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Vertex Operators, Semiclassical Limit for Soliton S-matrices and the Number of Bound States in Affine Toda Field Theories

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Abstract

Soliton time delays and the semiclassical limit for soliton S-matrices are calculated for non-simply laced Affine Toda Field Theories. The phase shift is written as a sum over bilinears on the soliton conserved charges. The results apply to any two solitons of any Affine Toda Field Theory. As a by-product, a general expression for the number of bound states and the values of the coupling in which the S-matrix can be diagonal are obtained. In order to arrive at these results, a vertex operator is constructed, in the principal gradation, for non-simply laced affine Lie algebras, extending the previous constructions for simply laced and twisted affine Lie algebras.

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1 Introduction.

In 1+1 dimensions, a well-known class of integrable theories are the Affine Toda Field Theories (ATFTs). For each affine Lie algebra $\widehat{g}$, we can associated an ATFT. For simplicity just the untwisted algebras will be considered.(The twisted cases were consided in [4].) If the coupling β is imaginary, there exist degenerate vacua and solitons interpolating these vacua. An N-soliton solution can be written as [1]

$$e^{-\beta\lambda_j\cdot\phi} = \frac{<\Lambda_j|g(t)|\Lambda_j>}{<\Lambda_0|g(t)|\Lambda_0>^{m_j}} \tag{1}$$

with

$$\begin{aligned} g(t) &= \prod_{k=1}^{N} e^{Q_{i(k)}W_{i(k)}\widehat{F}^{i(k)}(\theta)}\,, &(2)\\ W_{i(k)}(x,t) &= e^{\mu_{i(k)}\cosh\theta_k(x-v_kt)}\,, &(3)\\ Q_{i(k)} &= e^{i\psi_k}e^{-x_k^0\mu_{i(k)}\cosh\theta_k}\,, &(4)\end{aligned}$$

where θ_k is the rapidity of the $k^{\rm th}$ soliton, x_k^0 its position at $t=0$, ψ_k is a phase and $\mu_{i(k)}$is the mass particle of species $i(k)$. Let g be the underlying Lie algebra. For each dot of the Dynkin diagram of g a soliton species can be associated. Each species may have various different topological charges. For example, sine-Gordon, which is associated with $\widehat{g}=a_1^{(1)}$, has one soliton species with two topological charges: the well-known soliton and anti-soliton solutions. The mass for each species is[1]

$$M_i = \frac{4h\mu_i}{\alpha_i^2|\beta^2|}\,, \tag{5}$$

where h is the Coxeter number of g. From (1), we can equivalently write the N-soliton solution as

$$\phi = -\frac{1}{\beta}\sum_{j=0}^{r}\alpha_j^{\rm V}\ln <\Lambda_j|g(t)|\Lambda_j>$$

where α_0 is the negative of the hightest root. Written in this form, it becomes clear that $<\Lambda_j|g(t)|\Lambda_j>$ are the τ-functions which appear in the Hirota method, used in[7]-[9].

Quite a lot of work has been done in order to obtain the S-matrix for the particles[10]-[18] and for the solitons[19]-[23] of ATFT. As usual, the proposed S-matrix must be constructed in agreement with the S-matrix axioms. However, in order to confirm that the proposed S-matrix is associated with a given theory, one must check if it has the correct semiclassical limit.

The highest non-vanishing power of $\widehat{F}^i(\theta)$ has a vertex operator construction in the principal gradation which was first proven for level one representations [24][25][2] and then extended for any representation of simply laced [3] and twisted[4] affine Lie algebras. From this construction, it was shown[5] that any given soliton regains its original shape, after having collisions with other solitons, as expected by a soliton solution. The only effect is a time delay which was calculated. From the time delay, the semiclassical limit for the transmission amplitude of the simply laced soliton S-matrix was obtained[5] .

In the present paper, these results are extended for the remaining case of non-simply laced affine Lie algebras obtaining a semiclassical expression which holds for any affine Lie algebra. The paper is divided as follows: in section 2, after a brief review, it is shown that the non-simply laced case, the last non-vanishing power of $\widehat{F}^i(\theta)$ also has a vertex construction. Then, the asymptotic behavior of the solitons is analyzed in section 3 and the time delay resulting from the collision of two or more solitons is calculated in section 4. From the time delay, the semiclassical limit of the soliton S-matrix which holds for any ATFT is calculated, in section 5. The phase shift is written in terms of the soliton conserved charges. In section 6 a general expression for the number of bound states in the direct and cross channel of the S-matrix is obtained along with the values of the coupling in which the S-matrix can be purely elastic. Our conclusions are stated in section 7.

2 Vertex operator construction for non-simply laced affine Lie algebras.

In order to deal with soliton solitons in ATFT, it is convenient to consider an alternative basis for an affine Lie algebra $\widehat{g}$ (for simplicity, let us consider $\widehat{g}$ untwisted) with generators $\widehat{E}_M$, M being an exponent of $\widehat{g}$(i.e $M = mh + \mu$ where μ are exponents of g and $m \in Z$) , $\widehat{F}^i_N$, $N \in Z$ and the level x. Using the conventions of [1]-[5], they satisfy the commutation relations

$$\begin{aligned}\left[\widehat{E}_M, \widehat{E}_N\right] &= Mx\delta_{M+N,0}\,, & (6)\\ \left[\widehat{E}_M, \widehat{F}^i_N\right] &= \gamma_i \cdot q([M])\widehat{F}^i_{M+N}\,. & (7)\end{aligned}$$

where $[M]$ means M mod h, $q([M])$ is the eigenvector of the Coxeter element associated with the eigenvalue $\exp 2\pi i[M]/h$ and $\gamma_i = c_i\alpha_i$ with $c_i = \pm 1$ depending on the "color" of α_i. The generators which ad-diagonalize the principal Heisenberg subalgebra are

$$\widehat{F}^i(z) = a_i \sum_{N=-\infty}^{\infty} z^{-N}\widehat{F}^i_N\,, \tag{8}$$

z being a complex variable. The numbers a_i are normalization constants which can be fixed by imposing $< \Lambda_0|\widehat{F}^i(z)|\Lambda_0 >= 1$, implying that [4]

$$< \Lambda_j|\widehat{F}^i(z)|\Lambda_j >= e^{-2\pi i\lambda_i\cdot\lambda_j}\,. \tag{9}$$

