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Quantization and classical limit of a linearly damped particle, a van der Pol system and a Duffing system*

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Abstract

Using a quantization process, independent of Lagrangians and Hamiltonians, we quantize a linearly damped particle, a van der Pol system and a Duffing system. In order to provide logical consistence to this quantization scheme we also evaluate the classical limit $\hbar \to 0$.

Key-worsds: Dissipative systems; Quantization; Classical limit.

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I. INTRODUCTION

The state of a dynamical system at time t is specified by N variables $z_1(t), z_2(t), ..., z_N(t)$ whose time evolution is given by a set of N first order ordinary differential equations dependent explicitly of time and generally nonlinear

$$\dot{z}_i \equiv \frac{dz_i}{dt} = K_i(z_1, ..., z_N, t), (i = 1, ..., N). \tag{1}$$

An important property of Eq.(1) lies in the existence and uniqueness theorem of its solutions which defines a dynamical system as deterministic [1]. Another property is the existence of a set of admissible transformations $(z_i) \mapsto (Z_i)$ holding invariant a given feature inherent in (1) [2].

Geometrically, the dynamical system (1) may be studied making use of the phase space Γ concept. Then, once defined the state (z_i), the dynamics $(\dot{z}_i = K_i)$ and the geometric arena (Γ) we can establish a criterion characterizing the conservative or nonconservative nature of a certain dynamical system. It is straightforward to show that the behaviour in time of a given region $\mathcal{D}$ of Γ is the responsible for the classification of the dynamical systems as conservative or nonconservative. So, the system (1) is conservative, for any region $\mathcal{D}$, if and only if

$$\nabla.\mathbf{K} \equiv \mathrm{div}\mathbf{K} \equiv \sum_{i=1}^{N} \frac{\partial K_i}{\partial z_i} = 0 \tag{2}$$

and nonconservative if and only if

$$\nabla.\mathbf{K} \equiv \mathrm{div}\mathbf{K} \equiv \sum_{i=1}^{N} \frac{\partial K_i}{\partial z_i} \neq 0. \tag{3}$$

In the case div $\mathbf{K} < 0$, on average at time t, that is,

$$\langle \mathrm{div}\mathbf{K} \rangle_t = \frac{1}{t} \int_0^t \mathrm{div}\mathbf{K} d\tau < 0,$$

the sistem is considered dissipative [3,4].

Now in order to endow the criterion (2) or (3) with an objective character we require that the admissible transformations $(z_i) \mapsto (Z_i)$, from variables z_i to Z_i, are those holding invariant the divergence sign, i.e.,

$$\text{sgn}(\text{div}\mathbf{K}(z_1, z_2, ..., z_N, t)) = \text{sgn}(\text{div}\bar{\mathbf{K}}(Z_1, Z_2, ..., Z_N, t)). \tag{4}$$

As an example of dynamical system (1) let us consider the Newtonian mechanical systems

$$\dot{p} = f(p, q, t) \tag{5}$$

$$\dot{q} = \frac{p}{m}, \tag{6}$$

where q is the position, p the linear momentum, m the mass and f an arbitrary force. The divergence associated with (5) and (6) is equal to $\partial f/\partial p$, so that when f is derived of a potencial function $V(q,t)$ we have a conservative system.

As a subclass of the Newtonian systems (5) and (6) there exist the Hamiltonian mechanical systems

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial \pi} \tag{7}$$

$$\dot{\pi} = -\frac{\partial \mathcal{H}}{\partial x}, \tag{8}$$

π being the generalized momentum canonically conjugated to the generalized coordinate x. Since the divergence of (7) and (8) is always null the Hamiltonian flux is essentially conservative, while the admissible transformations, which hold this property invariant, are the canonical transformations.

