and

$$\omega_0^2 - \Omega_r^2 = \eta^2 \sum_{k=1}^{N} \frac{\omega_k^2}{\omega_k^2 - \Omega_r^2}. \tag{10}$$

There are $N+1$ solutions Ω_r to Eq.(10), corresponding to the $N+1$ normal collective oscillation modes. To have some insight into these solutions, we take $\Omega_r = \Omega$ in Eq.(10) and transform the right hand term. After some manipulations we obtain

$$\omega_0^2 - N\eta^2 - \Omega^2 = \eta^2 \sum_{k=1}^{N} \frac{\Omega^2}{\omega_k^2 - \Omega^2} \tag{11}$$

It is easily seen that if $\omega_0^2 > N\eta^2$ Eq.(11) yelds only positive solutions for Ω^2, what means that the system oscillates harmonically in all its modes. Indeed, in this case the left hand term of Eq.(11) is positive for negative values of Ω^2. Conversely the right hand term is negative for those values of Ω^2. Thus there is no negative solution of that equation when $\omega_0^2 > N\eta^2$. On the other hand it can be shown that if $\omega_0^2 < N\eta^2$, Eq.(11) has a single negative solution Ω_-^2. In order to prove it let us define the function

$$I(\Omega^2) = (\omega_0^2 - N\eta^2) - \Omega^2 - \eta^2 \sum_{k=1}^{N} \frac{\Omega^2}{\omega_k^2 - \Omega^2} \tag{12}$$

Accordingly Eq.(11) can be rewritten as $I(\Omega^2) = 0$. It can be noticed that $I(\Omega^2) \to \infty$ as $\Omega^2 \to -\infty$ and

$$I(\Omega^2 = 0) = \omega_0^2 - N\eta^2 < 0 \tag{13}$$

Furthermore $I(\Omega^2)$ is a monotonically decreasing function in that interval. Consequently $I(\Omega^2) = 0$ has a single negative solution when $\omega_0^2 < N\eta^2$ as we have pointed out. This means that there is an oscillation mode whose amplitude varies exponentially and that does not allows stationary configurations. We will not care about this last situation. Thus we assume $\omega_0^2 > N\eta^2$ and define the *renormalized* oscillator frequency $\bar{\omega}$ [14],

$$\bar{\omega} = \sqrt{\omega_0^2 - N\eta^2}. \tag{14}$$

In terms of the renormalized frequency Eq.(10) becomes,

$$\bar{\omega}^2 - \Omega_r^2 = \eta^2 \sum_{k=1}^{N} \frac{\Omega_r^2}{\omega_k^2 - \Omega_r^2}. \tag{15}$$

From Eqs. (8), (9) and (15), a straightforward calculation shows the orthonormality relations for the transformation matrix (t_μ^r).

We get the transformation matrix elements for the oscillator-field system by taking the limit $N \to \infty$ in the above equations. Recalling the definition of η from Eqs. (3) and (4), we obtain after some algebraic manipulations, from Eqs. (15), (8) and (9), the matrix elements in the limit $N \to \infty$,

$$t_0^r = \frac{\Omega_r}{\sqrt{\frac{R}{2\pi gc}(\Omega_r^2 - \bar{\omega}^2)^2 + \frac{1}{2}(3\Omega_r^2 - \bar{\omega})^2 + \frac{\pi gR}{2c}\Omega_r^2}} \tag{16}$$

and

$$t_k^r = \frac{\eta\omega_k}{\omega_k^2 - \Omega_r^2} t_0^r. \tag{17}$$

2.2 - The eigenfrequencies spectrum

Let us now return to the coupling oscillator-field by taking the limit $N \to \infty$ in the relations of the preceeding subsection. In this limit it becomes clear the need for the frequency renormalization in Eq.(14). It is exactly the analogous of a mass renormalization in field theory, the infinite ω_0 is chosen in such a way as to make the renormalized frequency $\bar{\omega}$ finite. Remembering Eq.(15) the solutions with respect to the variable Ω of the equation

$$\bar{\omega}^2 - \Omega^2 = \frac{2\pi gc}{R} \sum_{k=1}^{\infty} \frac{\Omega^2}{\omega_k^2 - \Omega^2}, \tag{18}$$

give the collective modes frequencies. We remember $\omega_k = k\frac{\pi c}{R}$, $k = 1, 2, ...$, and take a positive x such that $\Omega = x\frac{\pi c}{R}$. Then using the identity,

$$\sum_{k=1}^{\infty} \frac{x^2}{k^2 - \Omega^2} = \frac{1}{2}(1 - \pi x \cot \pi x), \tag{19}$$

