Quantization of the Relativistic Fluid in Physical Phase Space on Kähler Manifolds

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Abstract

We discuss the quantization of a class of relativistic fluid models defined in terms of one real and two complex conjugate potentials with values on a Kähler manifold, and parametrized by the Kähler potential $K(z, \overline{z})$ and a real number λ. In the hamiltonian formulation, the canonical conjugate momenta of the potentials are subjected to second class constraints which allow us to apply the symplectic projector method in order to find the physical degrees of freedom and the physical hamiltonian. We construct the quantum theory for that class of models by employing the canonical quantization methods. We also show that a semiclassical theory in which the Kähler and the complex potential are not quantized has a highly degenerate vacuum. Also, we define and compute the quantum topological number (quantum linking number) operator which has non-vanishing contributions from the Kähler and complex potentials only. Finally, we show that the vacuum and the states formed by tensoring the number operators eigenstates have zero linking number and show that linear combinations of the tensored number operators eigenstates which have the form of entangled states have non-zero linking number.

Key-words: Relativistic fluids; Kähler manifolds; Hamiltonian quantization.

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1   Introduction

The recent interest in the lagrangian and hamiltonian formulation of the relativistic fluid mechanics with non-zero vorticity has been motivated by a large range of applications such as: the neutron star dynamics [1], the representation of the massless axion field in cosmology [2], the Kalb-Ramond and global string induced superfluid vorticity [3, 4], and the relationship between the superstrings, supermembranes and the superfluid dynamics [5, 6, 7, 8].

The analysis of the vorticity in the lagrangian formalism requires a particular parametrization of the fluid potentials. In the case of the non-relativistic fluid, this is achieved by the Clebsch parametrization of the velocity potential $\mathbf{v} = \nabla\theta + \alpha\nabla\beta$, where α, β (the Gauss potentials) and θ are scalar fields [9]. In the hamiltonian formulation, the Clebsch parametrization allows to express the kinematic helicity as a surface integral and to realize canonically the non-vanishing brackets between the variables ρ (matter density field) and $\mathbf{v}$ [10]. In [6], it was proposed a different parametrization of the fluid potentials in which the vortex dynamics is expressed in terms of grassmannian coordinates and the vortex is associated with the spin. That parametrization has been used for the generalization of the Chaplygin gas model in $d = 2+1$ and $d = 3+1$ dimensions [11](see for an introduction and reviews [12, 13].)

Another very interesting proposal for a parametrization of the fluid potentials was put forward in [14], where the potentials are complex coordinates on an arbitrary Kähler manifold. That approach allows to construct interesting models of supersymmetric fluids which, on their turn, shade some light on the problem of the superfluid phases of certain supersymmetric systems [15, 16, 17]. The parametrization of the potentials in terms of complex coordinates preserves the property of the Clebsch parametrization, namely to elliminate the obstruction to building a consistent lagrangian which could exist due to a Chern-Simons term that is necessary in order to consider the non-zero vorticity [12]. The complex parametrization can be introduced instead of the Clebsch parametrization already at the non-supersymmetric level and the full structure of the divergence free currents and the topological charges associated to non-zero vorticity can be obtained in this way [14]. An important feature of the complex parametrization is that the hamiltonian dynamics is governed by a set of simple second-class constraints among the fluid degrees of freedom. In this paper, we propose a quantized theory for a large set of non-supersymmetric relativistic fluid models on Kähler manifolds.

The fluid properties of the system under consideration are described by a set of fields that represent the fluid potentials and which, together with their canonically conjugate momenta, can be organized to form a canonical phase space. The potentials and their momenta are subjected to second class constraints which form a set of simple algebraic equations. As a first result, we will obtain the physical degrees of freedom of the largest set of classical fluid models by applying the symplectic projector method developed in [18, 19, 20] and more recently employed in the study of the extended abelian Chern-Simons theory [21], the noncommutative open string [22], the Maxwell-Chern-Simons theory [23], the Lorentz-symmetry violation [24], quantum gravity potentials [25] and cosmological perturbations [26] (see for reviews [27, 28].) Althought equivalent to other methods used to solve the second class constraints, the symplectic projector method has the advantage of giving a geometrical picture to the relationship between the set of fields and their canonical conjugate momenta on one hand and the set of physical degrees of freedom defined by local coordinates on the constraint surface on the other hand. Compared with

the literature, our results reproduce exactly the general classical hamiltonian from [14]. Moreover, there are some new results consisting in the explicit identification of the physical degrees of freedom and of their relationship with the linearly dependent fields. The main result of this paper is the quantization of a smaller class of models parametrized by the Kähler potential $K(z, \overline{z})$ and a real parameter λ. That corresponds to choosing the arbitrary potential function on the fluid density $f(\rho)$ of the form $f(\rho) = \lambda \rho^2/2$ by which a set of quantizable fluid models is selected. The larger class of classical models is parametrized by $K(z, \overline{z})$ and the general potential functions $f(\rho)$ and, in general, it is not suitable for quantization due to the arbitrary form each of these two functions can take. Our approach to the quantum fluid is based on the canonical quantization performed in the physical phase space of the relativistic fluid. However, since the Kähler and the complex potentials determine a $d = 3$ vector field $\mathbf{A}(z, \overline{z})$ associated to the conserved currents, we quantize the field $\mathbf{A}$ rather than the complex fields z and $\overline{z}$. Due to the large arbitrariness in choosing the Kähler potential $K(z, \overline{z})$, there is no simple relationship between the quantized fields $\mathbf{A}$ and z and $\overline{z}$. That suggest that one could also consider a semiclassical situation in which the potential $\mathbf{A}$ is kept classical and just the canonical pair θ and π_θ is quantized. However, it turns out that in this case the Fock space is highly degenerate having all one-particle states proportional to the vacuum. Therefore, the quantization of the complex potentials is imposed, and quantizing the field $\mathbf{A}$ represents the simplest approach to this problem. The last new result of this paper is the calculation of the quantum topological charge (or quantum linking number) operator $\widehat{\omega}$. This is constructed from the normal ordered expression of the classical topological charge. The normal ordering is necessary because there are terms in $\widehat{\omega}$ which have the general form $\delta^3(0) k^n$, where k is any component of the momentum and $n = 1, 2, 3$. These terms are undetermined in the low momentum limit and vanish otherwise. We discuss the structure of the Fock space with respect to $\widehat{\omega}$ and show that the vacuum and the number operator eigenstates have zero linking number while the linear combination of tensored number operators eigenstates have non-zero topological charge [1].