On the basis of the criterion (2-4) we can criticize some procedures which aim to describe nonconservative systems using the Hamiltonian formalism. The Bateman Hamiltonian

$$\mathcal{H}(x, \pi, t) = \frac{\pi^2}{2m} e^{-\beta t} + V(x) e^{\beta t}, \tag{9}$$

where $\pi = p e^{\beta t}$ and $x = q$, does never describe a dissipative system because $\nabla.\mathbf{K}(x, \pi, t) = 0$, even though it is erroneously interpreted in the literature as the Hamiltonian of a nonconservative system [5-9]. Indeed, the Hamiltonian (9) describes a conservative system either with variable frequency [10], or variable mass [11] or a system at a non-inertial frame [9]. This ambiguity (null divergence) inherent in Eq.(9) is not a particular case, it also occurs for its equivalent [8] $\mathcal{H} = -(\beta q\pi)/2 + (1/2)\ln[(q^2/4)(4k^2 - \beta^2)]$ with $\pi = (2/q)(4k^2-\lambda^2)^{-1/2}\mathrm{tg}^{-1}[(2\dot{q}/q+\lambda)(4k^2-\lambda^2)^{-1/2}]$, for a damped harmonic oscillator ($m = 1$) $V = k^2q^2/2$, k^2 =const. . In fact, it occurs for all Hamiltonians built after the method of integrating factors developed by Havas [6].

A correct way of studying dissipation analitically is to begin with the Hamiltonian equations modified by a dissipative term [7,12]

$$\dot{q} = \frac{\partial H}{\partial p} \tag{10}$$

$$\dot{p} = -\frac{\partial H}{\partial q} - \mathcal{F}(q,p,t) \tag{11}$$

which can be derived from a modified Hamilton principle or from the Newton equations together with the d'Alembert principle. Here, $H = T + V$ (kinetic energy plus potential energy) and the canonical momentum coincides with the kinetic momentum. It is easy to verify that the divergence asssociated with (10) and (11) is given by

$$\nabla.\mathbf{K} = -\frac{\partial \mathcal{F}}{\partial p}. \tag{12}$$

This shows that we are really dealing with a dissipative system in the Hamiltonian formalism. However the main feature of this formalism is lost: p and q do not necessarily possess any link of canonicity.

The Dekker procedure [13,14] of generalizing the Hamiltonian theory in terms of complex variables

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial \pi}$$

$$\dot{\pi} = -\frac{\partial \mathcal{H}^*}{\partial x}$$

generates a divergence different from zero, since $\mathcal{H}^* \neq \mathcal{H}$. Enz [15], in turn, generalizes the sympletic structure

$$\dot{q} = D_{11}\frac{\partial H}{\partial q} + D_{12}\frac{\partial H}{\partial p}$$

$$\dot{p} = D_{21}\frac{\partial H}{\partial q} + D_{22}\frac{\partial H}{\partial p}$$

so that the divergence is also not null.

The Caldirola [16] and Tarasov [17] procedures are not correct. The former does introduce an inadmissible transformation in order to convert an initially nonconservative system into a conservative one, while Tarasov uses a generalized form of the least action principle

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial \pi} - \frac{\partial \omega}{\partial \pi}$$

$$\dot{\pi} = -\frac{\partial \mathcal{H}}{\partial x} + \frac{\partial \omega}{\partial x},$$

where $\mathcal{H} \equiv H = T + V$ and $\omega = \omega(x, \pi)$, with divergence equal to zero.

In brief, *the Hamiltonian formalism is not unique for describing nonconservative systems.* This fact makes a possible quantization of these systems, using conventional methods, entirelly blurred. On the contrary, our aim in this paper is to quantize a linearly damped particle, a Duffing system and a van der Pol system without using any Lagrangian or Hamiltonian function. We call *dynamic quantization* such quantization process, starting directly from the equations of motion [18-21]. Thus, we organize our article as follows. In Section II, by starting with the Liouvillian formalism, we introduce the Wigner representation of classical mechanics from which we define quantization conditions. In Section III we calculate the classical limit of quantum dynamics and we make our concluding remarks in Section IV.

II. DYNAMIC QUANTIZATION

In the Liouvillian formulation of classical mechanics the state is specified by the probability density $F(z_1, ..., z_N, t)$ and the dynamics is given by the deterministic Liouville equation [4,22]

$$\frac{\partial F}{\partial t} + \sum_{i=1}^{N} K_i \frac{\partial F}{\partial z_i} = -F \sum_{i=1}^{N} \frac{\partial K_i}{\partial z_i} \tag{13}$$

generated by the dynamical system (1). In the case of the Newtonian system (5) and (6), we have

$$\frac{\partial F}{\partial t} + \frac{p}{m}\frac{\partial F}{\partial q} - \frac{\partial V}{\partial q}\frac{\partial F}{\partial p} = -\frac{\partial}{\partial p}(\mathcal{F}F). \tag{14}$$

where $f(p, q, t)$ was split into a conservative force $-\partial V(q, t)/\partial q$ and a nonconservative force $\mathcal{F}(q, p, t)$.