Eq.(18) may be rewritten in the form,

$$cotg\pi x = \frac{c}{Rg}x + \frac{1}{\pi x}(1 - \frac{R\bar{\omega}^2}{\pi gc}). \tag{20}$$

The secant curve corresponding to the right hand side of the above equation cuts only once each branch of the cotangent in the left hand side. Thus we may label the solutions x_r as $x_r = r + \epsilon_r$, $0 < \epsilon_r < 1$, $r = 0, 1, 2...$, and the collective eigenfrequencies are,

$$\Omega_r = (r + \epsilon_r)\frac{\pi c}{R}, \tag{21}$$

the ϵ's satisfying the equation,

$$cot(\pi\epsilon_r) = \frac{\Omega_r^2 - \bar{\omega}^2}{\Omega_r \pi g} + \frac{c}{\Omega_r R}. \tag{22}$$

The field $\phi(\mathbf{r}, t)$ can be expressed in terms of the normal modes. We start from its expansion in terms of spherical Bessel functions,

$$\phi(\mathbf{r}, t) = c \sum_{k=1}^{\infty} q_k(t)\phi_k(\mathbf{r}), \tag{23}$$

where

$$\phi_k(\mathbf{r}) = \frac{sin\frac{\omega_k}{c}|\mathbf{r}|}{\mathbf{r}\sqrt{2\pi R}}. \tag{24}$$

Using the principal axis transformation matrix together with the equations of motion we obtain an expansion for the field in terms of an orthonormal basis associated to the collective normal modes,

$$\phi(\mathbf{r},t) = c\sum_{s=0}^{\infty} Q_s(t)\Phi_s(\mathbf{r}), \tag{25}$$

where the normal collective Fourier modes

$$\Phi_s(\mathbf{r}) = \sum_k t_k^s \frac{sin\frac{\omega_k}{c}|\mathbf{r}|}{\mathbf{r}\sqrt{2\pi R}} \tag{26}$$

satisfy the equation

$$(-\frac{\Omega_s^2}{c^2} - \Delta)\phi_s(\mathbf{r}) = 2\pi\sqrt{\frac{g}{c}}\delta(\mathbf{r})t_0^s, \tag{27}$$

which has a solution of the form

$$\phi(\mathbf{r},t) = -\sqrt{\frac{g}{c}}\frac{t_0^s}{2|\mathbf{r}|sin\delta_s}sin(\frac{\Omega_s}{c}|\mathbf{r}| - \delta_s). \tag{28}$$

To determine the phase δ_s we expand the right hand term of Eq.(28) and compare with the formal expansion (26). This imply the condition

$$sin(\frac{\Omega_s}{c}R - \delta_s) = 0. \tag{29}$$

Remembering from Eq.(21) that there is $0 < \epsilon_s < 1$ such that $\Omega_s = (s+\epsilon_s)\frac{\pi}{R}$, it is easy to show from the condition in Eq.(29) that the phase $0 < \delta_s < \pi$ has the form

$$\delta_s = \epsilon_s\pi. \tag{30}$$

Comparing Eqs.(24) and (26) and using the explicit form (16) of the matrix element t_0^s we obtain the expansion for the field in terms of the normal collective modes,

$$\phi(\mathbf{r},t) = -\frac{\sqrt{gc}}{2}\sum_s \frac{Q_s sin(\frac{\Omega_s}{c}|\mathbf{r}| - \delta_s)}{|\mathbf{r}|\sqrt{sin^2\delta_s + (\frac{\eta R}{2c})^2(1 - \frac{sin\delta_s cos\delta_s}{\Omega_s R/c})}} \tag{31}$$

3 The limit $R \to \infty$ - mathematical aspects

3.1 - Discussion of the mathematical problem

Unless explicitly stated, in the remaining of this paper the symbol $R \to \infty$ is to be understood as the situation of a cavity of fixed, arbitrarily large radius. In order to compare the behaviour of the system in a very large cavity to that it would be in free space, let us firstly consider the system embedded in an *a priori* infinite Euclidean space; in this case to compute the quantities describing the system means essentially to replace by integrals the discrete sums appearing in the confined problem, taking direcltly $R = \infty$. An alternative procedure is to compute the quantities describing the system confined in a sphere of radius R and take the limit $R \to \infty$ afterwards. This last approach to describe the system in free space should keep in some way the "memory" of the confined system. To be physically equivalent one should expect that the two approachs give the same results. We will see that at least from a mathematical point of view this is not exactly the case. We remark that a solution to the problem of a system composed of an oscillator coupled to a field in free space, is already known since a long time ago [15] in the context of Bownian motion. This solution is quite different from ours, in the sense that it not concerns the system confined to a box and also that it is limited to the dipole term from the multipolar expansion to the field.