From the commutation relation (7), it follows directly that

$$\left[\widehat{E}_M, \widehat{F}^i(z)\right] = \gamma_i \cdot q([M])z^M\widehat{F}^i(z)\,. \tag{10}$$

For a *simply-laced* $\widehat{g}$, in the representation with highest weight Λ_j of level m_j, the highest nonvanishing power of $\widehat{F}^i(z)$ has a vertex operator construction[2][3]

$$\widehat{V}^i(z) \equiv \frac{\widehat{F}^i(z)^{m_j}}{m_j!} = e^{-2\pi i\lambda_i\cdot\lambda_j}Y^i_-Y^i_+ \tag{11}$$

where

$$Y_{\pm}^{i} = \exp\left(\sum_{M\in\mathcal{E}} \frac{\gamma_i \cdot q([\pm M])z^{\mp M}}{\mp M} E_{\pm M}\right) .$$

with the sun running over the positive exponents $\mathcal{E}$ of $\widehat{g}$. The normal ordering of two vertex operators is [2][3]

$$\widehat{V}^i(z)\widehat{V}^k(w) = X_{ik}(z,w)^{m_j} : \widehat{V}^i(z)\widehat{V}^k(w) : \tag{12}$$

where

$$: \widehat{V}^i(z)\widehat{V}^k(w) := e^{-2\pi i(\lambda_i+\lambda_k)\cdot\lambda_j} Y_{-}^{i} Y_{-}^{k} Y_{+}^{i} Y_{+}^{k}$$

$$X_{<i><k>}(z_i, z_k) = \prod_{p=1}^{h} \left(1 - \omega^p \frac{z_k}{z_i}\right)^{\sigma^{-p}(\gamma_{<i>})\cdot\gamma_{<k>}} . \tag{13}$$

Let us now prove for non-simply laced affine Lie algebras, the highest non-vanishing power of $\widehat{F}^i(z)$ also admits a vertex operator construction. As is well-known [6], all non-simply laced Lie algebras can be obtained from a simply-laced Lie algebra g as a fixed subalgebra g_τ under an outer automorphism τ of g. The Dynkin diagram of g_τ, $\Delta(g_\tau)$, is obtained by identifying the vertices on each separate orbit of τ. The set of vertices in the orbit containing the vertex i will be denoted by $< i >$ and p_i stands for the number of vertices. Denoting by α_i and λ_i, the simple roots and fundamental weights of g, the g_τ simple roots are[26]

$$\alpha_{<i>} = \frac{1}{p_i} \sum_{i\in<i>} \alpha_i \tag{14}$$

and the g_τ fundamental coweights

$$\lambda_{<i>}^{\vee} = \sum_{i\in<i>} \lambda_i \tag{15}$$

have the correct inner products with the simple roots. From (14), it follows that

$$\frac{2}{\alpha_{<i>}^2} = p_i \;\;\Rightarrow\;\; \alpha_{<i>}^{\vee} = \sum_{i\in<i>} \alpha_i \, . \tag{16}$$

Turning to the affine Lie algebras, one can once more obtain all the non-simply laced $\widehat{g}_\tau$ as fixed subalgebras of simply-laced $\widehat{g}$ by an outer automorphism $\widehat{\tau}$. One notes that the exponents of $\widehat{g}_\tau$ are a subset of the exponents of $\widehat{g}$ and it was proven[2] that for this subset of exponents,

$$\widehat{\tau}(\widehat{E}_M) = \widehat{E}_M \, . \tag{17}$$

It was also shown that

$$\widehat{\tau}(\widehat{F}_N^i) = \widehat{F}_N^{\tau(i)} \, . \tag{18}$$

Therefore, the generators of $\widehat{g}$ which belong to the fixed subalgebra $\widehat{g}_\tau$ are

$$\left\{ \begin{array}{cc} \widehat{E}_M & \text{if M is an exponent of } \widehat{g}_\tau \\ \widehat{F}_N^{<i>} := \sum_{i\in<i>} \widehat{F}_N^i & N \in Z \end{array} \right. . \tag{19}$$

From (17), (18) and (10) it follows that

$$\gamma_{\tau(i)} \cdot q([M]) = \gamma_i \cdot q([M])\,. \tag{20}$$

Using this relation, (14) and the fact that all the roots in the orbit $< i >$ the have same color, it follows that for the generators (19),

$$\left[\widehat{E}_M, \widehat{F}_N^{<i>}\right] = \gamma_{<i>} \cdot q([M])\widehat{F}_{M+N}^{<i>}\,. \tag{21}$$

Then,

$$\widehat{F}^{<i>}(z) := \sum_{i\in<i>} \widehat{F}^i(z) \tag{22}$$

are in $\widehat{g}_\tau$ and

$$\left[\widehat{E}_M\,,\, \widehat{F}^{<i>}(z)\right] = \gamma_{<i>} \cdot q([M]) z^M \widehat{F}^{<i>}(z)\,, \tag{23}$$

for $\widehat{E}_M \in \widehat{g}_\tau$.