The paper is organized as follows. In Section 2 we briefly review the general relativistic fluid on Khäler manifolds in the hamiltonian formulation from [14]. Also, we determine the physical degrees of freedom which are the local coordinates on the constraint manifold by the symplectic projector method and write down the expression of the physical hamiltonian and the conserved charges in term of these. The physical hamiltonian obtained by applying the symplectic method coincides with the one given in the literature. In Section 3 we quantize the particular class of models parametrized by $K(z, \overline{z})$ and λ. We give the structure of the Fock space and show that if the vector potential $\mathbf{A}$ is considered as a classical field, the Fock space of the semiclassical quantized fluid is highly degenerate. In Section 4 we discuss the quantum topological operator and determine the general form of the states with non-vanishing linking number. The last section is devoted to conclussions.

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[1] The quantum fluid states with non-zero linking number are actually entangled number operators eigenstates.

2 Relativistic Fluid in the Hamiltonian Formulation

In this section we are going to review the non-supersymmetric perfect relativistic fluid studied in [14]. Also, we are going to derive the physical degrees of freedom in a new way by applying the symplectic projector method developped in [27, 28]. That is equivalent, of course, to solving the constraint equations in a purely algebraic way as was done in [14]. However, the symplectic projector method provides a simple geometrical picture for the relationship among potentials, physical degrees of freedom and constraint surface.

In the class of models under consideration, the relativistic fluid is characterized by the equations of state for the local physical quantities p, ε, and ρ which are the pressure, the energy density and the local fluid density, respectively. The dynamics conserves the energy-momentum tensor $T_{\mu\nu}$ and the fluid density current j^μ

$$\partial^\mu T_{\mu\nu} = 0, \qquad \partial_\mu j^\mu = 0, \tag{1}$$

where

$$T_{\mu\nu} = p\eta_{\mu\nu} + (\varepsilon + p)u_\mu u_\nu, \qquad j^\mu = \rho u^\mu. \tag{2}$$

Here, $\eta_{\mu\nu} = (-, +, +, +)$ is the Minkowski metric, $u^\mu = dx^\mu/d\tau$ is the velocity four-vector with $u_\mu^2 = -1$ and τ is the proper time along the flow line of the current. From the current conservation given in the equations (1), one can see that there are three independent current components to which one can assign three fluid potentials $(\theta, z, \bar{z})$ that play the role of the Lagrange multipliers in the lagrangian formulation of the theory. As was argued in [14], one can take θ real, z complex and $\bar{z}$ the complex conjugate of z. One can choose the complex potentials to parametrize an arbitrary Kähler manifold characterized by the Kähler potential $K(z, \bar{z})$ which is a real function on z and $\bar{z}$. The above conservation equations can be obtained from the following lagrangian density [14]

$$\mathcal{L}[j^\mu, \theta, \bar{z}, z] = -j^\mu \left(\partial_\mu \theta + i\partial K \partial_\mu z - i\overline{\partial} K \partial_\mu \bar{z} \right) - f(\rho), \tag{3}$$

where $\partial K = \partial_z K$, $\overline{\partial} K = \partial_{\bar{z}} K$ and $f(\rho)$ is some potential function on ρ. Also, one can see from the definition of the fluid density current that $\rho = \sqrt{-j^2}$. The action is invariant under the spacetime translations and the potential fields reparametrizations. The corresponding conservation laws are: the conservation of the energy-momentum tensor and the conservation of the fluid density current as given in the equations (1), and the conservation of an infinity of reparametrization currents $J_\mu[G] = -2G(\bar{z}, z)j_\mu$, where $G(\bar{z}, z)$ are arbitrary analytic functions on z and $\bar{z}$. Beside the reparametrization currents given above, there are conserved axial currents generated by conserved topolgical charges defined by the following relation [14]

$$\omega = -2i \int d^3x \, \partial_i \left[\varepsilon^{ijk} \theta \partial \overline{\partial} K \partial_j \bar{z} \, \partial_k z \right]. \tag{4}$$

The charges ω can be interpreted as the linking number of vertices formed in the fluid.

The fluid flow and the non-zero vorticity can be described in the hamiltonian formalism, too. Following [14], we define the canonically conjugate momenta as

follows

$$\pi_\mu = \left.\frac{\partial\mathcal{L}}{\partial u^\mu}\right|_\rho = \rho\left(\partial_\mu\theta + i\partial K\partial_\mu z - i\overline{\partial}K\partial_\mu\bar{z}\right), \qquad \pi_\theta = \frac{\partial\mathcal{L}}{\partial\partial_0\theta} = j_0, \tag{5}$$

$$\pi_z = \frac{\partial\mathcal{L}}{\partial\partial_0 z} = i\partial K j_0, \qquad \pi_z = \frac{\partial\mathcal{L}}{\partial\partial_0\bar{z}} = -i\overline{\partial}K j_0. \tag{6}$$

Note that the currents j_μ do not appear dynamically in the theory and that the axial current depends locally on π_μ. Therefore, the relevant phase space for the physical degrees of freedom is the reduced phase space $(\theta, z, \bar{z}, \pi_\theta, \pi_z, \pi_z)$. The equations (6) represent a set of two second class constraints in the reduced phase space

$$\Omega_1 = \pi_z - i\partial K\pi_\theta = 0, \qquad \Omega_2 = \pi_{\bar{z}} + i\overline{\partial}K\pi_\theta = 0. \tag{7}$$