In the continuous formalism of free space the field normal modes Fourier components (analogous to the components ϕ_s in Eq.(26)) are,

$$\phi_\Omega = h(\Omega) \int_0^\infty d\omega \frac{\omega}{\omega^2 - \Omega^2} \frac{sin\frac{\omega}{c}|\mathbf{r}|}{|\mathbf{r}|}, \tag{32}$$

where

$$h(\Omega) = \frac{2g\Omega}{\sqrt{(\Omega^2 - \bar{\omega}^2)^2 + \pi g^2 \Omega^2}} \tag{33}$$

and where the we have taken the appropriate continuous form of Eqs.(16) and (17). Splitting $\omega/(\omega^2-\Omega^2)$ into partial fractions we get

$$\phi_\Omega = h(\Omega)\int_{-\infty}^{+\infty} d\omega \frac{1}{\omega-\Omega}\frac{sin\frac{\omega}{c}|\mathbf{r}|}{|\mathbf{r}|}. \tag{34}$$

The pole at $\omega=\Omega$ prevents the existence of the integral in Eq.(34). The usual way to circumvect this difficulty is to replace the integral by one of the quantities,

$$Lim_{\epsilon\to 0}\int_{-\infty}^{+\infty} d\omega \frac{1}{\omega-(\Omega\pm i\epsilon)}\frac{sin\frac{\omega}{c}|\mathbf{r}|}{|\mathbf{r}|} \equiv \int_{-\infty}^{+\infty} d\omega\delta_\pm(\omega-\Omega)\frac{sin\frac{\omega}{c}|\mathbf{r}|}{|\mathbf{r}|}, \tag{35}$$

where

$$\delta_\pm(\omega-\Omega) = \frac{1}{\pi}P(\frac{1}{\omega-\Omega}) \pm i\delta(\omega-\Omega), \tag{36}$$

with P standing for principal value. In our case this redefinition of the normal modes Fourier components may be justified by the fact that both integrals in Eq.(35) are solutions of the equations of motion (1) and (2) for $\mathbf{r}\neq 0$, and so the solution should be a linear combination of them. The situation is different if we adopt the point of view of taking the limit $R\to\infty$ in the solution of the confined problem. In this case the Fourier component ϕ_Ω is obtained by taking the limit $R\to\infty$ in the expression for the field, Eq(28), what allows to obtain an uniquely defined expression to the normal modes Fourier components, to each ϕ_Ω corresponding a phase δ_Ω (the limit $R\to\infty$ of δ_s in Eq.(22) given by

$$cot\delta_\Omega = \frac{1}{\pi g}\frac{\Omega^2-\bar{\omega}^2}{\Omega}. \tag{37}$$

Also, comparing Eqs.(35), (36) and (26) we see that the adoption of the continuous formalism is equivalent to assume that in the limit $R\to\infty$ the elements t_i^s of the transformation matrix should be replaced by $\delta_+(\omega-\Omega)$ or by $\delta_-(\omega-\Omega)$. This procedure is, from a mathematical point of view, perfectly justified but at the price of loosing uniqueness in the definition of the field components.

If we take the solution of the confined problem and we compute the matrix elements t_i^s for R arbitrarily large, we will see in subsection **3.2** that these elements do not tend to distributions in this limit. As R becomes larger and larger the set of non-vanishing elements t_i^s concentrate for each i in a small neighbourhood of ω_i. In the limit $R \to \infty$ the whole set of the matrix elements t_i^s contains an arbitrarily large number of elements quadratically summables [16]. For the matrix elements t_0^s we obtain a quadratically integrable expression.

In the continuous formulation the unit matrix, corresponding to the absence of coupling, has elements $E_\omega^\Omega = \delta(\omega - \Omega)$, while if we start from the confined situation, it can be verified that in the limit $g \to 0$, $R \to \infty$, the matrix $T = (t_\mu^s)$ tends to the usual unit matrix of elements $E_{\omega,\Omega} = \delta_{\omega,\Omega}$.

The basic quantity describing the system, the transformation matrix $T = (t_\mu^s)$ has, as we will see, different properties in free space, if we use the continuous formalism or if we adopt the procedure of taking the limit $R \to \infty$ from the matrix elements in the confined problem . In the first case we must define the matrix elements t_ω^Ω linking free field modes to normal modes, as distributions. On the other side adopting the second procedure we will find that the limiting matrix elements $Lim_{R\to\infty}\, t_i^s$ are not distributions, but well defined finite quantities. The two procedures are not equivalent, the limit $R \to \infty$ does not commute with other operations. In this note we take as physically meaningfull the second procedure, we solve first the problem in the confined case (finite R) and take afterwards the limit of infinite (in the sense of arbitrarily large) radius of the cavity. In the next subsection we perform a detailed analysis of the limit $R \to \infty$ of the transformation matrix (t_μ^r).