As it has been explained in [4], under a folding procedure, the inequivalent fundamental representations of $\widehat{g}$ with highest weights $\Lambda_j\,,\ \Lambda_{\tau(j)}\,,\ \Lambda_{\tau^2(j)}\,,$ ... become identified as a single fundamental representation of $\widehat{g}_\tau$ whose highest weight is denoted by $\Lambda_{<j>}$ and with level $m_{<j>} = m_j = m_{\tau(j)} = \ \dots$.

Recall that as $\widehat{g}$ is simply laced, in the $\widehat{g}$ representation with highest weight Λ_j, the highest non-vanishing power of $\widehat{F}^i(z)$ has the vertex operator construction (11). This remains true in the $\Lambda_{<j>}$ representation of $\widehat{g}_\tau$ as does $(F^i(z))^{m_{<j>}+1} = 0$. Since[2]

$$\left[\widehat{F}^i(z), \widehat{F}^{\tau(i)}(z)\right] = 0\,, \tag{24}$$

it follows that in the $\Lambda_{<j>}$ representation,

$$\left(\widehat{F}^{<i>}(z)\right)^{p_i m_{<j>}+1} = 0\,,$$

$$\widehat{V}^{<i>}(z) \equiv \frac{\left(\widehat{F}^{<i>}(z)\right)^{p_i m_{<j>}}}{(p_i m_{<j>})!} = \prod_{i\in<i>} \frac{\left(\widehat{F}^i(z)\right)^{m_{<j>}}}{m_{<j>}!}\,. \tag{25}$$

But each factor in the above product can be written as a vertex operator. Performing the normal ordering (12), using (15) and (16), it follows that

$$\widehat{V}^{<i>}(z) = (b_{<i>})^{m_j}\, e^{-2\pi i \lambda^{\mathrm{V}}_{<i>}\cdot\lambda_{<j>}} Y_-^{<i>} Y_+^{<i>} \tag{26}$$

where

$$Y_\pm^{<i>} = \exp\left(\sum_{M\in\mathcal{E}} \frac{\gamma^{\mathrm{V}}_{<i>} \cdot q([\pm M]) z^{\mp M}}{\mp M} \widehat{E}_{\pm M}\right)\,, \tag{27}$$

$$b_{<i>} = \prod_{0\leq p<q\leq p_i} X_{\tau^{p}(i)\tau^{q}(i)}(z,z)\,. \tag{28}$$

The sum in (27) runs over the positive exponents of $\widehat{g}$. However, in order for the vertex operator to make sense in the non-simply laced algebra $\widehat{g}_\tau$, the sum must be over its exponents. Let us see that this indeed happens. For the cases in which τ is of order two and M is not a degenerate exponent, $\widehat{\tau}(\widehat{E}_M) = -\widehat{E}_M$ [2], and therefore

$$\gamma_i \cdot q([M]) = -\gamma_{\tau(i)} \cdot q([M])\,. \tag{29}$$

This implies that $\gamma^{\rm V}_{<i>}.q([M])$ will vanish if $\widehat{E}_M$ does not belong to $\widehat{g}_\tau$. The same result can be obtained by explicit computation for the order three automorphism of $d_4^{(1)}$ and for the degenerate exponents of $d_{2n}^{(1)}$, by using the fact that $\gamma_i \cdot q([M])$ are proportional to the components of the eigenvectors of the Cartan matrix, which are well-known. Therefore, the above vertex operator construction makes sense in the non-simply laced algebra $\widehat{g}_\tau$, since the sum is restricted to its exponents.

With the above result, we conclude that for *any* affine Lie algebra, in a level m_j representation, the highest non-vanishing power of $\widehat{F}^i(z)$ has a vertex operator given by (26), with $p_i = 1$ and $b_i = 1$ for the simply laced and twisted algebras.

Using the definition (25), it is straightforward to check that, in the $\Lambda_{<j>}$ representation,

$$\left[\widehat{E}_M, \widehat{V}^{<i>}(z)\right] = m_{<j>}\gamma^{\rm V}_{<i>} \cdot q([M]) z^M \widehat{V}^{<i>}(z)\,. \tag{30}$$

The vertex operator (26) indeed satisfies this commutation relation. It is interesting to note that $\widehat{E}_M/m_{<j>}$ and $\widehat{V}^{<i>}(z)$ in the $\Lambda_{<j>}$ representation satisfy the same commutation relation as $\widehat{E}_M$ and $\widehat{F}^{<i>}(z)$ satisfy for the dual algebra $(\widehat{g}_\tau)^{\rm V}$.

Using (6), the normal ordering of two vertex operators can be performed, in the $\Lambda_{<j>}$ representation, giving

$$\widehat{V}^{<i>}(z_i)\widehat{V}^{<k>}(z_k) = X_{<i><k>}(z_i,z_k)^{m_{<j>}} : \widehat{V}^{<i>}(z_i)\widehat{V}^{<k>}(z_k) : \tag{31}$$

where

$$X_{<i><k>}(z_i,z_k) = \exp\left(-\sum_{N\in\mathcal{E}} \frac{\gamma^{\rm V}_i \cdot q([N\})^* \gamma^{\rm V}_k \cdot q([N])}{N} \left(\frac{z_k}{z_i}\right)^N\right) \tag{32}$$