We now introduce the Wigner representation of classical mechanics using the following Fourier transform [20]

$$\chi(q + \frac{\ell\eta}{2}, q - \frac{\ell\eta}{2}, t) = \int F e^{\imath p\eta} dp. \tag{15}$$

Because the Wigner factor $e^{\imath p\eta}$ is an adimensional term and $\ell\eta$ has dimension of position, it follows that ℓ should have dimension of action. Inserting the classical Wigner function (15) into Eq.(14), we obtain

$$\imath\ell\frac{\partial\chi}{\partial t} + \frac{\ell^2}{2m}\left[\frac{\partial^2\chi}{\partial q_1^2} - \frac{\partial^2\chi}{\partial q_2^2}\right] - [\, V(q_1, t) - V(q_2, t) - O(q_1, q_2, t)]\,\chi = -\imath\ell\Omega\chi \tag{16}$$

where

$$O(q_1, q_2, t) = -\sum_{n=3,5,7,...}^{\infty} \frac{2}{n!}\left(\frac{q_1 - q_2}{2}\right)^n \left(\frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}\right)^n V(q_1, q_2, t), \tag{17}$$

$$\Omega\chi = \int \frac{\partial}{\partial p}(\mathcal{F}F)e^{\imath p\eta} dp \tag{18}$$

and

$$q_1 = q + \frac{\ell\eta}{2} \quad ; \quad q_2 = q - \frac{\ell\eta}{2}. \tag{19}$$

In order to quantize Eq.(16) we impose the conditions [20]

$$(q_1 - q_2)^4 \ll (q_1 - q_2) \tag{20}$$

$$\ell \to \hbar = \frac{h}{2\pi}. \tag{21}$$

Thus, we arrive at the nonconservative von Neumann equation

$$\imath\hbar\frac{\partial\rho}{\partial t} + \frac{\hbar^2}{2m}\left[\frac{\partial^2\rho}{\partial q_1^2} - \frac{\partial^2\rho}{\partial q_2^2}\right] - [\, V(q_1,t) - V(q_2,t)]\,\rho = -\imath\hbar\tilde{\Omega}\rho \tag{22}$$

with $\tilde{\Omega}$ being obtained from Ω after using (20) and (21), and ρ the called "density matrix". (Let us note that for conservative systems Eq.(22) is the usual von Neumann equation giving rise to a Schrödinger equation at point q_1 and its complex-conjugate at point q_2).

For conservative systems the admissible mathematical procedure (20) can be physically justified by using the concept of equilibrium entropy [32]. However for nonconservative systems we do not know still the physical reason behind Eq.(20). Together (20) and (21) have to imply $\ell > \hbar$ so that the quantum domain is characterized by the smallness of the Planck constant with respect to the classical actions.

Below we apply the *dynamic quantization* method to the following deterministic non-conservative systems: a particle with linear friction, a van der Pol system and a Duffing system.

(i) *Linearly damped particle.* The Newton equations for this dissipative system are given by

$$\dot{p} = -\frac{\partial V}{\partial q} - \beta p \tag{23}$$

$$\dot{q} = \frac{p}{m} \tag{24}$$

with divergence equal to $-\beta$. The correspondent generalized Liouville equation is

$$\frac{\partial F}{\partial t} + \frac{p}{m}\frac{\partial F}{\partial q} - \left(\beta p + \frac{\partial V}{\partial q}\right)\frac{\partial F}{\partial p} = \beta F, \tag{25}$$

while in terms of the classical Wigner function we have

$$\imath\ell\frac{\partial\chi}{\partial t}+\frac{\ell^2}{2m}\left[\frac{\partial^2\chi}{\partial q_1^2}-\frac{\partial^2\chi}{\partial q_2^2}\right]+\frac{\imath\ell\beta}{2}(\,q_1-q_2)\left(\frac{\partial\chi}{\partial q_1}-\frac{\partial\chi}{\partial q_2}\right)-\mathcal{A}\chi=0 \tag{26}$$

where

$$\mathcal{A}=V(q_1,t)-V(q_2,t)-\sum_{n=3,5,7,\ldots}^{\infty}\frac{2}{n!}\left(\frac{q_1-q_2}{2}\right)^n\left(\frac{\partial}{\partial q_1}+\frac{\partial}{\partial q_2}\right)^n V(q_1,q_2,t). \tag{27}$$