3.2 - The transformation matrix in the limit $R \to \infty$

From Eqs. (16) and (17) we obtain for R arbitrarily large,

$$t_0^r \to Lim_{\Delta\Omega\to 0}\, t_{\bar{\omega}}^{\Omega}\sqrt{\Delta\Omega} = Lim_{\Delta\Omega\to 0}\frac{\sqrt{2g}\Omega\sqrt{\Delta\Omega}}{\sqrt{(\Omega^2-\bar{\omega}^2)^2+\pi^2 g^2\Omega^2}}. \tag{38}$$

and

$$t_k^r = \frac{2g\omega_k\Delta\omega}{(\omega_k+\Omega_r)(\omega_k-\Omega_r)}\frac{\Omega_r}{\sqrt{(\Omega_r^2-\bar{\omega}^2)^2+\pi^2 g^2\Omega_r^2}}, \tag{39}$$

where we have used the fact that in this limit $\Delta\omega = \Delta\Omega = \frac{\pi c}{R}$. The matrix elements $t_{\bar{\omega}}^{\Omega}$ are quadratically integrable to one, $\int (t_{\bar{\omega}}^{\Omega})^2\, d\Omega = 1$, as may be seen using Cauchy theorem.

For R arbitrarily large ($\Delta\omega = \frac{\pi c}{R} \to 0$), the only nonvanishing matrix elements t_i^r are those for which $\omega_i - \Omega_r \approx \Delta\omega$. To get explicit formulas for these matrix elements in the limit $R \to \infty$ let us consider R large enough such that we may take $\Delta\omega \approx \Delta\Omega$ and consider the points of the spectrum of eigenfrequencies Ω inside and outside a neighbourhood η (defined in Eqs.(3) and (4) of ω_i. We note that $R > \frac{2\pi c}{g}$ implies $\frac{\eta}{2} > \Delta\omega$, then we may consider R such that the right (left) neighbourhood $\frac{\eta}{2}$ of ω_i contains an integer number, κ, of frequencies Ω_r,

$$\kappa\Delta\omega = \frac{\eta}{2} = \sqrt{\frac{g\Delta\omega}{2}}. \tag{40}$$

If R is arbitrarily large we see from (40) that $\frac{\eta}{2}$ is arbitrarily small, but κ grows at the same rate, what means firstly that the difference $\omega_i - \Omega_r$ for the Ω_r's outside the neighbourhood η of ω_i is abitrarily larger than $\Delta\omega$, implying that the corresponding matrix elements t_i^r tend to zero (see Eq.(39)). Secondly all frequencies Ω_r inside the neighbourhood η of ω_i are arbitrarily close to ω_i, being in arbitrarily large number. Only the matrix elements t_i^r corresponding to these frequencies Ω_r inside the neighbourhood η of ω_i are different from zero. For these we make the change of labels,

$$r = i - n\,(\omega_i - \frac{\eta}{2} < \Omega_r < \omega_i)\,;\; r = i + n\,(\omega_i > \Omega_r > \omega_i + \frac{\eta}{2}), \tag{41}$$

$i = 1, 2, ...$ We get, from Eq.(39)

$$t_i^i = \frac{g\omega_i}{\sqrt{(\Omega_r^2 - \bar{\omega}^2)^2 + \pi^2 g^2 \omega_i^2}} \frac{1}{\epsilon_i} \tag{42}$$

and

$$t_i^{i\pm n} = \frac{\mp g\omega_i}{\sqrt{(\Omega_r^2 - \bar{\omega}^2)^2 + \pi^2 g^2 \omega_i^2}} \frac{1}{n \pm \epsilon_i}, \tag{43}$$

where ϵ_i satisfies Eq.(22) in this case,

$$cot(\pi\epsilon_i) = \frac{\omega_i^2 - \bar{\omega}^2}{\omega_i \pi g}. \tag{44}$$

Using the formula

$$\pi^2 cosec^2(\pi\epsilon_i) = \frac{1}{\epsilon_i} + \sum_{n=1}^{\infty} \left[\frac{1}{(n+\epsilon_i)^2} + \frac{1}{(n-\epsilon_i)^2} \right], \tag{45}$$

it is easy to show the normalization condition for the matrix elements (42) and (43),

$$(t_i^i)^2 + \sum_{n=1}^{\infty} (t_i^{i-n})^2 + (t_i^{i+n})^2 = 1 \tag{46}$$

and also the orthogonality relation,

$$\sum_r t_i^r t_k^r = 0 \, (i \neq k) \tag{47}$$

in the limit $R \to \infty$.