Similarly to the simply laced case[2], $X_{<i><k>}$ can be also written as

$$X_{<i><k>}(z_i,z_k) = \prod_{p=1}^{h}\left(1-\omega^p \frac{z_k}{z_i}\right)^{\sigma^{-p}\left(\gamma^{\rm V}_{<i>}\right)\cdot\gamma^{\rm V}_{<k>}}\,, \tag{33}$$

where $\omega = \exp(2\pi i/h)$. Moreover, from the commutation relation (23) we find that

$$Y_+^{<i>}(z_i)\widehat{F}^{<k>}(z_k) = \widehat{F}^{<k>}(z_k)Y_+^{<i>}(z_i)X^{1/p_k}_{<i><k>}(z_i,z_k) \tag{34}$$

$$\widehat{F}^{<k>}(z_k)Y_-^{<i>}(z_i) = Y_-^{<i>}(z_i)\widehat{F}^{<k>}(z_k)X^{1/p_k}_{<k><i>}(z_k,z_i) \tag{35}$$

which will be very useful in the next sections. Using the fact that

$$\sum_{p=1}^{h} \sigma^p \gamma_k^{\vee} = 0$$

and

$$\sum_{p=1}^{h} p\sigma^p \gamma_k = h\left(\lambda_k + \Lambda_R(g)\right) ,$$

we find that $X_{<i><k>}$ has the symmetry property

$$X_{<i><k>}(z_i, z_k)^{1/p_k} = X_{<k><i>}(z_k, z_i)^{1/p_k}$$

It is very important to note that all results in this section also hold for simply laced algebras, just remembering that for these algebras the coroots (coweights) coincide with the roots(weights) and that the p_i's are one. For notational simplicity, from now on the brackets $< ... >$ will no longer be used.

3 Asymptotic behavior of soliton solutions.

Having obtained the vertex operator, the asymptotic behavior of the soliton solutions (1) can be checked. In order that one is able to put the one-soliton solution at rest, the variable z_k must take the form[1]

$$z_k = ie^{-\theta_k} e^{-i\pi\frac{(1+c(k))}{2h}} \tag{36}$$

where θ_k is the rapidity of the k^{th} soliton and $c(k)$ is the color associated to the k dot of the Dynkin diagram. Then, X_{ik} can be expressed as a function of $\theta_{ik} \equiv \theta_i - \theta_k$:

$$X_{ik}(\theta_{ik}) = \prod_{p=1}^{h} \left(e^{\theta_{ik}} - e^{\frac{i\pi}{2h}(4p+c(i)-c(k))}\right)^{\sigma^{-p}\gamma_i^{\vee}\cdot\gamma_k^{\vee}}$$

Now, since $\widehat{E}_M|\Lambda_j >= 0$ for $M > 0$, the *expectation value* for the vertex operator is

$$< \Lambda_j|\widehat{V}^i(z)|\Lambda_j >= b_i^{m_j} e^{-2\pi i\lambda_i^{\vee}\cdot\lambda_j} . \tag{37}$$

Then, the one-soliton solution *created* by the group element[1] $g(t) = exp(A_i\widehat{F}^i(\theta))$, $A_i \equiv Q_iW_i$, will take the form

$$e^{-\beta\lambda_j\cdot\phi} = \frac{1 + \cdots + b_i^{m_j} e^{-2\pi i\lambda_i^{\vee}\cdot\lambda_j} (A_i)^{m_j p_i}}{[1 + \cdots + b_i (A_i)^{p_i}]^{m_j}} , \tag{38}$$

where the dots indicate intermediate powers of A_i. These intermediate powers, whose coefficients we have not calculated, do not affect the asymptotic limits $x \to +\infty$ or $x \to -\infty$, which are equivalent to $A_i \to \infty$ or $A_i \to 0$, respectively. Thus:

$$e^{-\beta\lambda_j\cdot\phi} = \begin{cases} e^{-2\pi i\lambda_j\cdot\lambda_i^{\vee}} & x \to \infty \\ 1 & x \to -\infty \end{cases}$$

[1]No sum is assumed for the repeated indices.

In particular, this shows that, asymptotically, ϕ does approach one of the degenerate vacua. We can also conclude that the topological charge for the soliton *created* by $\widehat{F}^i(\theta)$ will satisfy

$$Q_{top} \equiv \phi(+\infty) - \phi(-\infty) = \frac{2\pi i}{\beta}\left(\lambda_i^{\mathrm{V}} + \Lambda_R(g_\tau^{\mathrm{V}})\right) .$$

Using the normal ordering (31), we obtain that the two-soliton solution created by $g(t) = \exp A_i\widehat{F}^i(z_i)\,\exp A_k\widehat{F}^k(z_k)$, takes the form

$$e^{-\beta\lambda_j\cdot\phi} = \frac{1 + \cdots + (b_i b_k)^{m_j} e^{-2\pi i(\lambda_i^{\mathrm{V}}+\lambda_k^{\mathrm{V}})\cdot\lambda_j}\left(A_i^{p_i}\, A_k^{p_k}\right)^{m_j} X_{ik}(\theta_{ik})^{m_j}}{\left[1 + \cdots + b_i b_k\, A_i^{p_i}\, A_k^{p_k} X_{ik}(\theta_{ik})\right]^{m_j}} ,$$

with the asymptotic limits $\exp\left[-2\pi i(\lambda_i^{\mathrm{V}} + \lambda_k^{\mathrm{V}})\cdot\lambda_j\right]$ and 1, confirming the expected result that the solution interpolates degenerate vacua, and with the topological charge satisfying

$$Q_{top} = \frac{2\pi i}{\beta}\left(\lambda_i^{\mathrm{V}} + \lambda_k^{\mathrm{V}} + \Lambda_R(g_\tau^{\mathrm{V}})\right) .$$

This argument is readily extended to more solitons.

4 Soliton time delays for non-simply laced ATFT.

In [5], the time delay or lateral displacement of the soliton trajectories arising from collisions has been obtained for simply laced ATFT. Now, we are in position to extend this result to the non-simply laced ATFT.

