Area Law for Localization-Entropy in Local Quantum Physics

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Abstract

The previously developed algebraic lightfront holography is used in conjunction with the tensor splitting of the chiral theory on the causal horizon. In this way a universal area law for the entanglement entropy of the vacuum relative to the split (tensor factorized) vacuum is obtained. The universality of the area law is a result of the kinematical structure of the properly defined lightfront degrees of freedom.

We consider this entropy associated with causal horizon of the wedge algebra in Minkowski spacetime as an analog of the quantum Bekenstein black hole entropy similar to the way in which the Unruh temperature for the wedge algebra may be viewed as an analog in Minkowski spacetime of the Hawking thermal behavior. My more recent preprint hep-th/20202085 presents other aspects of the same problem.

Key-words: Quantum field theory; Mathematical physics; Quantum gravity.

1 Introductory remarks

"Localization Entropy" [1][2] is an entropy which has its origin in the fact that in local quantum physics[1] (in distinction to quantum mechanics) the vacuum, if restricted to a causally closed algebra with a nontrivial causal complement, behaves as a thermal state [7]. The best known illustration is the vacuum restricted to the algebra of quantum matter localized in a Rindler wedge which leads to the planar Unruh situation i.e. a thermal state with a Hawking temperature (which depends on the horizon-generating acceleration in the Unruh Gedankenexperiment) which is accompanied by thermal radiation [6]. Whereas the thermal aspect of such a situation is related to the general observation that spacetime restrictions of globally pure states in QFT to regions with a nontrivial causal disjoint lead inevitably to impure states (not true in QM), the specific understanding of an entropy generated by "quantum" localization and its relation with the classical Bekenstein-Hawking area behavior is a more challenging task [2].

In this work we use the recently proposed algebraic lightfront holography [3] in order to derive a transverse tensor factorization structure of degrees of freedom. This means that the previously obtained generalized chiral operator algebra on the lightfront $\mathcal{A}(LF)$, whose main distinction from a standard chiral algebra is a huge transverse degeneracy, possesses a transverse tensor product foliation into nonoverlapping chiral longitudinal strip algebras $\mathcal{A}(\mathcal{S}_i)$ (tiling of LF into strips of unit transverse width) which are also generalized chiral conformal algebras

$$\mathcal{A}(LF) = \overline{\bigotimes_i} \mathcal{A}(\mathcal{S}_i) \tag{1}$$

$$H = \overline{\bigotimes_i} H(\mathcal{S}_i)$$

The restriction of this tiling to the causal horizon of a wedge W (the Unruh-Rindler situation) defines a tensor product tiling on the half-lightfront LF_+ (the upper causal horizon of W)

$$\mathcal{A}(LF_+) = \overline{\bigotimes_i} \mathcal{A}(\mathcal{S}_{+,i}) \tag{2}$$

$$H_i = \overline{\mathcal{A}(\mathcal{S}_{+,i})\Omega}$$

In the next section we will review the derivation of the chiral nature of the lightfront algebra and explain how from its additional transverse structure one arrives at this factorization.

Quantities as the entropy (which behave additively under tensoring into independent i.e noninteracting subalgebras) will then be naturally described in terms of an entropy density per unit $d-2$ dimensional transverse cell (taking the $d-2$ dimensional transverse instead of a naively expected d-1 dimensional horizontal density). However it is well-known among algebraic quantum field theorist [7] that the very nature of these algebras (the $\mathcal{A}(\mathcal{S}_{+,i})$ are hyperfinite type III$_1$ von Neumann factors) prevents a direct assignment of entropy. To be more specific, although the vacuum tensor-factorizes in transversal direction (i.e. there is no transverse vacuum polarization), a factorization in lightray direction is not possible i.e. there is no isomorphism of operator algebras

$$\mathcal{A}(\mathcal{S}) \nsimeq \mathcal{A}(\mathcal{S}_{-,i}) \overline{\bigotimes} \mathcal{A}(\mathcal{S}_{+,i}) \tag{3}$$

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[1]This is the formulation of QFT without the use of field-coordinatizations.

The physical reason are the uncontrollable vacuum fluctuation which the "tearing apart" of a type I global algebra $\mathcal{A}(\mathcal{S})$ into two type III_1 algebras causes at the common boundary. To arrive at a situation in which a longitudinal tensor factorization can be defined, we must find a way to control the vacuum polarization with the help of a more careful "splitting procedure". Buchholz and Wichmann have shown that each QFT with a reasonable thermal behavior fulfills the split property [7]. Its adaptation to the strip algebras is the subject of the third section in which it is shown that the original vacuum becomes highly entangled in terms of a suitable tensor product split which leaves a small finite lightlike separation δ between the two half-strips $\mathcal{S}_{\pm,i}$. There the reader finds also a formula for the overlap between the vacuum and its split version as a function of δ. The looked for split- or localization-entropy per unit d-2 dim. transverse volume (area for d=1+3) which measures the entanglement is the relative entropy between the vacuum and its split version considered as states on the strip algebra $\mathcal{A}(\mathcal{S}_+) \subset \mathcal{A}(LF_+)$. Its calculation is reduced to computations within "kinematical" ((half)integer scale dimensions) chiral theories, a fact which accounts for the insensitivity of the result on the original quantum matter content. In the concluding remarks we compare our results to previous attempts and mention some open problems.

Since the phenomenon of vacuum polarization (which separates quantum mechanics from local quantum field theory) and in particular its behavior under holographic lightfront projection is the main cause for transverse alignment of degrees of freedom on the lightfront, we will use the remainder of this introduction to recall some historical facts which help to understand the setting of this paper.