3.3 - The transformation matrix in the limit $g = 0$

From Eq. (16) we get for arbitrary R,

$$Lim_{g\to 0}\, t_0^r = \begin{cases} 1, & \text{if } \Omega_r = \bar{\omega}; \\ 0, & \text{otherwise.} \end{cases} . \tag{48}$$

From Eqs.(42) and (43) we see that the matrix elements t_i^r for $i \neq r$ all vanish for $g = 0$. Also, using Eqs.(21) and (**??**) we obtain for small g,

$$t_i^i \approx \frac{2g\Omega_i\omega_i}{(\Omega_i^2 - \bar{\omega}^2)(\omega_i + \Omega_i)}\frac{1}{\epsilon_i}, \tag{49}$$

or, expanding ϵ_i for small g from Eq.(44)

$$t_i^i(g = 0) = 1 \tag{50}$$

We see from the above expressions that in the limit $R \to \infty$ the matrix (t_μ^r) remains an orthonormal matrix in the usual sense as for finite R. With the choice of the procedure of taking the limit $R \to \infty$ from the confined solution, the matrix elements do not tend to distributions in the free space limit as it would be the case using the continuous formalism. All non- vanishing matrix elements t_i^r are concentrated inside a neighbourhood η of ω_i, their set is a quadratically summable enumerable set. The elements (t_0^r) tend to a quadratically integrable expression.

4 The Radiation Process

We start this section defining some coordinates q_0', q_i' associated to the "dressed" mechanical oscillator and to the field. These coordinates will reveal themselves to be suitable to give an appealling non-perturbative description of the oscillator-field system. The general conditions that such coordinates must satisfy, taking into account that the system is rigorously described by the collective normal coordinates modes Q_r, are the following:

- In reason of the linear character of our problem the coordinates q'_0, q'_i should be linear functions of the collective coordinates Q_r

- They should allow to construct ortogonal configurations corresponding to the separation of the system into two parts, the dressed oscillator and the field.

- The set of these configurations should contain the ground state, Γ_0.

The last of the above conditions restricts the transformation between the coordinates q'_μ, $\mu = 0, i = 1, 2, ...$ and the collective ones Q_r to those leaving invariant the quadratic form,

$$\sum_r \Omega_r Q_r^2 = \bar{\omega}(q'_0)^2 + \sum_i \omega_i (q'_i)^2 \tag{51}$$

Our configurations will behave in a first approximation as independent states, but they will evolve as the time goes on, as if transitions among them were being in progress, while the basic configuration Γ_0 represents a rigorous eigenstate of the system and does not change with time. The new coordinates q'_μ describe dressed configurations of the oscillator and field quanta.

4.1 - The dressed coordinates q'_μ

The eigenstates of our system are represented by the normalized eigenfunctions,

$$\phi_{n_0 n_1 n_2 ...}(Q,t) = \prod_s \left[N_{n_s} H_{n_s}(\sqrt{\frac{\Omega_s}{\hbar}} Q_s) \right] \Gamma_0 e^{-i\sum_s n_s \Omega_s t}, \tag{52}$$

where H_{n_s} is the n_s-th Hermite polynomial, N_{n_s} is a normalization coefficient,

$$N_{n_s} = (2^{-n_s} n_s!)^{-\frac{1}{2}} \tag{53}$$

and Γ_0 is a normalized representation of the ground state,

$$\Gamma_0 = exp\left[-\sum_s \frac{\Omega_s Q_s^2}{2\hbar} - \frac{1}{4} ln \frac{\Omega_s}{\pi\hbar} \right]. \tag{54}$$

To describe the radiation process, having as initial condition that only the mechanical oscillator, q_0 be excited, the usual procedure is to consider the interaction term in the

Hamiltonian written in terms of q_0, q_i as a perturbation, which induces transitions among the eigenstates of the free Hamiltonian. In this way it is possible to treat approximatelly the problem having as initial condition that only the bare oscillator be excited. But as is well known this initial condition is physically not consistent due to the divergence of the bare oscillator frequency if there is interaction with the field. The traditional way to circumvect this difficulty is by the renormalization procedure, introducing perturbativelly order by order corrections to the oscillator frequency. Here we adopt an alternative procedure, we do not make explicit use of the concepts of interacting bare oscillator and field, described by the coordinates q_0 and $\{q_i\}$, we introduce "dressed" coordinates q_0' and $\{q_i'\}$ for, respectivelly the "dressed" oscillator and the field, defined by,