Consider a solution (1) with two solitons created by the group element

$$g(t) = e^{A_{i(2)}(\theta_2)\widehat{F}^{i(2)}(\theta_2)}e^{A_{i(1)}(\theta_1)\widehat{F}^{i(1)}(\theta_1)} \tag{39}$$

where we consider that $\theta_1 > \theta_2$. In order to obtain the time delay, we shall be *tracking* each soliton in time. By tracking a soliton "i" we mean remain in its vicinity, which is near $x = x_i^0 + v_i t$. Following this tracking procedure for soliton 1, which is described in detail in section 4 of [5], and using (34), (35), it follows that in the past $t \to \infty$, in the vicinity of soliton 1,

$$e^{-\beta\lambda_j\cdot\phi} \to \frac{<\Lambda_j|e^{A_{i(1)}\widehat{F}^{i(1)}(\theta_1)}|\Lambda_j>}{<\Lambda_0|e^{A_{i(1)}\widehat{F}^{i(1)}(\theta_1)}|\Lambda_0>^{m_j}} \tag{40}$$

which corresponds to an one-soliton solution of species $i(1)$, velocity v_1, initial position x_1^0 and phase ψ_1. On the other hand, in the future $t \to \infty$, in the vicinity of soliton 1,

$$e^{-\beta\lambda_j\cdot\phi} \to e^{-2\pi i\lambda_{i(2)}^{\mathrm{V}}\cdot\lambda_j}\frac{<\Lambda_j|e^{X_{i(1)i(2)}^{1/p_{i(1)}}(\theta_{12})A_{i(1)}\widehat{F}^{i(1)}(\theta_1)}|\Lambda_j>}{<\Lambda_0|e^{X_{i(1)i(2)}^{1/p_{i(1)}}(\theta_{12})A_{i(1)}\widehat{F}^{i(1)}(\theta_1)}|\Lambda_0>^{m_j}} . \tag{41}$$

Again, we recognize once more an one-soliton of species $i(1)$, velocity v_1 and phase ψ_1. However, the factor $X_{i(1)i(2)}^{1/p_{i(1)}}$, which is real and positive, changes the modulus of $A_{i(1)}$ and hence x_1^0 (see (4)). The effect is

$$\mu_{i(1)}\cosh\left(x - x_1^0 - v_1 t\right) \to \mu_{i(1)}\cosh\left(x - x_1^0 - v_1 t\right) + \frac{1}{p_{i(1)}}\ln X_{i(1)i(1)}(\theta_{12}) . \tag{42}$$

So, soliton 1 regains its original shape after the collision, and the only effect of the collision is that solution (40) differs from (41) by a translation in space-time. More precisely, the lateral displacement of soliton 1, due to its collision with soliton 2, $\Delta_{12}x$, satisfies

$$E_1\Delta_{12}x = -\frac{M_{i(1)}}{p_{i(1)}\mu_{i(1)}}\ln X_{i(1)i(2)}(\theta_{12}) = -\frac{2h}{|\beta^2|}\ln X_{i(1)i(2)}(\theta_{12})\,, \tag{43}$$

where $E_1 = M_{i(1)}\cosh\theta_1$ is the energy of soliton 1 and (16) has been used as well as the mass formula (5). The time delay $\Delta_{12}t$ of soliton 1 with momentum P_1, is obtained from

$$P_1\Delta_{12}t = -E_1\Delta_{12}x\,.$$

Following [5], one can repeat the procedure by tracking the slower soliton 2, from which it follows that

$$E_2\Delta_{21}x = \frac{2h}{|\beta^2|}\ln X_{i(1)i(2)}(\theta_{12}) = -E_1\Delta_{12}x\,, \tag{44}$$

as expected, where $\Delta_{21}x$ is the lateral displacement of soliton 2 due to its collision with soliton 1. Therefore, we can combine both results as

$$P_i\Delta_{ik}t = \mathrm{sign}(\theta_i - \theta_k)\frac{2h}{|\beta^2|}\ln X_{ik}(\theta_{ik})\,. \tag{45}$$

It is not difficult to extend this procedure from the collision of two solitons to the collision of any number of solitons. Like [5], it results for the m^{th}soliton that

$$E_m\Delta_m x = \frac{2h}{|\beta^2|}\left(\sum_{v_k>v_m}\ln X_{i(m)i(k)}(\theta_{mk}) - \sum_{v_k<v_m}\ln X_{i(m)i(k)}(\theta_{mk})\right)$$

where $\Delta_m x$ is the total spatial displacement for the m^{th} soliton. The left summation are the contributions from the solitons faster than m and the right summation are the contributions from the slower solitons.

5 Semiclassical limit for the soliton S-matrix.

In 1+1 dimensions, the leading term, in semiclassical approximation, of the 2-state transmission S-matrix elements, can be obtained from the time delays, as was shown in [27]. They proved that for a collision of states α and β with total energy E in the centre of momentum frame, the transmission S-matrix element has the form

$$S_{\alpha\beta} = \exp\left(\frac{i\phi_{\alpha\beta}}{\hbar} + O(\hbar)\right) \tag{46}$$

with

$$\frac{d\phi_{\alpha\beta}}{dE} = \Delta_{\alpha\beta}t\,.$$

Rewriting this derivative in terms of the relative rapidity, $\theta \equiv |\theta_\alpha - \theta_\beta|$ yields

$$\frac{d\phi_{\alpha\beta}}{d\theta} = \mathrm{sgn}(\theta_\alpha - \theta_\beta) P_\alpha \Delta_{\alpha\beta} t \,.$$

Consider the collision of two solitons of species "i" and "k" with respective topological charges "a" and "b". Putting (45) in the last equation, one gets

$$\phi_{ia,kb}(\theta) = \phi_{ia,kb}(0) + \frac{2h}{|\beta^2|} \int_0^\theta d\eta \ln X_{ik}(\eta) \,. \tag{47}$$

Note that since the time delay does not depend on the soliton topological charges, then only the integration constant $\phi_{ia,kb}(0)$ may depend on them.