Historically vacuum polarization was first observed by Heisenberg, Weisskopf and others [9] while trying (in contemporary terminology) to define global Noether charges as limits of "partial charges" in open systems. Partial charges associated to sharply defined spatial regions diverge because the vacuum fluctuations which such an object creates if applied to the vacuum become uncontrollably big. Although the intuitive remedy in form of a "soft" boundary was quite clear, careful mathematical definition in terms of smearing functions for partial charges only appeared when (as a result of the appearance of the notion of a spontaneously broken symmetry) there was a need for a higher precision on this point [7][8]. In a way the split property may be viewed as the algebraic version of this taming of vacuum fluctuation. Although we have not found a direct connection between Noether currents and the problem of localization-entropy (since this entropy is a pure local quantum physics notion, it would be difficult to imagine that it arises from quantization of a classical Noether current), it may be interesting to point out that Wald in his earlier work within a more classical gravity setting [10] has given a description in terms of Noether currents. In addition a Noether formalism is used in the third section as an auxiliary device to implement a representation for the split vacuum state. Whether these observations are coincidental or deeply connected remains a problem of future research.

The fact that causality requires the presence of both frequencies, and in this way causes vacuum polarization, is also the mechanism behind the Reeh-Schlieder theorem. In the standard literature known under the name of "state-algebra relation" (i.e. the unique relation between operators of a local operator algebra and a dense set of vectors created by these operators when they act on the vacuum), it contrasts local quantum physics from quantum mechanics in such a dramatic way that it even attracted a lot of attention among philosophers [5]. The Reeh-Schlieder theorem is also the starting point for the use of the Tomita-Takesaki modular theory for the analysis of local algebras together with the vacuum (or other cyclic and separating state vectors). Operator algebraist call a Reeh-Schlieder pair $(\mathcal{A}, \Omega)$ a "standard" pair. The first application of the modular theory of operator algebras in QFT was elaborated

by Bisognano and Wichmann [11] for the standard pair $(\mathcal{A}(W), \Omega)$ where W denotes a (Rindler) wedge. The relation between that theory and Unruh's thermal observations about Rindler wedges (creation of a horizon by uniform acceleration) was first seen in [12]. The content of the present paper is best viewed as a refinement with the help of the lightfront algebra. Although the lightfront LF is the linear extension of the upper causal horizon $h_+(W) = LF_+$ of the wedge W and the two algebras have been shown to be equal i.e. $\mathcal{A}(W) = \mathcal{A}(LF_+)$, the local alignment of degrees of freedom in the $\mathcal{A}(LF_+)$ description is better suited to see the transverse factorization and to describe the ensuing localization entropy caused by splitting the vacuum along the bifurcation.

2   Review of lightfront holography

The algebraic lightfront holography [3] extends the old lightfront quantization (or $p \to \infty$ frame method) to the realm of interacting renormalizable QFT. Whereas the old method associates pointlike fields on the lightfront by a suitably defined restriction of the free (or superrenormalizable) fields, the lightfront holography reprocesses the d-spacetime dimensional local algebras (generated by smeared pointlike fields in case one starts from pointlike fields) of a wedge region[2] (which is the causal forward/backward shadow cast by a semi-lightfront) into a net of d-1 dimensional subalgebras of an algebra localized on the (upper) horizon (i.e. the lightfront half-plane) of the right wedge. This algebraic lightfront holography is applicable to all local field theories, whereas the old lightfront approach only works for theories in which the integral over the Kallen-Lehmann spectral function is finite (which is the same severe restrictions as the one required by the validity of the canonical formalism), which leaves only the free[3] models in d=1+3 [3]. The algebraic approach simply liberates the causal propagation picture from its narrow canonical limitation in terms of lightfront-restricted pointlike fields.

Although this reprocessing maintains the longitudinal localization (in the plane spanned by the two generating lightlike defining vectors of the wedge), one looses the localization in the transversal direction. Of course one may get this information by considering the intersection of all double cone algebras which touch the lightfront, but since regions on the lightfront with finite transversal/longitudinal extension do not cast a "causal shadow", one has to study the properties of d-dim. regions which become infinitely thin in the direction perpendicular to the lightfront, which makes the construction questionable. A better way, more in the spirit of algebraic QFT is to recover the transversal net structure by "Lorenz-tilting" the wedge around its upper defining lightlike vector i.e. the tilting belongs to the Wigner little group of that lightlike vector. By forming algebraic intersections between the original and the transformed strips in transversal direction one obtains a net structure on the lightfront [3].

The mathematical basis of the operator-algebraic holography is a theorem on modular inclusion [27] of two von Neumann algebras $\mathcal{N} \subset \mathcal{M}$ with a common cyclic and separating vector Ω such that the modular group $\sigma_{t,\mathcal{M}}$ of the pair $(\mathcal{M}, \Omega)$ upon restriction to $\mathcal{N}$ compresses i.e. $\sigma_{t,\mathcal{M}}(\mathcal{N}) \subset \mathcal{N}$, $t < 0$ (in this case it was called a +halfsided modular inclusion in [27]). Modular inclusions for which the relative commutant $\mathcal{N}' \cap \mathcal{M}$ is also standard with respect to Ω are isomorphic to chiral conformal theories [27]

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[2]We need an algebra in "standard position" relative to the vacuum vector Ω i.e. one for which the algebra acts cyclic and has no annihilators of Ω; hence we cannot take the full algebra on Minkowski space. Whenever the localization region has a nontrivial spacelike complement as in the case of a wedge, the "standardness" is guarantied.