$$\sqrt{\frac{\bar{\omega}_\mu}{\hbar}}q'_\mu = \sum_r t^r_\mu \sqrt{\frac{\Omega_r}{\hbar}}Q_r, \tag{55}$$

valid for arbitrary R, which satisfy the condition to leave invariant the quadratic form (51) and where $\bar{\omega}_\mu = \bar{\omega}$, $\{\omega_i\}$. In terms of the bare coordinates the dressed coordinates are expressed as,

$$q'_\mu = \sum_\nu \alpha_{\mu\nu} q_\nu, \tag{56}$$

where

$$\alpha_{\mu\nu} = \frac{1}{\sqrt{\bar{\omega}_\mu}} \sum_r t^r_\mu t^r_\nu \sqrt{\Omega_r}. \tag{57}$$

As R becomes larger and larger we get for the various coefficients α in Eq.(57):

a) from Eq.(38),

$$Lim_{R\to\infty}\, \alpha_{00} = \frac{1}{\sqrt{\bar{\omega}}} \int_0^\infty \frac{2g\Omega^2\sqrt{\Omega}d\Omega}{(\Omega^2 - \bar{\omega}^2)^2 + \pi^2 g^2 \Omega^2} \equiv A_{00}(\bar{\omega}, g). \tag{58}$$

b) To evaluate α_{0i} and α_{0i} in the limit $R \to \infty$, we remember from the discussion in subsection **3.2** that in the the limit $R \to \infty$, for each i the only non-vanishing matrix elements t^r_i are those for which the corresponding eigenfrequencies Ω_r are arbitrarily near

the field frequency ω_i. We obtain from Eqs. (38), (42) and (43),

$$Lim_{R\to\infty}\,\alpha_{i0} = Lim_{\Delta\omega\to 0}\frac{1}{\sqrt{\omega_i}}\frac{(2g^2\omega_i^5\Delta\omega)^{\frac{1}{2}}}{(\omega_i^2-\bar{\omega}^2)^2+\pi^2g^2\omega_i^2}(\sum_{n=1}^{\infty}\frac{2\epsilon_i}{n^2-\epsilon_i^2}-\frac{1}{\epsilon_i}) \tag{59}$$

and

$$Lim_{R\to\infty}\,\alpha_{0i} = Lim_{\Delta\omega\to 0}\frac{1}{\sqrt{\bar{\omega}}}\frac{(2g^2\omega_i^5\Delta\omega)^{\frac{1}{2}}}{(\omega_i^2-\bar{\omega}^2)^2+\pi^2g^2\omega_i^2}(\sum_{n=1}^{\infty}\frac{2\epsilon_i}{n^2-\epsilon_i^2}-\frac{1}{\epsilon_i}) \tag{60}$$

c) Since in the limit $R\to\infty$ the only non-zero matrix elements t_i^r corresponds to $\Omega_r=\omega_i$, the product $t_i^r t_k^r$ vanishes for $\omega_i\neq\omega_k$. Then we obtain from Eqs.(57) and (46)

$$Lim_{R\to\infty}\,\alpha_{ik} = \delta_{ik}. \tag{61}$$

Thus, from Eqs.(56), (61), (59), (60) and (58) we can express the dressed coordinates q'_μ in terms of the bare ones, q_μ in the limit $R\to\infty$,

$$q'_0 = A_{00}(\bar{\omega},g)q_0, \tag{62}$$

$$q'_i = q_i. \tag{63}$$

It is interesting to compare Eqs.(56) with Eqs.(62), (63). In the case of Eqs.(56) for finite R, the coordinates q'_0 and $\{q'_i\}$ are all dressed, in the sense that they are all collective, both the field modes and the mechanical oscillator can not be separeted in this language. In the limit $R\to\infty$, Eqs.(62) and (63) tells us that the coordinate q'_0 describes the mechanical oscillator modified by the presence of the field in a indissoluble way, the mechanical oscillator is always dressed by the field. On the other side, the dressed harmonic modes of the field, described by the coordinates q'_i are identical to the bare field modes, in other words, the field keeps in the limit $R\to\infty$ its proper identity, while the mechanical oscillator is always accompanied by a cloud of field quanta. Therefore we identify the coordinate q'_0 as the coordinate describing the mechanical oscillator dressed by its proper field, being the whole system divided into dressed oscillator and field, without

appeal to the concept of interaction between them, the interaction being absorbed in the dressing cloud of the oscillator. In the next subsections we use the dressed coordinates to describe the radiation process.