The physical degrees of freedom of the relativistic fluid can be obtained from the reduced phase space potentials by applying the symplectic projector method by which the reduced phase space is projected on to the constraint surface defined by the relations (7) [20, 27, 28]. Let us introduce the following notation for the potentials and their momenta

$$\{\xi_i\} = \{\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6\} = \{\theta, z, \bar{z}, \pi_\theta, \pi_z, \pi_z\}. \tag{8}$$

The local coordinates on the constraint surface $\{\xi_i^*\}$ are obtained by applying the symplectic projector Λ to the reduced phase space fields $\{\xi_i\}$

$$\xi_i^*(x^0, \mathbf{x}) = \int d^3\mathbf{y}\sum_{j=1}^{6}\Lambda_i^j(x^0, \mathbf{x}, \mathbf{y})\xi_j(x^0, \mathbf{y}), \tag{9}$$

where the general form of the symplectic projector is given by the following relation

$$\Lambda_j^i(x^0, \mathbf{x}, \mathbf{y}) = \delta_j^i\delta^3(\mathbf{x} - \mathbf{y}) - J^{ik}\int d^3\mathbf{z}d^3\mathbf{w}\frac{\delta\Omega_\alpha(x^0, \mathbf{z})}{\delta\xi_k(x^0, \mathbf{x})}D_{\alpha\beta}^{-1}(x^0, \mathbf{z}, \mathbf{w})\frac{\delta\Omega_\beta(x^0, \mathbf{w})}{\delta\xi_j(x^0, \mathbf{y})}. \tag{10}$$

Here $J^{ik} = -J^{ki}$ is the symplectic matrix of the reduced phase space and $D_{\alpha\beta}^{-1}$, $\alpha, \beta = 1, 2$, is the inverse of the Dirac matrix of the constraint brackets computed at equal times. It is a simple exercise to show that the symplectic projector has the following expression

$$\Lambda(x^0, \mathbf{x}, \mathbf{y}) = \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{\overline{\partial}K}{2\partial\overline{\partial}K\xi_4} & \frac{\partial K}{2\partial\overline{\partial}K\xi_4} \\ 0 & \frac{1}{2} & 0 & -\frac{\overline{\partial}K}{2\partial\overline{\partial}K\xi_4} & 0 & \frac{i}{2\partial\overline{\partial}K\xi_4} \\ 0 & 0 & \frac{1}{2} & -\frac{\partial K}{2\partial\overline{\partial}K\xi_4} & -\frac{i}{2\partial\overline{\partial}K\xi_4} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}\delta^3(\mathbf{x} - \mathbf{y}). \tag{11}$$

From the above relation, one concludes that the number of local coordinates $\{\xi_i^*\}$ on the constraint surface is equal to the number of unprojected fields $\{\xi_i\}$. However, since now the system is on the constraint surface, one can use the constraints $\{\Omega_\alpha\}$ to express the coordinates ξ_5^* and ξ_6^* as functions of ξ_4^*. This lefts us with the correct number of six physical degrees of freedom ξ^*'s which have been obtained from the

ten real fields ξ's acted upon by two complex constraint equations Ω_α. The physical hamiltonian density is given in terms of the linearly independent fields from $\{\xi_i^*\}$

$$\mathcal{H}^* = \mathbf{j} \cdot \left(\boldsymbol{\nabla}\xi_1^* + i\partial K \boldsymbol{\nabla}\xi_2^* - i\overline{\partial}K\boldsymbol{\nabla}\xi_3^* \right) + f\left(\sqrt{(\xi_4^*)^2 - \mathbf{j}^2} \right). \tag{12}$$

By using the field redefinition (8) one can easily see that the expression (12) is exactly the hamiltonian obtained in [14] by a different method. Also note that the currents $J^\mu[G]$ generate now the reparametrization symmetry of the physical subspace.

3   Quantization of the Relativistic Fluid

The theory presented in the previous section describes a large class of relativistic fluid models parametrized by the Kähler potential $K(z,\overline{z})$ and the function $f\left(\sqrt{(\xi_4^*)^2 - \mathbf{j}^2} \right)$, respectively. Both of these functions can have a quite general form which makes addressing the issue of quantization difficult, if not impossible, in the general case. A less general but still interesting class of relativistic fluid models can be obtained by choosing a simple form for $f(\rho)$ while leaving the Kähler manifold arbitrary. The equations of state of the reduced set of models have the following form [14]

$$p = \varepsilon = \frac{\lambda}{2}\rho^2 \quad , \quad f(\rho) = \frac{\lambda}{2}\rho^2. \tag{13}$$

These models are parametrized by the Kähler potential $K(z,\overline{z})$ and the real number λ. From the equations of motion for j_l, $l = 1, 2, 3$, one can determine the explicit form of these currents in terms of the reduced phase space fields as follows

$$\frac{\partial \mathcal{H}^*}{\partial j_l} = 0 \implies j_l = \frac{1}{\lambda}\left(\partial_l \xi_1^* + \frac{i}{2}\partial K \partial_l \xi_2^* - \frac{i}{2}\overline{\partial}K\partial_l\xi_3^* \right). \tag{14}$$

By applying the symplectic projector formalism employed in the previous section and by using the equations (14), the following physical hamiltonian is obtained

$$\mathcal{H}^* = \frac{1}{2\lambda}\left(\boldsymbol{\nabla}\xi_1^* + \frac{i}{2}\partial K \boldsymbol{\nabla}\xi_2^* - \frac{i}{2}\overline{\partial}K\boldsymbol{\nabla}\xi_3^* \right)^2 + \frac{\lambda}{2}(\xi_4^*)^2. \tag{15}$$

Note that $\mathcal{H}^*$ is identical to the hamiltonian calculated in [14] for the models parametrized by $\{K(z,\overline{z}),\lambda\}$.