[3]In d=1+1 there are also superrenormalizable interacting models (e.g.polynomially coupled scalar fields) which are canonical, but they are not very interesting for particle physics.

and hence the transversely unresolved lightfront holographic projection is a chiral theory. In fact it is a "kinematical" chiral theory in the sense that its pointlike generating fields (which have no direct relation to the original fields) have (half)integer scale dimensions.

For the case at hand $\mathcal{M} = \mathcal{A}(W)$, $\mathcal{N} = \mathcal{A}(W_{e_+}) = AdU(e_+)\mathcal{N}$ with $e_+ = (1,1,0,0)$. We collect the two most important results of these assumptions

- The wedge algebra is equal to its upper horizon (lightfront half-plane) algebra [3]

$$\mathcal{A}(W) = \mathcal{A}(R_+) \tag{4}$$

  It is enough to know the operator algebra for the standard x-t wedge W because the full net of local algebras may then be obtained by covariance and intersections of algebras. The family of subalgebras $\mathcal{A}([a,\infty])$ of $\mathcal{A}(R_+)$ which is indexed by semi-infinite intervals $[a,\infty] \subset R_+$ is isomorphic to the family $\left\{\mathcal{A}(W_{ae_+})\right\}_{a>0}$ whereas the original double cone subalgebras $\mathcal{A}(W_{ae_+-be_-})$ $a,b>0$ have a fuzzy image in $\mathcal{A}(R_+)$ (i.e. no geometrically characterizable position in $\mathcal{A}(R_+)$). Vice versa the interval-localized algebras $\mathcal{A}(I), I \subset R_+$ are holographic images of fuzzy localized subalgebras in $\mathcal{A}(W)$. The holographic relation can be extended to the full algebra $\mathcal{A}$ which then becomes holographically encoded into the full lightfront plane algebra $\mathcal{A}(R) = B(H)$

- The diffeomorphism group of the chiral $\mathcal{A}(R) = \mathcal{A}(S^1)$ is that of the circle (its infinitesimal generators obey the Virasoro-algebra commutation relations), whose modular origin has been recently established [18]. It has a holographic pullback to fuzzy acting automorphisms on the original algebra in d spacetime dimensions. The subgroup of the circular diffeomorphism group which acts in a local manner both on the original and the holographically projected algebra is the group generated by dilation and lightray translation into the e_+ direction. The rigid circular rotation generated by Virasoro's L_0 belongs to the symmetries which act fuzzy on the original algebra. Those Poincaré transformations which act locally on the lightfront form a 7-parametric subgroup.

Some more comments are in order.

Although the original as well as the projected theory may have a conventional description in terms of pointlike field generators, the algebraic holographic reprocessing of degrees of freedom in the presence of interactions cannot be formulated in terms of field coordinates, but rather needs the concepts of the operator-algebraic approach to QFT. The reason is that these holographic maps involve steps which change the spacetime indexing of operators in such a way that a geometrically localized algebra goes into one with a "fuzzy" localization and vice versa, i.e. a geometric localization in the holographic image may come from a fuzzy localized subalgebra of the original theory [3]. The physical intuitive content (but not the conceptual framework) of the present approach is close to 't Hooft's area-law inspired holography [24].

In order to have an easy geometric visualization of conformal theories associated with lightfronts, we sketch the important step of the modular inclusion method for d=1+2. Let W be the standard x-t wedge and consider the inclusion (Ad denotes the adjoint action)

$$\mathcal{A}(W_{e_+}) \subset \mathcal{A}(W) \tag{5}$$

$$\mathcal{A}(W_{e_+}) \equiv AdU(e_+)\mathcal{A}(W)$$

$$e_+ = \frac{1}{\sqrt{2}}(1,1,0)$$

The modular group σ_t of $\mathcal{A}(W)$ is implemented by the W-fixing x-t Lorentz boost $\Lambda_{x-t}(-2\pi t)$; it evidently compresses W_{e_+} for $t < 0$ which is the defining property of a modular inclusion. The relative commutant $\mathcal{A}(W_{e_+})' \cap \mathcal{A}(W)$ of the shifted algebra $\mathcal{A}(W_{e_+})$ in $\mathcal{A}(W)$ is the building block of the transversely unresolved lightfront algebra $\mathcal{A}(R_+)$

$$\mathcal{A}(R_+) \equiv \bigvee_t \sigma_t(\mathcal{A}(W_{e_+})' \cap \mathcal{A}(W)) \tag{6}$$

$$\mathcal{A}(R) \equiv \mathcal{A}(R_+) \vee AdJ\mathcal{A}(R_+)$$

Of course one must argue that the nontriviality of the relative commutant is not tied to the existence of a generating field with canonical commutation relations (decreasing Lehmann-Kallen spectral function) and that the relative commutant has a cyclic action on the vacuum. For this we refer to previous works [3]. The upshot is the relation (4) , which is the quantum analog of the classical statement that (with the exception of conformal covariant d=1+1 theories) the characteristic lightfront data on the upper (lower) wedge horizon determine the data within the wedge.