4.2 - Dressed configurations and the radiation process

Let us define for a fixed instant the complete orthonormal set of functions,

$$\psi_{\kappa_0\kappa_1\ldots}(q') = \prod_{\mu}\left[N_{\kappa_\mu}H_{\kappa_\mu}(\sqrt{\frac{\bar{\omega}_\mu}{\hbar}}q'_\mu)\right]\Gamma_0, \tag{64}$$

where $q'_\mu = q'_0, q'_i$, $\bar{\omega}_\mu = \bar{\omega}, \omega_i$ and N_{κ_μ} and Γ_0 are as in Eq.(52). Using Eq.(55) the functions (64) can be expressed in terms of the normal coordinates Q_r. But since (52) is a complete set of orthonormal functions, the functions (64) may be written as linear combinations of the eigenfunctions of the coupled system (we take $t = 0$ for the moment),

$$\psi_{\kappa_0\kappa_1\ldots}(q') = \sum_{n_0n_1\ldots} T^{n_0n_1\ldots}_{\kappa_0\kappa_1\ldots}(0)\phi_{n_0n_1n_2\ldots}(Q,0), \tag{65}$$

where the coefficients are given by,

$$T^{n_0n_1\ldots}_{\kappa_0\kappa_1\ldots}(0) = \int dQ\,\psi_{\kappa_0\kappa_1\ldots}\phi_{n_0n_1n_2\ldots}, \tag{66}$$

the integral extending over the whole Q-space.

We consider the particular configuration ψ in which only one dressed oscillator q'_μ is in its N-th excited state,

$$\psi_{0\ldots N(\mu)0\ldots}(q') = N_N H_N(\sqrt{\frac{\bar{\omega}_\mu}{\hbar}}q'_\mu)\Gamma_0. \tag{67}$$

The coefficients (66) can be calculated in this case using Eqs.(66), (64) and (55) with the help of the theorem [17],

$$\frac{1}{m!}\left[\sum_r (t^r_\mu)^2\right]^{\frac{m}{2}} H_N\left(\frac{\sum_r t^r_\mu\sqrt{\frac{\Omega_r}{\hbar}}Q_r}{\sqrt{\sum_r (t^r_\mu)^2}}\right) = \sum_{m_0+m_1+\ldots=N}\frac{(t^0_\mu)^{m_0}(t^1_\mu)^{m_1}\ldots}{m_0!m_1!\ldots}H_{m_0}(\sqrt{\frac{\Omega_0}{\hbar}}Q_0)H_{m_1}(\sqrt{\frac{\Omega_1}{\hbar}}Q_1)\ldots \tag{68}$$

We get,

$$T^{n_0 n_1 ...}_{0...N(\mu)0...} = (\frac{m!}{n_0! n_1! ...})^{\frac{1}{2}} (t^0_\mu)^{n_0} (t^1_\mu)^{n_1} ..., \tag{69}$$

where the subscripts $\mu = 0, i$ refer respectivelly to the dressed mechanical oscillator and the harmonic modes of the field and the quantum numbers are submited to the constraint $n_0 + n_1 + ... = N$.

In the following we study the behaviour of the system with the initial condition that only the dressed mechanical oscillator q'_0 be in the N-th excited state. We will study in detail the particular cases $N = 1$ and $N = 2$, which will be enough to have a clear understanding of our approach.

- $N = 1$: Let us call Γ^μ_1 the configuration in which only the dressed oscillator q'_μ is in the first excited level. The initial configuration in which the dressed mechanical oscillator is in the first excited level is Γ^0_1. We have from Eq.(67), (65) (69) and (55) the following expression for the time evolution of the first-level excited dressed oscillator q'_μ,

$$\Gamma^\mu_1 = \sum_\nu f^{\mu\nu}(t) \Gamma^\nu_1(0), \tag{70}$$

where the coefficients $f^{\mu\nu}(t)$ are given by

$$f^{\mu\nu}(t) = \sum_s t^s_\mu t^s_\nu e^{-i\Omega_s t}, \tag{71}$$

That is, the initially excited dressed oscillator naturally distributes its energy among itself and all others dressed oscillators, as time goes on. If the mechanical dressed oscillator is in its first excited state at $t = 0$, its decay rate may evaluated from its time evolution equation,

$$\Gamma^0_1 = \sum_\nu f^{0\nu}(t) \Gamma^\nu_1(0). \tag{72}$$

In Eq.(72) the coefficients $f^{0\nu}(t)$ have a simple interpretation: remembering Eqs.(62) and (63), $f^{00}(t)$ and $f^{0i}(t)$ are respectivelly the probability amplitudes that at time t the

dressed mechanical oscillator still be excited or have radiated a field quantum of frequency ω_i. We see that this formalism allows a quite natural description of the radiation process as a simple exact time evolution of the system. Let us for instance evaluate the oscillator decay probability in this language. From Eqs.(38) and (71) we get

$$f^{00}(t) = \int_0^\infty \frac{2g\Omega^2 e^{-i\Omega t}\, d\Omega}{(\Omega^2 - \omega^2)^2 + \pi^2 g^2 \Omega^2}. \tag{73}$$