Let us denote by $x_i^{(\nu)}$ and $y_i^{(\nu)} = \alpha_i^2 x_i^{(\nu)}/2$ the components of the left and right eigenvectors of the Cartan matrix of g_τ associated to the common eigenvalue $4\sin^2(\pi\nu/2h)$ (ν being a exponent of g_τ) and satisfying

$$x_i^{(\nu)} y_i^{(\mu)} = \delta_{\mu\nu} \quad , \quad x_i^{(\nu)} y_j^{(\nu)} = \delta_{ij} \,. \tag{48}$$

The vectors $x_i^{(\nu)}$ satisfy[2]

$$c_i x_i^{(\nu)} = x_i^{(h-\nu)} \tag{49}$$

and the inner product $\gamma_i^{\mathrm{V}} \cdot q(\nu)$ can be written in terms of $x_i^{(\nu)}$ as [15]

$$\gamma_i^{\mathrm{V}} \cdot q(\nu) = -i\sqrt{2h}\, e^{\frac{i\pi\nu}{2h}(1+c(i))} x_i^{(\nu)} \,. \tag{50}$$

Using this result, (36) and the exponential form (32) for the $X_{ik}(\eta)$ function, the above integral can be written as

$$\phi_{ia,kb}(\theta) = \phi_{ia,kb}(0) + \frac{4h^2}{|\beta^2|} \sum_{N\in\mathcal{E}} \overline{x}_i^{(N)} \overline{x}_k^{(N)} \frac{\left(e^{-N\theta} - 1\right)}{N^2} \tag{51}$$

where $\overline{x}_i^{(\nu+nh)} \equiv [-c(i)]^n x_i^{(\nu)}$. This result can be substantially improved: using (49), the θ dependent term takes the form

$$\sum_\nu x_i^{(\nu)} x_k^{(\nu)} \left[\frac{e^{-\nu\theta}}{\nu^2} + \sum_{m=1}^{\infty} \left(\frac{e^{-(2mh+\nu)\theta}}{(2mh+\nu)^2} + \frac{e^{-(2mh-\nu)\theta}}{(2mh-\nu)^2} \right) \right] , \tag{52}$$

where ν are the exponents of the underlying Lie algebra. If we put $\theta = 0$, we recognize the term inside the brackets as the series expansion on simple fractions of the $\sin^{-2}$ function[28], resulting that

$$\sum_{N>0} \frac{\overline{x}_i^{(N)} \overline{x_k}^{(N)}}{N^2} = \frac{\pi^2}{h^2} \sum_\nu \frac{x_i^{(\nu)} x_k^{(\nu)}}{4\sin^2\frac{\pi\nu}{2h}} = \frac{\pi^2}{h^2} \lambda_i^{\mathrm{V}} \cdot \lambda_k^{\mathrm{V}} \,, \tag{53}$$

where the last equality can be checked directly just using the fact that $x^{(\nu)}$ are eigenvectors of the Cartan matrix.

[2] No sum is assumed for the repeated indices

The infinite series (52) can be written as a finite sum of polylogarithm functions, $\mathrm{Li}_m(y) \equiv \sum_{n=1}^{\infty} y^n/n^m$, $|y| < 1$, by using the identity[3]

$$\frac{1}{h}\sum_{p=1}^{h} e^{i\pi\frac{p\nu}{h}}\mathrm{Li}_m\left(e^{-\frac{i\pi p}{h}}x\right) = \sum_{n=1}^{\infty}\frac{x^{nh+\nu}}{(nh+\nu)^m}\,.$$

Then,

$$\begin{aligned}\phi_{ia,kb}(\theta) &= \phi_{ia,kb}(0) - \frac{4\pi^2}{|\beta^2|}\lambda_i^{\mathrm{V}}\cdot\lambda_k^{\mathrm{V}} + \\ &\quad + \frac{4h^2}{|\beta^2|}\sum_{\nu} x_i^{(\nu)}x_k^{(\nu)}\left(\frac{e^{-\nu\theta}}{\nu^2} + \frac{1}{2h}\sum_{p=1}^{h}\cos\frac{\pi p\nu}{2h}\mathrm{Li}_2\left(e^{-\frac{i\pi p}{h}-2\theta}\right)\right)\,. \end{aligned} \tag{54}$$

Alternatively, the phase $\phi_{ia,kb}(\theta)$ can be written in terms of the infinite soliton conserved charges. Indeed, the conserved charges for the one-soliton solution *created* by $expQ\widehat{F}^i(\theta)$ are [29]

$$P_i^{\pm N}(\theta_i) = -\frac{\mu^N}{|\beta^2|}\gamma_i^{\mathrm{V}}\cdot q([\pm N])z_i^{\pm N} = \mp\frac{\mu^N}{|\beta^2|}\sqrt{2h}\overline{x}_i^{(N)}e^{\pm N\theta_i}\,,\;\; N>0\,.$$