The gain of using the holographic projection onto the lightfront is that the transversely unresolved lightfront algebra turns out to be a *chiral theory with additional automorphisms acting on it.* The simplicity of our d=1+2 illustration lies in the fact that the transversal symmetries which act as automorphisms of the lightfront algebra are easily recognizable. They consist of the y-translation and the one-dimensional Wigner little group[4] of the lightlike vector e_+

$$x_+ \to x_+,\ y \to y + vx_+ \tag{7}$$

$$x_- \to x_- + 2vy + v^2 x_+$$

which acts as a kind of transversal Galilei transformation in the lightfront plane

$$y \to y - vx_+ \tag{8}$$

$$x_+ \to x_+$$

Hence the total symmetry group acting on the 2-dim. lightfront plane is a 4-parametric subgroup of the 6-parametric Poincaré group (in d=1+3 the lightfront symmetry group is 7-dimensional and there are 2 Wigner "translations" whose holographic projection looks like a transversal Galilei transformation in the 3-dim. lightfront plane.) The two remaining transformations which lead out of the 2-dim. lightfront plane are the spatial rotation and the lightlike translation perpendicular to the plane. The transformation (8) may be used to equip the lightfront with a net structure by intersecting the transverse strips corresponding to a longitudinal (lightlike) intervals with their images under (8) which are inclined strips. Our main interest is in the quantum physical behavior of longitudinal strips which may be obtained from the net by additivity of the lightfront net. Let us imagine that we have divided the lightfront plane into horizontal strips S_i $i \in \mathbb{Z}$ of unit transversal width. These strips have the special property of containing just one lightlike direction, all other directions are spacelike. This together with the fact that the lightlike generator is positive has the following well-known consequence

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[4]It consists of a t-y Lorentz boost which would turn the original edge of the wedge (of which the half lightfront is the upper horizon) outside of the lightfront combined with an appropriately chosen x-y rotation which turns the edge back into the lightfront but into a tilted position with respect to its original location.

Proposition 1 *([19]) The weakly closed algebras $\mathcal{A}(S_i)$ localized in the strips are type I_∞ tensor factors whose tensor product is the (weak closure of the) global algebra*

$$\bigotimes_i \mathcal{A}(S_i) = B(H) \tag{9}$$

The transversal causal decoupling is "quantum mechanical" i.e. in the sense of transversal factorization of the vacuum

$$\Omega = \prod_i \otimes \Omega_i$$

The crucial property used in the proof is the existence of a positive generator lightlike translation $U_{x_+}(a)$ which leaves each $\mathcal{A}(S_i)$ invariant. This leads to

$$\left\langle AdU_{x_+}(a)A_i \cdot A_j \right\rangle = \left\langle A_j \cdot AdU_{x_+}(a)A_i \right\rangle, \ i \neq j \tag{10}$$

which together with the analytic properties in a following from the positivity of the lightlike momentum leads to the independence on a which in turn combined with the cluster property yields the vacuum factorization property. This property also holds for the half strips which partition the half lightfront (the horizon of W) since the associated algebras are subalgebras of the S_i.

This shows a somewhat unexpected (complete absence of transverse vacuum fluctuations) behavior know from a collection of quantum mechanical systems without relative interactions between them. Since the lightfront degrees of freedom factorize in this perfect way we encounter an "area law" in the sense that if we would be able to measure the degrees of freedom in a strip in terms of a strip entropy, then this entropy would have the interpretation of the entropy per unit edge size of the edge of the Unruh-Rindler-wedge W.

It has been noted previously [3] that those local Poincaré automorphisms which lead out of the lightfront become fuzzy (non-geometric) under holographic projection. In the other direction there are transformations which in the sense of the lightfront net structure are local and which if reprocessed into the higher dimensional geometric setting will be fuzzy: an example is the conformal rotation (its generator is often referred to as the "conformal Hamiltonian L_0). As a result of the modular origin[5] of the diffeomorphisms of the circle [18][20], this diffeomorphism group also acts locally on the strip algebras and lifts to fuzzy symmetries in the original higher dimensional setting. For this reason the lightfront holography represents a more radical reprocessing than Rehren's AdS-CQFT isomorphism [23] (which is the rigorous field-theoretic version of Maldacena's conjecture).

Clearly there is nothing in our d=1+2 sketch which has no counterpart in d>1+2 spacetime dimensions. The transverse transformations in the general case consist of ordinary transverse translations and translations within the Wigner little group of e_+ which is the Euclidean group in d-2 dimensions (containing d-2 transverse "Galilei" transformations). With the help of the latter one can resolve the transversal net structure [3].

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[5]The modular origin substitutes the existence of the energy-momentum tensor. As a result of the transversal structure the strip algebras are much larger chiral algebras than those standard chiral algebras resulting from d=1+1 conformal QFT.

3   The split isomorphism and its implementations

Since the chiral strip algebras restricted to the horizon of W are not ordinary quantum mechanical algebras (i.e. type I von Neumann algebras with minimal projectors associated with pure states corresponding to best possible measurements), a direct attempt to associate to it a von Neumann entropy[6] is meaningless. As already indicated in the last part of the introduction, the non quantum mechanical nature of local operator algebra (hyperfinite von Neumann factor of type III_1) does not allow the definition of a von Neumann entropy; in fact such operator algebras do not even permit tracial states/weights. Since there are no pure states, the notion of entanglement is also void of meaning.

The remedy of passing to localized algebra with a fuzzy boundaries of arbitrary small but finite thickness by splitting was already mentioned in the introduction [13][7]. Let us present it first in a more general setting before adapting it to the kind of chiral theory which represent the strip algebras.