The quantization of the relativistic fluid described by the hamiltonian (15) can be performed in the canonical approach. Let us return to the original field notations $\{\theta, \pi_\theta, z, \overline{z}\}$ where the fields are from the physical phase subspace. The canonical conjugate variables are θ and π_θ, while z and $\overline{z}$ do not propagate. Let us introduce the following real vector potential field

$$\mathbf{A}(K, z, \overline{z}) \equiv \mathbf{A}(x) = \frac{i}{2}\partial K \boldsymbol{\nabla}z - \frac{i}{2}\overline{\partial}K\boldsymbol{\nabla}\overline{z}. \tag{16}$$

From the Hamilton equations, we obtain the following equation of motion for the θ potential

$$\left(\partial^0\partial_0 + \boldsymbol{\nabla}^2 \right)\theta(x^0, \mathbf{x}) = -\boldsymbol{\nabla} \cdot \mathbf{A}(x^0, \mathbf{x}). \tag{17}$$

This equation shows that the physical phase subspace of the relativistic fluid is equivalent to that of a massless scalar field θ moving inside the potential created by $K(z, \overline{z})$ and the fluid potentials z and $\overline{z}$. Actually, by assuming that $\left|\theta(x^0, \mathbf{x})\mathbf{A}(x^0, \mathbf{x})\right| \rightarrow 0$ as $|x^0| \rightarrow \infty$, one can write the hamiltonian (15) as

$$\mathcal{H}^* = \int dx^0 d\mathbf{x} \left[\frac{\lambda}{2} \pi_\theta^2 + \frac{1}{2\lambda} (\boldsymbol{\nabla}\theta)^2 + V(\theta, \mathbf{A}) \right], \tag{18}$$

where

$$V(\theta, \mathbf{A}) = -\frac{1}{\lambda} \left[\theta \boldsymbol{\nabla} \cdot \mathbf{A} - \frac{1}{2} (\mathbf{A})^2 \right]. \tag{19}$$

For $\boldsymbol{\nabla} \cdot \mathbf{A} \neq 0$, the function V has a zero for each field configuration that satisfies

$$\theta_0 = \frac{1}{2} \frac{(\mathbf{A})^2}{\boldsymbol{\nabla} \cdot \mathbf{A}}. \tag{20}$$

At the points from the physical phase subspace where the equations (20) is satisfied, the relativistic fluid is described by the scalar potential θ only, and it is equivalent to a free massless scalar field. Actually, the vanishing of the gradient of the vector potential signals an extremum of V. The extrema of V in the θ direction and in the $\mathbf{A}$ directions, respectively, are given in terms of physical degrees of freedom by the following equations

$$\partial_j \left(\partial K \partial_j z - \overline{\partial} K \partial_j \overline{z} \right) = 0, \qquad i \left(\partial K \partial_j z - \overline{\partial} K \partial_j \overline{z} \right) = 2\partial_j \theta. \tag{21}$$

The potential V takes at these points the following values

$$V_1 = \frac{1}{2\lambda} (\mathbf{A})^2 \quad, \quad V_2 = -\frac{1}{2\lambda} \left(\theta \partial_j \partial^j \theta - \partial_j \theta \partial^j \theta \right). \tag{22}$$

If the extremum value is obtained in all directions of physical phase subspace simultaneously, then by equating V_1 and V_2 we obtain the following relationship among the fluid potentials

$$(\partial K)^2 \nabla z \cdot \nabla z + \left(\overline{\partial} K \right)^2 \nabla \overline{z} \cdot \nabla \overline{z} - 2\partial K \overline{\partial} K \nabla z \cdot \nabla \overline{z} + 2\theta (\nabla)^2 \theta - 2\nabla \theta \cdot \nabla \theta = 0. \tag{23}$$

The above equation describes the fluid configurations for which the contribution to the fluid energy from the Kähler and the complex fluid potentials and from their interaction with the scalar fluid potential is extreme.

For the classical models described above, the reparametrization charges $Q[G]$ have a simple form in terms of physical phase space coordinates

$$Q[G(\xi^*)] = \int d^3x\, G(z, \overline{z}) \pi_\theta. \tag{24}$$

Therefore, it is easy to show that the charges $Q[G]$ are conserved provided that the following relation holds

$$\int_\Sigma d\mathbf{s} \cdot (\nabla \theta + \mathbf{A}) = 0, \tag{25}$$

where $d\mathbf{s}$ is the area element of Σ which is a spacelike surface at the spatial infinity.

Since the theory is formulated in the hamiltonian formalism, it is possible to study the quantum fluctuation of the relativistic fluid in the canonical quantization. To this end, we interpret θ and $\mathbf{A}$ as field operators with the dynamics given by the equation (17) and replace the Poisson brackets on the constraint surface by the

corresponding commutators. In what follows we are going to use the symbol $\widehat{}$ to denote the operators. Then the equation of motion of the quantum field $\widehat{\theta}$ is

$$\left(\partial^0 \partial_0 + \mathbf{\nabla}^2\right) \widehat{\theta}(x^0, \mathbf{x}) = -\mathbf{\nabla} \cdot \widehat{\mathbf{A}}(x^0, \mathbf{x}). \tag{26}$$

The operators $\widehat{\theta}$ and $\widehat{\mathbf{A}}$ can be decomposed in terms of plane waves in the usual fashion

$$\widehat{\theta}(x^0, \mathbf{x}) = \int d^3 k N_{\mathbf{k}}^{\theta} e^{i\mathbf{k}\cdot\mathbf{x}} \widehat{a}_{\mathbf{k}}(x^0), \tag{27}$$

$$\widehat{\mathbf{A}}(x^0, \mathbf{x}) = \int d^3 k N_{\mathbf{k}}^{\mathbf{A}} e^{i\mathbf{k}\cdot\mathbf{x}} \widehat{\mathbf{b}}_{\mathbf{k}}(x^0). \tag{28}$$