Then,

$$\phi_{ia,kb}(\theta) = \phi_{ia,kb}(0) - \frac{4\pi^2}{|\beta^2|}\lambda_i^{\mathrm{V}}\cdot\lambda_k^{\mathrm{V}} - 2h|\beta^2|\sum_{N\in\mathcal{E}}\frac{P_i^{-N}(\theta_i)P_k^N(\theta_k)}{\mu^{2N}N^2} \tag{55}$$

for $\theta_i > \theta_k$. This shows an intimate relationship between soliton S-matrix elements and soliton conserved charges of ATFTs. Indeed, as pointed out in [11][30], the phase shift of any purely elastic exact S-matrix for an integrable theory could be written as a sum over bilinears on the *quantum* conserved charges. In the next section, the values of the coupling in which the soliton S-matrix can be purely elastic are obtained. So for these values of the coupling, one could expect that an expression like this for the phase shift would be exact, with P_i^N being the quantum conserved charges and making use of the substitution (58), given in the next section. For example, for sine-Gordon, the integral representation of *exact* phase shift of the soliton-soliton transmission amplitude[19] is

$$\begin{aligned}\frac{\phi_{ss}(\theta)}{\hbar} &= -\frac{1}{2i}\int_{-\infty}^{\infty}\frac{dt}{t}e^{\frac{2i\theta t}{\pi}}\frac{\sinh(1-\gamma)t}{\sinh\gamma t\cosh t} \\ &= \frac{1}{2i}\left[\int_{-\infty}^{\infty}\frac{dt}{t}e^{\frac{2i\theta t}{\pi}} - \int_{-\infty}^{\infty}\frac{dt}{t}e^{\frac{2i\theta t}{\pi}}\tanh t\coth\gamma t\right]\,,\end{aligned}$$

where

$$\gamma = \frac{\hbar|\beta^2|}{4\pi}\left[1-\frac{\hbar|\beta^2|}{4\pi}\right]^{-1}\,. \tag{56}$$

[3]The author thanks D. Olive for pointing out this identity.

Using the residue theorem it can be written as

$$\frac{\phi_{ss}(\theta)}{\hbar} = \epsilon\frac{\pi}{2}\left(1-\frac{1}{\gamma}\right) - \epsilon\left(\sum_{k=1}^{\infty}\frac{2e^{-\epsilon(2k-1)\theta}}{(2k-1)}\cot\frac{\gamma(2k-1)\pi}{2} - \sum_{k=1}^{\infty}\frac{e^{-\epsilon\frac{2k\theta}{\gamma}}}{k}\tan\frac{k\pi}{\gamma}\right)$$

where $\epsilon = \mathrm{sgn}\ \mathrm{Re}\,\theta$. One clearly sees that for $\gamma = 1/n$, $n = 2, 3, ...$, when the exact reflection amplitude vanishes, the last summation vanishes and the *exact* phase shift takes a form similar[4] to (51) or (55) , with the first term of the series expansion of the cotangent given by the semiclassical term. From this expression of the phase shift one gets for free the sine Gordon soliton quantum charges.

It is interesting to note that the ATFT particle S-matrix, which is purely elastic for any value of the coupling, has a phase shift [17] similar to (55) but with the conserved charges being proportional to the right eingenvector $y_i^{(\nu)}$instead of $x_i^{(\nu)}$. Therefore, for the values of the coupling in which the soliton S-matrix is purely elastic, there is an intriguing similarity between the $\hat{g}$-ATFT soliton S-matrix and the $\hat{g}^{\mathrm{V}}$-ATFT particle S-matrix. Clearly for each particle species there is a soliton species with many possible topological charges.

6 Number of bound states and purely elastic regime.

From the above expression for the phase shift, it follows that

$$\begin{aligned} \phi_{ia,\overline{kb}}(0) + \phi_{ia,kb}(0) - \phi_{ia,\overline{kb}}(\infty) - \phi_{ia,kb}(\infty) &= \frac{4\pi^2}{|\beta^2|}\lambda_i^{\mathrm{V}}\cdot\left(\lambda_k^{\mathrm{V}} + \lambda_{\bar{k}}^{\mathrm{V}}\right) \\ &\equiv n_{ik}(\beta^2)\pi\hbar\,, \end{aligned} \tag{57}$$

where $\overline{kb}$ means the antisoliton of the soliton of species k and topological charge b. Following the super-Levinson theorem [27], the largest integer smaller than $n_{ik}(\beta^2)$ gives the number of bound states in both, the direct and the cross channels. Note that for sine-Gordon ($g_\tau = a_1^{(1)}$) $\lambda_1 = 1/\sqrt{2}$, and we recover the well-known result[5] that the number of bound states is the largest integer smaller than $n_{11} = 4\pi/\hbar|\beta^2|$, in the semiclassical approximation. In order to obtain the exact result, one should replace[31] $\hbar|\beta^2|/4\pi$ by γ given in (56). In view of the arguments in [33], one would expect that a similar substitution

$$\frac{\hbar|\beta^2|}{4\pi} \to \gamma \equiv \frac{\hbar|\beta^2|}{4\pi}\left[\frac{h}{h^{\mathrm{V}}} - \frac{\hbar|\beta^2|}{4\pi}\right]^{-1}. \tag{58}$$

in (57) would give the exact number of bound states.

These bound states should correspond to poles in the exact S-matrix associated not just to breathers but also for the fusing of soliton solutions, which appear as poles of $X_{ik}(\theta)$ [5] and are governed by Dorey's fusing rule[13]. Some of the bound states were analysed[22] for some particular algebras and particular colliding soliton species.

[4]Remembering that the exponents of a_1^1 are the odd integers.

[5]The fact that we are adopting the convention $\alpha_i^2 = 2$ for the longest simple roots results that our expression differ by a factor 1/2 from the standard expression.