We start from an inclusion of two double cones $\mathcal{C}_1, \mathcal{C}_2$, but different from the physical realizations of modular inclusion where the causal horizons touch each other, the causal boundaries of a split inclusion are separated by a "collar" of thickness δ

$$\mathcal{C}_1 \subset\subset_\delta \mathcal{C}_2$$

In theories fulfilling the Buchholz-Wichmann nuclearity (a phase space property which insures a reasonable thermal behavior) such split inclusions [7] lead to the existence of an intermediate type I factor which is contained in the bigger and contains the smaller operator algebra, but has no sharp boundaries within the collar. This existence of an intermediate type I factor (without geometric interpretation) constitutes the definition of a split inclusion whose mathematical aspects have been investigated in detail in [13]. In fact there it was shown that if the inclusion $\mathcal{A}(\mathcal{C}_1) \subset \mathcal{B}(\mathcal{C}_2)$ splits in this fashion, there exists even a distinguished intermediate canonical type I factor $\mathcal{N}$ functorially related to $\mathcal{A}(\mathcal{C}_1), \mathcal{B}(\mathcal{C}_2)$. Suppose now that the reference state ω is also faithful on the algebra $\mathcal{A}(\mathcal{C}_1) \vee \mathcal{B}(\mathcal{C}_2)'$ i.e. the operator algebra generated by the algebra $\mathcal{A}(\mathcal{C}_1)$ and the commutant of $\mathcal{B}(\mathcal{C}_2)$. Such a state retains this property upon restriction to the generating subalgebras and it is easy to prove that if one forms the product state $\omega \cdot \omega$ by eliminating the correlations between these two subalgebras, this new product state is implemented by a vector η in the common Hilbert space (by using an appropriate natural cone representation, η will even be unique) [7]

$$\omega \cdot \omega(AB') \equiv \omega(A)\omega(B'), \ A \in \mathcal{A}(\mathcal{C}_1) \subset \mathcal{B}(\mathcal{C}_2), \ B' \in \mathcal{B}(\mathcal{C}_2)' \tag{11}$$
$$\curvearrowright \exists \, \eta \in H \ s.t. \, \omega(\cdot) = \langle \eta \, | \cdot | \, \eta \rangle, \ \ \mathcal{A}(\mathcal{C}_1) \subset \mathcal{N} \subset \mathcal{B}(\mathcal{C}_2)$$
$$B(H) \simeq \mathcal{N} \bar{\otimes} \mathcal{N}', \ H = H_\mathcal{N} \overline{\otimes} H_{\mathcal{N}'}, \ H_\mathcal{N} \equiv \overline{\mathcal{A}(\mathcal{C}_1)\eta} = P_\mathcal{N} H$$
$$\mathcal{N} = P_\mathcal{N} B(H) P_\mathcal{N}, \ \mathcal{A}(\mathcal{C}_1) \subset \mathcal{N}, \ \mathcal{B}(\mathcal{C}_2)' \subset \mathcal{N}'$$

Hence the desired split is accomplished by the type I algebra[7] $\mathcal{N}$ and the state vector η is a tensor product vector without entanglement in the tensor product description $\mathcal{N} \otimes \mathcal{N}'$ and hence a fortiori on

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[6]Even if the absolute entropy is infinite, the relative entropy between two different states on the same algebra can be finite.

[7]There exists a concrete formula for the type I factor $\mathcal{N}$ in terms of $\mathcal{A}(\mathcal{C}_1)$ and the modular involution J_{collar} of the collar algebra $\mathcal{A}(\mathcal{C}_1)' \cap \mathcal{B}(\mathcal{C}_2)$ [13]. Physical intuition suggests that the dominating behavior in the limit of small collar size is independent of the chosen type I interpolation.

$\mathcal{A}(\mathcal{C}_1) \otimes \mathcal{B}(\mathcal{C}_2)' \subset \mathcal{N} \otimes \mathcal{N}'$. On the other hand the original non-split vacuum is highly entangled with the consequence that it is not only a thermal state (with the Hawking temperature) on $\mathcal{A}(\mathcal{C}_1) \otimes \mathcal{B}(\mathcal{C}_2)'$, but even remains thermal on the type I quantum mechanical algebra $\mathcal{N}$. The localization of this "relativistic box quantization", contrary to a quantum mechanical box, is fuzzy inside the collar $\mathcal{C}_1 \setminus \mathcal{C}_2$. But it needs to be emphasized that contrary to the inside/outside nonrelativistic box quantization no degrees of freedom have been dumped or cut-off in the present case; they were only somewhat spatially displaced within the collar region in order to avoid uncontrollable vacuum fluctuations from the sharp splitting.

For the case at hand the necessary splitting is in a *generalized* chiral theory. Here "generalized" means that there is a large transverse multiplicity which is encoded into the finite width of the strip $S = \dot{R} \times width$, whereas standard chiral theories, which result from the tensor-product decomposition of 2-dim. conformal models, are depicted as being localized on the compactified line $\dot{R} \simeq S^1$. The splitting into two $S_{\pm,\delta} = R_{\pm,\delta} \times width$ with $S_{\pm,\delta} \subset LF_{\pm}$, where $S_{\pm,\delta}$ correspond to the above $\mathcal{A}(\mathcal{C}_1), \mathcal{B}(\mathcal{C}_2)'$, is done by splitting $\dot{R} \simeq S^1$ symmetrically at lightlike zero and infinity by a small δ; the two small disconnected splitting intervals correspond to the above collar. For the explicit description of the split vacuum η it is important to notice that the chiral theories which originate through lightfront holography are "kinematical" in the sense that they only contain canonical ((half)integer) scale dimensions. In fact such chiral algebras always possess pointlike field generators of (half)integer scale dimensions [21].