Since the equation of motion of $\widehat{\theta}$ in the absence of the vector potential $\widehat{\mathbf{A}}$ is the same as for a free massless scalar field, we consider that $\widehat{\theta}$ is massless. Also, since $\widehat{\mathbf{A}}$ is constructed from non-propagating fields [2], we consider the simplest situation where $\mathbf{A}$ is a massless field, too. Then it is possible to set the normalization constants $N_{\mathbf{k}}^{\theta} = N_{\mathbf{k}}^{\mathbf{A}} = N_{\mathbf{k}}$. By plugging the relations (27) and (28) into (26), we obtain the following set of equations for the operators $\widehat{a}_{\mathbf{k}}(x^0)$ and $\widehat{\mathbf{b}}_{\mathbf{k}}(x^0) = \left\{ \widehat{b}_{n\mathbf{k}}(x^0) \right\}, n = 1, 2, 3,$

$$\left(\partial_0^2 + \mathbf{k}^2\right) \widehat{a}_{\mathbf{k}}(x^0) = i\mathbf{k} \cdot \widehat{\mathbf{b}}_{\mathbf{k}}(x^0), \tag{29}$$

for all $\mathbf{k}$. The general solution of the equation (29) can be written as

$$\widehat{a}_{\mathbf{k}}(x^0) = \widehat{a}_{\mathbf{k}}^{(1)}(x^0) e^{-\frac{i\omega_{\mathbf{k}}}{c} x^0} + \widehat{a}_{\mathbf{k}}^{(2)}(x^0) e^{\frac{i\omega_{\mathbf{k}}}{c} x^0}, \tag{30}$$

$$\widehat{\mathbf{b}}_{\mathbf{k}}(x^0) = \widehat{\mathbf{b}}_{\mathbf{k}}^{(1)}(x^0) e^{-\frac{i\omega_{\mathbf{k}}}{c} x^0} + \widehat{\mathbf{b}}_{\mathbf{k}}^{(2)}(x^0) e^{\frac{i\omega_{\mathbf{k}}}{c} x^0}. \tag{31}$$

Since the classical fluid potential are real functions, i. e. $\theta = \overline{\theta}$ and $\mathbf{A} = \overline{\mathbf{A}}$, it follows that the corresponding quantum fields are hermitian. Therefore, the following relations hold

$$\left(\widehat{a}_{\mathbf{k}}^{(1)}\right)^{\dagger} = \widehat{a}_{-\mathbf{k}}^{(2)}, \left(\widehat{\mathbf{b}}_{\mathbf{k}}^{(1)}\right)^{\dagger} = \widehat{\mathbf{b}}_{-\mathbf{k}}^{(2)}. \tag{32}$$

By using the hermiticity condition (32), one can write the final form of the plane wave expanssion for the physical field operators

$$\widehat{\theta}(x^0, \mathbf{x}) = \int d^3 k N_{\mathbf{k}} \left[\widehat{a}_{\mathbf{k}} e^{i\left(\mathbf{k}\cdot\mathbf{x} - \frac{\omega_{\mathbf{k}}}{c} x^0\right)} + \widehat{a}_{\mathbf{k}}^{\dagger} e^{-i\left(\mathbf{k}\cdot\mathbf{x} - \frac{\omega_{\mathbf{k}}}{c} x^0\right)} \right], \tag{33}$$

$$\widehat{\pi}_{\theta}(x^0, \mathbf{x}) = -\frac{i}{\lambda c} \int d^3 k N_{\mathbf{k}} \left[\widehat{a}_{\mathbf{k}} e^{i\left(\mathbf{k}\cdot\mathbf{x} - \frac{\omega_{\mathbf{k}}}{c} x^0\right)} - \widehat{a}_{\mathbf{k}}^{\dagger} e^{-i\left(\mathbf{k}\cdot\mathbf{x} - \frac{\omega_{\mathbf{k}}}{c} x^0\right)} \right], \tag{34}$$

$$\widehat{A}_n(x^0, \mathbf{x}) = \int d^3 k N_{\mathbf{k}} \left[\widehat{b}_{n\mathbf{k}} e^{i\left(\mathbf{k}\cdot\mathbf{x} - \frac{\omega_{\mathbf{k}}}{c} x^0\right)} + \widehat{b}_{n\mathbf{k}}^{\dagger} e^{-i\left(\mathbf{k}\cdot\mathbf{x} - \frac{\omega_{\mathbf{k}}}{c} x^0\right)} \right], \tag{35}$$

where $\widehat{A}_{n\mathbf{k}}$ is the n-th component of $\widehat{\mathbf{A}}$. The normalization constant $N_{\mathbf{k}}$ can be determined by postulating the canonical equal-time commutators

$$\left[\widehat{\theta}(x^0, \mathbf{x}), \widehat{\pi}_{\theta}(x^0, \mathbf{x}') \right] = i\hbar \delta^3(\mathbf{x} - \mathbf{x}'), \tag{36}$$

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[2] The field $\mathbf{A}(K, z, \overline{z})$ has rather a geometric character since it constains the information about the Kähler space parametrized by the complex potentials z and $\overline{z}$.

and by defining the usual commutators among the creation and annihilation operators

$$\left[\widehat{a}_{\mathbf{k}}, \widehat{a}_{\mathbf{k}'}^{\dagger}\right] = \delta^3(\mathbf{k} - \mathbf{k}'), \left[\widehat{b}_{n\mathbf{k}}, \widehat{b}_{m\mathbf{k}'}^{\dagger}\right] = \delta_{nm}\delta^3(\mathbf{k} - \mathbf{k}') \tag{37}$$

$$\left[\widehat{a}_{\mathbf{k}}, \widehat{b}_{n\mathbf{k}'}\right] = [\widehat{a}_{\mathbf{k}}, \widehat{a}_{\mathbf{k}'}] = \left[\widehat{b}_{n\mathbf{k}}, \widehat{b}_{m\mathbf{k}'}\right] = 0. \tag{38}$$

By using the relations (36), (37) and (38), one can show that the normalization constant has the following form

$$N_{\mathbf{k}} = \left[\frac{\lambda \hbar c}{2\omega_{\mathbf{k}}(2\pi)^3}\right]^{\frac{1}{2}}. \tag{39}$$