By direct inspection, one can conclude that

$$\lambda^{\vee}_{\bar{k}} = -\lambda^{\vee}_{k} + \Lambda_R(g_\tau)\,. \tag{59}$$

This result is due to the fact that[3]

$$\lambda^{\vee}_{\bar{k}} = -\sigma_0(\lambda^{\vee}_{k})$$

where σ_0 is the very special element of the Weyl group of $g^{\vee}_{\tau}$, which is the longest one and maps the positive Weyl chamber into its negative. Note that if we consider

$$\hbar|\beta^2| = \frac{4\pi}{n}\,, \tag{60}$$

where n is a natural number then, as a consequence of (59), n_{ik} are natural numbers for any i and k (and if $\hbar|\beta^2| > 4\pi$, $0 > n_{ik} > 1$, all bound states become unbound). These are exactly the values of the coupling in which the soliton S-matrix for sine-Gordon becomes purely elastic[32] in the semiclassical approximation. Now we shall show that same is true for the other ATFTs.

In two dimensions, a two-state S-matrix must satisfy the unitarity and crossing relations

$$\begin{aligned} S^{\mu\nu}_{\alpha\beta}(\theta) &= \left[S^{-1}(-\theta)\right]^{\mu\nu}_{\alpha\beta}\,, & (61)\\ S^{\mu\nu}_{\alpha\beta}(i\pi-\theta) &= S^{\mu\overline{\beta}}_{\alpha\overline{\nu}}(\theta)\,. & (62)\end{aligned}$$

Equivalently, one can combine both relations and obtain

$$S^{\mu\nu}_{\alpha\beta}(i\pi+\theta) = \left[S^{-1}(\theta)\right]^{\mu\overline{\beta}}_{\alpha\overline{\nu}}\,. \tag{63}$$

In the semiclassical approximation, we must use the crossing relation involving the analytic continuation $\theta \to \theta + i\pi$ rather than $\theta \to i\pi - \theta$, since the semiclassical approximation breaks down on the imaginary axis, as was pointed out by Coleman[34].

Considering that the S-matrix is purely elastic, conditions (61) and (63) become

$$\begin{aligned} S_{\alpha\beta}(\theta)S_{\alpha\beta}(-\theta) &= 1\,, & (64)\\ S_{\alpha\beta}(i\pi+\theta) &= S^{-1}_{\alpha\overline{\beta}}(\theta)\,, & (65)\end{aligned}$$

where $S_{\alpha\beta}(\theta) \equiv S^{\alpha\beta}_{\alpha\beta}(\theta)$. Taking $\theta = 0$ in the first relation, one finds that $S_{\alpha\beta}(0) = \pm 1$. Then, using the semiclassical expression (46), it follows that

$$\phi_{\alpha\beta}(0) = N_{\alpha\beta}\pi\hbar \tag{66}$$

where $N_{\alpha\beta}$ must be an integer number and gives the number of bound states in the direct channel[27].

Let us see the constraint the crossing relation imposes on the phase shift of ATFT. Before doing this, recall the fact that[15] $c_{\overline{k}} = (-1)^h c_k$ and[13] $x^{(\nu)}_{\overline{k}} = (-1)^{\nu+1}x^{(\nu)}_k$. Then, it follows from the definition of $\overline{x}^{(N)}_k$, that

$$\overline{x}_k^{(N)} e^{-i\pi N} = -\overline{x}_{\overline{k}}^{(N)}\,.$$

This relation implies that

$$\phi_{ia,kb}(\theta + i\pi) = \phi_{ia,kb}(0) - \frac{4\pi^2}{|\beta^2|}\lambda_i^{\rm V} \cdot \lambda_k^{\rm V} - \frac{4h^2}{|\beta^2|}\sum_{N\in\mathcal{E}} \frac{\overline{x}_i^{(N)}\overline{x}_{\overline{k}}^{(N)}}{N^2} e^{-N\theta}\,.$$

Then, the crossing condition (65) results that

$$\phi_{ia,\overline{kb}}(0) = -\phi_{ia,kb}(0) + \frac{4\pi^2}{|\beta^2|}\lambda_i^{\rm V} \cdot \left(\lambda_k^{\rm V} + \lambda_{\overline{k}}^{\rm V}\right) \quad ({\rm mod} 2\pi\hbar) \tag{67}$$

Using (59) and the fact that $\phi_{ia,\overline{kb}}(0)$ and $\phi_{ia,kb}(0)$ must satisfy (66), it implies that the above equation can only be consistent if the coupling satisfies the condition (60). Therefore, only for these values of the coupling the unitarity and crossing relations for purely elastic S-matrix is fulfilled. So, the S-matrix can only be purely elastic for these values in the semiclassical approximation. Once more, this result holds at the semiclassical approximation and in order to obtain the exact result one should probably perform a substitution like (58). Comparing relation (67) with the super-Levinson relation (57) implies that $\phi_{ia,\overline{kb}}(\infty) + \phi_{ia,kb}(\infty) = 0 \ {\rm mod}(2\pi\hbar)$. It would be interesting if one could calculate the constants $\phi_{ia,kb}(0)$, which gives the number of bound states in the direct channel[27].

7 Conclusions.

In this paper, we have extended our vertex operator construction for non-simply laced affine Lie algebras from which we obtained the time delay resulting from the collision of two (or more) solitons. From the time delay, we obtained the semiclassical limit for the transmission amplitude of the two soliton S-matrix which holds for any ATFT. The semiclassical phase shift appeared as a sum over bilinears on the soliton conserved charges. Using the super-Levison theorem, an universal expression for the number of bound states in the direct and cross channels of the S-matrix was obtained. We also obtained the values of the coupling in which the S-matrix can be purely elastic. For these values, one would expect that the form of the exact phase shift would be like (55), with P_N^i being the *quantum* soliton conserved charges.

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