Let us first look at a standard chiral situation i.e. let us ignore the transverse extension. For simplicity assume that the generating field has scale dimension $d = \frac{1}{2}$

Let us furthermore consider the special case that the chiral theory has a generating field of scale dimension $d = \frac{1}{2}$. The two δ-split even operator algebras are then of the simple form

$$\mathcal{A}((R_{\pm,\delta})) = alg\left\{\psi(f)\psi^*(g) \,|\, supp f, g \in R_{\pm,\delta}\right\} \tag{12}$$

and the split isomorphism Φ acts as

$$\Phi\left(\mathcal{A}(R_{-,\delta}) \vee \mathcal{A}(R_{+,\delta})\right) = \mathcal{A}(R_{-,\delta}) \bar{\otimes} \mathcal{A}(R_{+,\delta}) \tag{13}$$

We are interested in the relation of the original vacuum Ω to the split vacuum η

$$\eta \simeq \Omega \bar{\otimes} \Omega \tag{14}$$
$$\langle \eta \,|\, \mathcal{A}(R_{-,\delta}) \vee \mathcal{A}(R_{+,\delta} \,|\, \eta \rangle =$$
$$\langle \Omega \,|\, \mathcal{A}(R_{-,\delta}) \,|\, \Omega \rangle \cdot \langle \Omega \,|\, \mathcal{A}(R_{+,\delta}) \,|\, \Omega \rangle$$

in the limit of shrinking split size δ. The choice of the implementing vector η is not unique; a mathematically preferred choice is obtained by taking η in the natural cone of the standard pair [13][7] $\mathcal{A}((-\infty + a, -a)) \vee \mathcal{A}((a, \infty - a)), \Omega)$. Any other choice η' is related to the canonically preferred η by the following inequality [14]

$$\|\eta - \Omega\|^2 = 2\,|1 - (\eta, \Omega)| \tag{15}$$
$$= inf\left\{\|\eta' - \Omega\|^2, \eta' \,is\, split\right\}$$
$$= \|\omega_{split} - \omega_\Omega\| \leq \|\eta' - \Omega\|^2$$

where the last line uses the so-called canonical Bures distance in the convex space of states. Bures distance 2 is an indication that the state ω_{split} belongs to an inequivalent folium (its GNS representation defines

an inequivalent representation of the chiral algebra). It is generally believed that in the limit $\delta \to 0$ all implementing state vectors show the same behavior. We will assume that this is true, and that there is no physically preferred implementation.

The main point of this section is now the proposition that thanks to the simple kinematical structure of the holographic projection, the "escape" into an inequivalent representation in the limit $\delta \to 0$ [6] as well as the entanglement of the vacuum and the resulting localization-entropy (and its divergence with shrinking collar size) can be studied quantitatively with the help of the "flip trick" which is an implementation of Φ in terms of concrete unitary operators. For the case at hand we notice that the fields in the different tensor product factors can be interpreted as a doublet i.e. $\psi_1 = \psi \bar{\otimes} 1$, $\psi_2 = 1 \bar{\otimes} \psi$. In the spirit of a $SO(2)$ Noether symmetry the implementation of the unitary flip operation can then be done in terms of a Noether current [14] formalism

$$\Phi(\psi(x)) = e^{ij(f)} \psi_2(x) e^{-ij(f)} = \begin{cases} \psi_1(x), \ x \in R_{+,\delta} \\ \psi_2(x), \ x \in R_{-,\delta} \end{cases} \tag{16}$$

$$j(x) = \psi_2^*(x)\psi_1(x), \quad f = \begin{cases} 1, \ x \in R_{-,\delta} \\ 0, \ x \in R_{+,\delta} \end{cases}$$

Clearly $U(f) = e^{ij(f)}$ acting on $H \bar{\otimes} H$ implements the product state (14) for $(\mathcal{A}(R_{-,\delta}) \vee \mathcal{A}(R_{+,\delta})$. As expected the state vector $\eta' = U(f)(\Omega \bar{\otimes} \Omega)$ becomes orthogonal on all vectors in $H \bar{\otimes} H$ for $\delta \to 0$. Let us check this for the vacuum $\Omega \bar{\otimes} \Omega = \Omega_{vac}$

$$\langle \Omega_{vac} | \eta' \rangle = \langle \Omega_{vac} | U(f) | \Omega_{vac} \rangle \tag{17}$$
$$= e^{-\frac{1}{2} \langle j(f), j(f) \rangle_0} \sim 0, \ \delta \to 0$$
$$\langle \eta | AB | \eta \rangle = \langle \Omega_{vac} | A | \Omega_{vac} \rangle \langle \Omega_{vac} | B | \Omega_{vac} \rangle$$
$$A \in \mathcal{A}(R_{-,\delta}), \ B \in \mathcal{A}(R_{+,\delta})$$

The norm square of the smeared current $j(f)$ can be explicitly computed from the known current two-point function

$$\langle j(f), j(f) \rangle_0 \overset{\delta \to 0}{\sim} -\log \delta$$

i.e. the resulting logarithmic dependence on the collar size δ leads to a positive power law for the after exponentiation. This is a quantitative expression for Wald's qualitative discussion of the inequivalence (orthogonality) of the two Hilbert spaces [6].

In the same vein the inner product with all basis vectors converges with a power law to zero for $\delta \to 0$

$$\langle \Omega_{vac} | U(f) | a_1^*(p_1)...a_n^*(p_n) \otimes a_2^*(k_1)...a_2^*(k_m)\Omega_{vac} \rangle \to 0 \tag{18}$$

i.e. the original vacuum becomes a highly entangled state on the split algebra which in the limit $\delta \to 0$ even leaves the Hilbert space i.e. belongs to an inequivalent representation of the algebra. By tracing out the first tensor factor one expects to obtains a density matrix in the second factor which represents the vacuum Ω_{vac} as a mixed state on the factor space $\overline{\mathcal{A}(R_{+,\delta})\eta'}$. In this cumbersome way one could try to compute the split entropy, but it would be difficult to see more than its logarithmic divergence which corresponds to power law which describes the approach to the inequivalent representations. Fortunately there is a neater way due to Kosaki [22] which presents this relative entanglement entropy of ω and

$\omega^2 \equiv \omega \cdot \omega$ in terms of a variational problem

$$S(\omega^2, \omega) = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (19)$$

$$sup_{y(t)} \int_0^\infty \left[\frac{\omega^2(1)}{1+t} - \omega^2(y^*(t)y(t)) - \frac{1}{t}\omega(x(t)x^*(t)) \right] \frac{dt}{t}$$

$$x(t) \equiv 1 - y(t)$$

here $y(t)$ is a path in the algebra $\mathcal{A}(R_{-,\delta}) \vee \mathcal{A}(R_{+,\delta})$ which in our model is a Weyl algebra. Note that this variational formula is completely independent of which vector realization η' one chooses for the product state ω^2.