The Fock space states can be constructed from the vacuum state that is annihilated to zero by all annihilation operators in the known way

$$\widehat{a}_{\mathbf{k}}\left|0\right\rangle = \widehat{b}_{n\mathbf{k}}\left|0\right\rangle = 0, n = 1, 2, 3, \tag{40}$$

for all $\mathbf{k}$. Since the classical fluid is invariant under spacetime translations, it is natural to impose the invariance of the vacuum under translations

$$\widehat{p^{\mu}}\left|0\right\rangle = p^{\mu}\left|0\right\rangle. \tag{41}$$

The physical states are obtained by acting with the creation operators on the vacuum state. For example, the one-particle excitations of the quantum fluid potentials are described by the following states

$$\left|\mathbf{k}\right\rangle_{\theta} = \widehat{a}_{\mathbf{k}}^{\dagger}\left|0\right\rangle \quad, \quad \left|k_n\right\rangle_{\mathbf{A}} = \widehat{b}_{n\mathbf{k}}^{\dagger}\left|0\right\rangle, n = 1, 2, 3. \tag{42}$$

In this way, the canonical quantization can be carried out straightforwardly to the relativistic fluid models described by the hamiltonian (15). However, there are some differences from the general field quantization due to the quantum equation of motion (26) that should be satisfied operatorially on all quantum states, and to some arbitrariety in defining the quantum structure of the vector potential $\widehat{\mathbf{A}}$. Indeed, the equation (26) takes the following form in terms of creation and annihilation operators

$$\left(-\frac{\omega_{\mathbf{k}}^2}{c^2} + \mathbf{k}^2\right)\left(\widehat{a}_{\mathbf{k}} + \widehat{a}_{-\mathbf{k}}^{\dagger}\right)\left|\psi\right\rangle = i\mathbf{k} \cdot \left(\widehat{\mathbf{b}}_{\mathbf{k}} + \widehat{\mathbf{b}}_{\mathbf{k}}^{\dagger}\right)\left|\psi\right\rangle. \tag{43}$$

Note that for $\mathbf{k} = 0$ either $\omega_{\mathbf{0}} = 0$ or $\left(\widehat{a}_{\mathbf{0}} + \widehat{a}_{\mathbf{0}}^{\dagger}\right)\left|\psi\right\rangle = 0$. If $\left|\psi\right\rangle = \left|0\right\rangle$ then (43) is equivalent to the following relation

$$\left(-\frac{\omega_{\mathbf{k}}^2}{c^2} + \mathbf{k}^2\right)\left|\mathbf{k}\right\rangle_{\theta} = i\sum_{n=1}^{3} k_n \left|k_n\right\rangle_{\mathbf{A}}. \tag{44}$$

That shows that for $\mathbf{k} \neq 0$ a different normalization of $\widehat{b}_{n\mathbf{k}}^{\dagger}\left|0\right\rangle$ can be considered

$$\widehat{b}_{n\mathbf{k}}^{\dagger}\left|0\right\rangle = -ik_n\left(\frac{\left|\mathbf{k}\right|^2 c^2 - \omega_{\mathbf{k}}^2}{\left|\mathbf{k}\right|^2 c^2}\right)\left|\mathbf{k}\right\rangle_{\mathbf{A}}, \tag{45}$$

where $\left|\mathbf{k}\right\rangle_{\mathbf{A}} = \sum_{n=1}^{3} \mathbf{e}_n \left|k_n\right\rangle_{\mathbf{A}}$ and $\{\mathbf{e}_n\}$, $n = 1, 2, 3$ are orthogonal unit vectors on the spacelike surface. The equations (44) with the normalization (45) is identically satisfied by the dispersion relation of the massless scalar field $\omega_{\mathbf{k}}^2 = \left|\mathbf{k}\right|^2 c^2$.

In the above quantization, the unpropagating complex fluid potentials z and $\overline{z}$ and the Kähler potential field $K(z,\overline{z})$ were not quantized directly, but rather under the form of the vector potential $\mathbf{A}$. Therefore, in principle one could also consider the possibility of the semiclassical quantized fluid in which the field $\mathbf{A}$ is a classical vector potential and the dynamical canonical pair (θ, π_θ) is quantized. However, by repeating the steps performed above in the canonical quantization formalism, but now with $\hat{\mathbf{A}} = \mathbf{A}\hat{\mathbf{1}}$, where $\hat{\mathbf{1}}$ is the identity operator, we arrive at the following relation among the one-particle excitations and the vacuum state

$$|\mathbf{k}\rangle_\theta = i\left(\mathbf{k}^2 - \frac{\omega^2}{c^2}\right)\mathbf{k}\cdot\mathbf{b_k}|0\rangle_\theta. \tag{46}$$

That shows that if the Fourier coefficients $b_{n\mathbf{k}}$ are classical functions on z and $\overline{z}$, then the Fock space is highly degenerate with all one-particle states proportional to the vacuum. Thus, we conclude that if the relativistic fluid model of the type discussed here is to be treated as a quantum system, the vector potential $\mathbf{A}$ should be quantized as before in order to avoid the infinite vacuum degeneracy.

4   Quantum Topological Charge

In this section we are going to construct the quantum topological charge (quantum linking number) operator which is the quantum counterpart of the classical linking number, and to discuss the properties of the Fock space states with respect to it.