The above integral can be evaluated by Cauchy theorem. For large t ($t >> \frac{1}{\bar{\omega}}$), but arbitrary coupling g, we obtain for the oscillator decay probability, the result,

$$|f^{00}(t)|^2 = e^{-\pi g t}(1 + \frac{\pi^2 g^2}{4\bar{\omega}^2}) + e^{-\pi g t}\frac{8\pi g}{\pi \bar{\omega}^4 t^3}(sin\tilde{\bar{\omega}}t + \frac{\pi g}{2 < \bar{\omega} >} cos\tilde{\bar{\omega}}t) + \frac{16\pi^2 g^2}{\pi^2 \bar{\omega}^8 t^6}, \tag{74}$$

where $\tilde{\bar{\omega}} = \sqrt{\bar{\omega}^2 - \frac{\pi^2 g^2}{4}}$. In the above expression the approximation $t >> \frac{1}{\bar{\omega}}$ plays a role only in the two last terms, due to the difficulties to evaluate exactly the integral in Eq. (73) along the imaginary axis. The first term comes from the residue at $\Omega = \tilde{\bar{\omega}} + i\frac{\pi g}{2}$ and would be the same if we have done an exact calculation. If we consider the case of weak coupling, $g << \bar{\omega}$, we obtain the well known perturbative exponential decay law for the harmonic oscillator[18],

$$|f^{00}(t)|^2 \approx e^{-\pi g t}, \tag{75}$$

but we emphasize that Eq.(74) is valid for all values of the coupling constant g, even large, it is an expression valid for weak as well as strong couplings.

- $N = 2$

Let us call $\Gamma_{11}^{\mu\nu}$ the configuration in which the dressed oscillators q'_μ and q'_ν are at their first excited level and Γ_2^μ the configuration in which q'_μ is at its second excited level. Taking as initial condition that the dressed mechanical oscillator be at the second excited level, the time evolution of the state Γ_2^0 may be obtained in an analogous way as in the

preceeding case,

$$\Gamma_2^0(t) = \sum_\mu [f^{\mu\mu}(t)]^2 \, \Gamma_2^\mu + \frac{1}{\sqrt{2}} \sum_{\mu \neq \nu} f^{0\mu}(t) f^{0\nu}(t) \Gamma_{11}^{\mu\nu}, \tag{76}$$

where the coefficients $f^{\mu\mu}$ and $f^{0\mu}$ are given by (71). Then it easy to get the following probabilities:

Probability that the dressed oscillator still be excited at time t:

$$P_0(t) = |f^{00}(t)|^4, \tag{77}$$

probability that the dressed oscillator have decayed at time t to the first level by emission of a field quantum:

$$P_1(t) = 2|f^{00}(t)|^2(1 - |f^{00}(t)|^2) \tag{78}$$

and probability that the dressed oscillator have decayed at time t to the ground state:

$$P_2(t) = 1 - 2|f^{00}(t)|^2 + |f^{00}(t)|^4. \tag{79}$$

Replacing Eq.(74) in the above expressions we get expressions for the probabilities decays valid for any value of the coupling constant. In the particular case of weak coupling we obtain the well known perturbative formulas for the oscillator decay [18],

$$P_0(t) \approx e^{-2\pi g t}, \tag{80}$$

$$P_1(t) \approx 2e^{-\pi g t}(1 - e^{-\pi g t}) \tag{81}$$

and

$$P_2(t) \approx 1 - 2e^{-\pi g t} + e^{-2\pi g t}. \tag{82}$$

5 Concluding Remarks

In this paper we have analysed a symplified version of an atom-electromagnetic field system and we have tried to give the more exact and rigorous treatment we could to the problem. We have adopted a general physicist' s point of view, in the sense that we have rennounced to approach very closely to the real behaviour of a complicated non-linear system, to study instead a simple linear model. As a counterpart, an exact solution has been possible. Our dressed coordinates give a description of the behaviour of the system that is exact and valid for weak as well as for strong coupling. If the coupling between the mechanical oscillator and the field is weak, we recover the well known behaviour from perturbation theory.

6 In Memoriam

This paper evolved from umpublished work we have done and discussions we have had, with Prof. Guido Beck when two of us (A.P.C.M. and N.P.A.) were his students at Instituto de Fisica Balseiro in Bariloche (Argentina), in the late sixties and the early seventies. We dedicate this article to his memory.

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