Such formulas have been used for estimates of split entropies in [1]. We hope to be able to return to a more direct calculation along the indicated line.

Our strip algebras are different from standard chiral algebras in that we are dealing with a collection of $\psi's$ indexed by points on the transversal interval. But this only means that the flip current involves an additional integration over a transverse unit cell which does not participate in its commutation relations with the ψ. Hence the auxiliary Noether trick is still applicable.

Even after having understood the insensitivity of the dependence of quantum area law on the concrete matter content in terms of the universality of the lightfront holography, there remains the problem of the logarithmic divergence with shrinking collar size which the classical Bekenstein formula does not show. In fact as shown in [10] the classical discussion relates the entropy with a classical Noether current without vacuum polarization effects. However in local quantum physics the dependence of quantum entropy on the spatial distribution (localization) of degrees of freedom and not only on their total number is an unavoidable consequence of local quantum physics (vacuum polarization). In fact it is inexorably linked to the transverse area behavior and it will not going away in curved spacetime as the result of presence of curvature. However if the localization entropy enters thermodynamic laws as the usual heat bath entropy does, one could expect that the other quantum matter dependent terms in such a law also show this vacuum polarization effect. In such a case it is conceivable that this δ-dependence could be scaled out of the equation.

4   Concluding remarks

In this note we have proposed concepts which explain the universality of a area law for localization quantum entropy in terms of the kinematical nature of holographically projected degrees of freedom. This was achieved by encoding as much as possible of the more complicated aspects into the actions of automorphisms in order to keep the maximal simplicity of the substrate (i.e. the lightfront) on which these automorphisms act.

Although this kinematical simplification aspect is very much in line with why at the beginning of the 70s particle physicist became interested in the use of lightfront and $p \to \infty$ methods, the present formulation uses quite different concepts. The underlying strategy is to *abstract from free fields only those properties which do not depend on short distance property.*

This is why we had to reject the traditional restriction of pointlike fields to the lightfront; it suffers from the same short distance limitation as the canonical equal time formalism i.e. in d=1+3 theories all interactions would be excluded; even asymptotically free theories (or supersymmetric theories) do

not allow a lightfront restriction since the asymptotic freedom property does not make the integral over the spectral Kallen-Lehmann function convergent. If on the other hand we generalize the *modular aspects* of operator algebras generated by free field to the realm of interactions there is no such problem. However this strategy which avoids field-coordinatizations has its price in that one looses the transverse localization properties. Fortunately the 7-parametric symmetry group of the lightfront algebra contains transformations from the Wigner little group of the unique lightray in the lightfront which allow to recreate the net structure of localized algebras on the lightfront, so that its transverse tensor product foliation can be studied. We believe that there is much more to these observations in that the structure of the holographic lightfront projection may turn out to be the starting point of a new constructive approach to QFT.

The new aspects of our calculation become more transparent if one compares with the approach of Bombelli et al. [15]. These authors do explicit entropy computations on a box-localized (in the spirit of nonrelativistic box quantization) zero mass free field, assuming that entropy is well defined. They then find that it isn't, which forces them to cutoff integrals. As far as the conservative field theoretic setting of these authors is concerned, the calculations have a lot in common. The main difference is that in the present approach no degrees of freedom have been thrown away by cutting off integrals or in any other way. The split inclusion method is a subtle but natural method which reprocesses the original situation into one in which the inside/outside causal factorization of degrees of freedom can take place.

An important step is to detach the degree of freedom issue from "field coordinatization" (and their associated short distance properties) and tie it directly to the structure of the restriction of the vacuum to the local algebra. This very step already removes one source through which ultraviolet divergencies usually appear, namely the short distance properties of the (singular) field-coordinatization. Without focussing on algebras (and their degrees of freedom) and shifting the emphasis away from field coordinates, it would not be possible to consider the vacuum as an entangled state vector with respect to a causal inside/outside division. Even if the original theory and its holographic projection are both generated by pointlike fields, in the presence of interactions there is still no direct relation between them; our algebraic holographic method avoids being lured into such wrong ideas where short distance conformal theories may get confused with conformal aspects of lightfront holography. Although the holographic projection is a conformal theory, it is a kinematical chiral theory which (in the presence of interactions) has nothing to do with the scaling limit of the original theory [3]. *The distinction between short distance universality classes and the present kinematical aspects of lightfront holography is an important issue.*

It would be important to test the transverse tensor factorization in rotational symmetric situations i.e. for the lightcone horizon of double cone algebras. Here the difficulty is the absence of a global Killing symmetry in the characterization of the causal horizon. Nevertheless there are interesting analogies. For this and other problems which were omitted in the present work, we refer to [16]. There the reader also finds more details concerning the mathematical aspects of the lightfront QFT.

The main effect of curved spacetime seems to be a "geometrization" of modular properties of operator algebras. Whereas in Minkowski spacetime the Rindler restriction is the only way to have a causal horizon in terms of a global Killing symmetry, curved spacetime creates such situations (together with singularities) even in case of compact regions of bifurcation as in the case of the Schwarzschild spacetime. If one views QFT as an abstract functor [17] between spacetimes and algebras, then the different spacetimes only give different localization textures to the same algebraic substrate (e.g. the degrees of freedom of

the Weyl algebra) and the properties of localization entropy for causal horizons with Killing symmetries become more visible on the classical side.