The starting point is the classical topological linking number ω defined in the relation (4). After some algebraic manipulations, it can be put in the following form

$$\omega = \int d^3x \varepsilon^{lmn}\left(\partial_l\theta\partial_m\partial_n\theta + 2\partial_l\theta\partial_m A_n + 2A_l\partial_m\partial_n\theta + 4A_l\partial_m A_n\right). \tag{47}$$

Note that the first and the third terms from the above relation vanish due to presence of the totally antisymmetric tensor in the integrand, while the second term vanishes up to a total derivative. Let us define the quantum topological charge operator $\widehat{\omega}$ by interpreting the fields in (47) as quantum operators and taking the normal ordering of the creation and annihilation operators in the mode expansion of the quantum fields inside the integral. The ordering prescription is necessary due to the presence of potentially divergent terms, e.g. $\delta^3(0)k_lk_m\varepsilon^{lmn}$, which for arbitrary value of k_n are zero and for $|k| \longrightarrow 0$ have an undetermined limit. They are a consequence of the commutation relation (37) among the oscillator operators of $\widehat{\theta}$ field. By using the field expansion relations (33), (34) and (35) inside the relation (47), the quantum topological charge operator can be decomposed in the following sum

$$:\widehat{\omega}(x^0): \ = 2(2\pi)^3\left(:\widehat{\omega}_2(x^0): \ + 2i:\widehat{\omega}_4(x^0):\right), \tag{48}$$

The operators $:\widehat{\omega}_a(x^0):$, $a = 2, 4$ from the r. h. s. of the above relation represent the operatorial counterpart of the second term (a total derivative contribution0 and the last term from the relation (47) in that order. After some calculations, one arrives at the following explicit form of the operators $:\widehat{\omega}_a(x^0):$ written in terms of

creation and annihilation operators

$$: \widehat{\omega}_2(x^0): \; = \int d^3k N_{\mathbf{k}}^2 \varepsilon^{lmn} \left[\widehat{b}^\dagger_{n\mathbf{k}} \widehat{a}_{\mathbf{k}} + \widehat{a}^\dagger_{\mathbf{k}} \widehat{b}_{n\mathbf{k}} + \widehat{a}_{\mathbf{k}} \widehat{b}_{n\mathbf{k}} e^{-\frac{2i\omega_{\mathbf{k}}}{c} x^0} + \widehat{a^\dagger}_{\mathbf{k}} \widehat{\mathbf{b}^\dagger}_{n\mathbf{k}} e^{\frac{2i\omega_{\mathbf{k}}}{c} x^0} \right] k_l k_m,$$
(49)

$$: \widehat{\omega}_4(x^0): \; = \int d^3k N_{\mathbf{k}}^2 \varepsilon^{lmn} \left[\widehat{b}^\dagger_{l\mathbf{k}} \widehat{b}_{n\mathbf{k}} - \widehat{b}^\dagger_{n\mathbf{k}} \widehat{b}_{l\mathbf{k}} - \widehat{b}_{l\mathbf{k}} \widehat{b}_{n\mathbf{k}} e^{-\frac{2i\omega_{\mathbf{k}}}{c} x^0} + \widehat{b}^\dagger_{l\mathbf{k}} \widehat{b}^\dagger_{n\mathbf{k}} e^{\frac{2i\omega_{\mathbf{k}}}{c} x^0} \right] k_m.$$
(50)

Since the components of the momentum k_n commute with any operator, one can see that $: \widehat{\omega}_2 :$ vanishes due to the presence of the totally antisymmetric tensor. The same symmetry arguments do not apply to $: \widehat{\omega}_4 :$ which is not zero. However, the time dependent terms vanish in $: \widehat{\omega}_4 :$. Indeed, by using the commutation relations (37) in the terms $\varepsilon^{lmn} \widehat{b}_{l\mathbf{k}} \widehat{b}_{n\mathbf{k}} k_m$ and $\varepsilon^{lmn} \widehat{b}^\dagger_{l\mathbf{k}} \widehat{b}^\dagger_{n\mathbf{k}} k_m$, one obtains vanishing operator coefficients for each component k_n. Therefore, the topological charge operator of the quantum relativistic fluid from the class of models discussed here has the form

$$: \widehat{\omega}: \; = 2i\lambda\hbar c \int \frac{d^3k}{\omega_{\mathbf{k}}} \varepsilon^{lmn} k_m \left(\widehat{b}^\dagger_{l\mathbf{k}} \widehat{b}_{n\mathbf{k}} - \widehat{b}^\dagger_{n\mathbf{k}} \widehat{b}_{l\mathbf{k}} \right),$$
(51)

where the dependence on x^0 has been dropped from $: \widehat{\omega}:$. Remark that the quantum topological number operator is parametrized by the parameter λ linearly and that it is expressed in $\hbar c$ units. Its dependence on the Kähler and complex potentials is hidden in the integrand and has a simple form due to the quantization of $\mathbf{A}$ field. The fields $\widehat{\theta}$ and $\widehat{\pi}_\theta$ do not contribute to the topological charge operator that is time independent.

In order to see how $: \widehat{\omega}:$ acts on the Fock space, we consider a general state of fixed momentum vector $\mathbf{k}' = \left(k_1', k_2', k_3' \right)$ of the form

$$\left| \Psi \left(k_1', k_2', k_3' \right) \right\rangle = \left| n_1 \left(k_1' \right), n_2 \left(k_2' \right), n_3 \left(k_3' \right) \right\rangle = \bigotimes_{q=1}^{3} \left| n_q \left(k_q' \right) \right\rangle.$$
(52)

Here, $n_q \left(k_q' \right)$ denotes the number n_q of excitations of b_q type which have the value of momentum k_q' in the $q = 1, 2, 3$ direction. The last equality in (52) shows that the state factorizes as a tensor product of number operator eigenstates in the corresponding directions [3]. Also, we consider that the number of excitations in each direction is fixed in the state (52) and that the number operator eigenstates are ortogonal and normalized to unity

$$\left\langle n_l (k_l) | m_s \left(k_s' \right) \right\rangle = \delta_{l,s} \delta_{n_l, m_s} \delta \left(k_l - k_s' \right).$$
(53)

It is easy to see that the expectation value of $: \widehat{\omega}:$ is zero in the states of the form (52) because of the fixed number of excitations in each direction one starts with. The vacuum state is of the form (52) and thus one concludes that the vacuum has zero linking number as expected.

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[3] The dependence of the state given in the relation (52) on the excitations of the scalar potential $\widehat{\theta}$ does not affect the calculation of the topological charge since $\widehat{\theta}$ does not contribute to the topological number operator.