The kinematic chiral theory of the lightfront holography possesses the Virasoro structure and as we have demonstrated recently, the diffeomorphism of the circle have a modular origin and correspond to symmetries in the common Hilbert space which act in a fuzzy way on the original non chiral degrees of freedom before they suffered the holographic projection [18]. This shows that the chiral structures used by Carlip [25] in his entropy discussion are available without making additional assumptions. However the identification of entropy with that of the temperature state of the global chiral L_0 circular rotation "Hamiltonian" (in order to be able to apply the Cardy formalism) appears somewhat ad hoc[8]. If the entanglement entropy of the chiral theory in the present treatment would be equal to the Cardy entropy obtained from rotational L_0 thermal states, then this would appear like an accident on the present level of understanding. To clarify this, one needs more detailed investigations. In any case the present explanation in terms of relative entanglement, unlike most other attempts, require no additional degrees of freedom than those one is dealing with in QFT in CST.

If there is any message about "Quantum Gravity" at all in the present approach, perhaps it should be looked for in a better understanding of possible relations between geometry and thermal behavior mediated by modular theory rather than to a quantization sub prima facie of classical general relativity. Although this message may sound pretty wild, recent results on the construction of external and internal symmetries and spacetime geometry from the relative position of operator algebras and in particular the emergence of infinite dimensional fuzzy analogs of diffeomorphism groups (including the Poincaré and conformal diffeomorphisms) from modular inclusions and intersections of algebras point into the same direction [27][28][29][2].

References

[1] H. Narnhofer, in *"The State of Matter"*, ed. by M. Aizenman and H. Araki (World-Scientific, Singapore) 1994

[2] B. Schroer, J. Math. Phys. **41**, (2000) 3801 and ealier papers of the author quoted therein

[3] B. Schroer, *Lightfront Formalism versus Holography&Chiral Scanning*, hep-th/0108203 and previous work cited therein

[4] C. D'Antoni and R. Longo, J. Funct. Anal. **15**, (1983) 199

[5] H. Halvorson, *Reeh-Schlieder Defeats Newton-Wigner: On alternative localization schemes in relativistic quantum field theories* and references therein, quant-ph/0007060

[6] R. M. Wald, *Quantum Field theory in Curved Spacetime and Black Hole Thermodynamics*, University of Chicago Press 1994

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[8] According to Rehren [23] the rotational conformal operator becomes the bona fide Hamiltonian if one re-processes the spacetime labeling of a conformal theory into a AdS description which has a compact time coordinate.

[7] R. Haag, *Local Quantum Physics*, Springer Verlag (1992)

[8] M. Requardt, Commun. Math. Phys. **50**, (1979) 259

[9] Selected papers on *Quantum Electrodynamics*, edited by J. Schwinger, Dover Publications Inc. New York 1958

[10] R. M. Wald, Phys.Rev. D48 (1993) 3427-3431

[11] J. J. Bisognano and E. H. Wichmann, J. Math. Phys. **17**, (1976) 303

[12] G. L. Sewell, Ann. Phys. (N.Y.) **141**, (1982) 201

[13] S. Doplicher and R. Longo, Invent. math. **75**, (1984) 493

[14] C. D'Antoni, S. Doplicher, K. Fredenhagen and R. Longo, Commun. Math. Phys. **110**, (1987) 325

[15] M.Bombelli, R. K. Kaul, J Lee and R. D. Sorkin, Phys. Rev.D **34**, (1986) 373

[16] B. Schroer, *The Paradigm of a transverse Area Structure for Lightfront Degrees of Freedom*, hep-th/0202085

[17] R. Brunetti, K. Fredenhagen and R. Verch, *The Generally Covariant Locality Principle-A New Paradigm for Local Quantum field Theory*, hep-th/01122000 and references on prior seminal work of S. Hollands and R. M. Wald.

[18] L. Fassarella and B. Schroer, *The Fuzzy Analog of Chiral Diffeomorphisms in higher dimensional Quantum Field Theories*, hep-th/0106064

[19] W. Driessler, Acta Physica Austriaca **46**, (1977) 163, and references therein

[20] K. Ebrahimi-Fard, *Comment on: Modular theory and Geometry*, math-ph/001149

[21] M. Jörss, Lett. Math. Phys. **38**, (1996) 252

[22] H. Kosaki, J. Operator Theory **16**, (1986) 335

[23] K.-H. Rehren, Ann. Henri Poincaré **1**, (2000) 607

[24] G. 't Hooft, in Salam-Festschrift, A. Ali et al. eds., World Scientific 1993, 284

[25] S. Carlip, Nucl.Phys.Proc. Suppl.**88**, (2000) 10, gr-qc/9912118

[26] Lucio Fassarella and Bert Schroer, *From Wigner Particle Theory to Local Quantum Physics*, hep-th/

[27] H.-W. Wiesbrock, Comm. Math. Phys. **158**, (1993) 537

    B. Schroer and H-W Wiesbrock, Rev. Math. Phys. **12**, (2000) 139

    B. Schroer and H.-W. Wiesbrock Rev. Math. Phys. **12**, (2000) 461

    R. Kaehler and H.-W. Wiesbrock, JMP **42**, (2000) 74

[28] D. Buchholz, O. Dreyer, M. Florig and S. J. Summers, Rev. Math. Phys. **12**, (2000) 475

[29] B. Schroer and H.-W. Wiesbrock Rev. Math. Phys. **12**, (2000) 461