Since the fixed number of excitations in the state (52) together with the ortogonality relation (53) make the topological charge number vanish, one could take instead the more general linear combination of states that have the following form

$$|\Psi\rangle = \int \frac{d^3k'}{(2\pi)^3} \sum_{n_1,n_2,n_3=1}^{3} C_{n_1,n_2,n_3}\left(k_1',k_2',k_3'\right) \left|n_1\left(k_1'\right), n_2\left(k_2'\right), n_3\left(k_3'\right)\right\rangle, \quad (54)$$

where the arbitrary complex coefficient functions $C_{n_1,n_2,n_3}\left(k_1',k_2',k_3'\right)$ can be normalized to make the integral covariant if necessary. The states from the above relations are entangled states of tensored number operators eigenstates. One can show that the expectation value $\langle :\widehat{\omega}: \rangle$ in the states of the form (54) is given by the following relation

$$\langle :\widehat{\omega}: \rangle = 4i\lambda\hbar c \int \frac{d^3k}{\omega_{\mathbf{k}}} \sum_{n_1,n_2,n_3=1}^{3} C_{n_1,n_2,n_3}(k_1,k_2,k_3)$$

$$\times \Big\{ k_1 \Big[\sqrt{n_2(n_3+1)}\overline{C}_{n_1,n_2-1,n_3+1}(k_1,k_2,k_3) - \sqrt{n_3(n_2+1)}\overline{C}_{n_1,n_2+1,n_3-1}(k_1,k_2,k_3) \Big]$$

$$+ k_2 \Big[\sqrt{n_1(n_3+1)}\overline{C}_{n_1+1,n_2,n_3-1}(k_1,k_2,k_3) - \sqrt{n_3(n_1+1)}\overline{C}_{n_1-1,n_2,n_3+1}(k_1,k_2,k_3) \Big]$$

$$+ k_3 \Big[\sqrt{n_1(n_2+1)}\overline{C}_{n_1-1,n_2+1,n_3}(k_1,k_2,k_3) - \sqrt{n_2(n_1+1)}\overline{C}_{n_1+1,n_2-1,n_3}(k_1,k_2,k_3) \Big] \Big\},$$

$$(55)$$

Some comments are in order now. Firstly, note that the quantum topological number given in the relation (55) is different from zero if the integral does not vanish. This condition should be satisfied by a large set of arbitrary coefficients $C_{n_1,n_2,n_3}\left(k_1',k_2',k_3'\right)$ which thus define the states with non-vanishing quantum topological linking number. Secondly, one can see from the classical state equations of the model (13) and the relation (55), that there is a large number of quantum states in which the quantum linking number is infinite due to the infinite value of the classical limit of the pressure density or the energy density, i. e. for $p \to \infty$ or $\varepsilon \to \infty$, respectively. On the other hand, for $p \to 0$ or $\varepsilon \to 0$, the topological number is zero, unless the integral and sum in the r. h. s. of (55) diverge in the corresponding state. Thirdly, note that in general either the quantum linking number is zero or it is time independent as expected. This result expresses the conservation of the expectation values of the linking number operator.

5   Conclussions

In this paper, we have investigated the quantization of the relativistic fluid on Kähler manifolds. The class of models considered here are parametrized by an arbitrary Kähler potential depending on two complex fluid potentials z and $\overline{z}$ and a real parameter λ. That type of models represents a subset of a larger set parametrized by $\{K(z,\overline{z}), f(\rho)\}$ that was firstly proposed in [14]. Due to the arbitrary dependence of the lagrangian on $\rho = \sqrt{\pi_\theta^2 - \mathbf{j}^2}$, the full set of relativistic fluid models is not suitable for quantization. As was shown in [14], the degrees of freedom of the full set $\{K(z,\overline{z}), f(\rho)\}$ are constrained by second class constraints. As a first result, we have obtained the physical degrees of freedom of the fluid described by $\{K(z,\overline{z}), f(\rho)\}$ by applying the symplectic projector method. We have concluded that the classical

theory on the physical surface displays the same topological charges as the original theory [14] since the symplectic projector does not include the conserved axial currents. Our results for the full set of classical relativistic fluids agree with the ones presented in the literature which revalidates the applicability of the symplectic projector method to second class constraints.

The main result of this paper is the quantization of the smaller set of models $\{K(z,\overline{z}),\lambda\}$. We have obtained the quantum theory by applying the canonical quantization methods to the pair of fields (θ,π_θ) as well as to the vector field $\mathbf{A}(K,z,\overline{z})$ which encodes the information about the Kähler and the complex fluid potentials, respectively, and we have constructed the Fock space of the relativistic fluid and the one-particle excitations of the relativistic potentials. Also, we have discussed the semiclassical quantization of the relativistic fluid in which the potential $\mathbf{A}(K,z,\overline{z})$ is a classical field. By analysing the one-particle spectrum, it has been shown that the vacuum of the semiclassical theory is infinitely degenerate. From that, one concludes that $\widehat{\mathbf{A}}$ should be treated as a quantum field.

The second important result of the present paper is the construction of the quantum linking number operator $\widehat{\omega}$ which was defined by taking the normal ordered field products in the r. h. s. of the relation (47). The operator $\widehat{\omega}$ is time independent and it is determined only by the Kähler and the complex potentials, with no contribution from the real potential. We have shown that the vacuum state has vanishing linking number, as well as the states formed by taking the tensor product of number operators eigenstates of $\widehat{\mathbf{A}}$ field. However, there are entangled number operator eigenstates with non-vanishing linking number.

As a final comment, we note that the classical topological number can be expressed as a surface term as in the relation (4). It would be certainly interesting to compare the approach presented in this paper with a different quantization method and to attempt a proper treatment of the boundaries in the quantum theory. That analysis and the application of the present method to the supersymmetric fluid [6] in the Kähler parametrization will hopefully be discussed elsewhere.

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