Quantizing according to Eqs.(20) and (21), we arrive at

$$\imath\hbar\frac{\partial\rho}{\partial t}+\frac{\hbar^2}{2m}\left[\frac{\partial^2\rho}{\partial q_1^2}-\frac{\partial^2\rho}{\partial q_2^2}\right]+\frac{\imath\hbar\beta}{2}(\,q_1-q_2)\left(\frac{\partial\rho}{\partial q_1}-\frac{\partial\rho}{\partial q_2}\right)-[\,V(q_1,t)-V(q_2,t)]\,\rho=0. \tag{28}$$

This equation describes quantally a particle with linear friction. It is irreducible to any Schrödinger type equation. In particular, a damped harmonic oscillator does not possess a wave function. This result is the same obtained by Dekker [14]. Note still that (28) is the Caldeira-Leggett equation [20,23] without the fluctuation term.

(ii) *The van der Pol system.* The van der Pol nonlinear differential equations are

$$\dot{p}=-\frac{\partial V}{\partial q}+\beta p(1-q2) \tag{29}$$

$$\dot{q}=\frac{p}{m}. \tag{30}$$

It is a system with nonlinear damping originally modelling an electric circuit [24,25]. Its divergence is $\beta(1-q2)$ whereas the Liouville equation has the form

$$\frac{\partial F}{\partial t}+\frac{p}{m}\frac{\partial F}{\partial q}+\left(\beta p(1-q2)-\frac{\partial V}{\partial q}\right)\frac{\partial F}{\partial p}=-\beta(1-q2)F. \tag{31}$$

In the Wigner representation this equation turns out to be

$$\imath\ell\frac{\partial\chi}{\partial t}+\frac{\ell^2}{2m}\left[\frac{\partial^2\chi}{\partial q_1^2}-\frac{\partial^2\chi}{\partial q_2^2}\right]-\frac{\imath\ell\beta}{2}(\,q_1-q_2)\left[1-\left(\frac{q_1-q_2}{2}\right)2\right]\left(\frac{\partial\chi}{\partial q_1}-\frac{\partial\chi}{\partial q_2}\right)-\mathcal{A}\chi=0, \tag{32}$$

$\mathcal{A}$ being the same expression (27). In the quantum domain we get

$$\imath\hbar\frac{\partial\rho}{\partial t}+\frac{\hbar^2}{2m}\left[\frac{\partial^2\rho}{\partial q_1^2}-\frac{\partial^2\rho}{\partial q_2^2}\right]-\frac{\imath\hbar\beta}{2}(\,q_1-q_2)\left[1-\left(\frac{q_1-q_2}{2}\right)2\right]\left(\frac{\partial\rho}{\partial q_1}-\frac{\partial\rho}{\partial q_2}\right)-\mathcal{B}\rho=0, \tag{33}$$

with

$$\mathcal{B}\rho=[\,V(q_1,t)-V(q_2,t)]\,\rho. \tag{34}$$

In the case of a harmonic potencial this Eq.(33) describes a quantum van der Pol oscillator.

(iii) *The Duffing system.* It is a linearly damped nonlinear system under the action of a generalized potencial $V+(\kappa/4)q4$. The nonconservative Liouville equation, associated with the Duffing equations

$$\dot{p}=-\frac{\partial V}{\partial q}-\beta p-\kappa q^3 \tag{35}$$

$$\dot{q}=\frac{p}{m}, \tag{36}$$

is

$$\frac{\partial F}{\partial t}+\frac{p}{m}\frac{\partial F}{\partial q}-\left(\beta p+\kappa q^3\frac{\partial V}{\partial q}\right)\frac{\partial F}{\partial p}=\beta F, \tag{37}$$

with divergence $-\beta$. The term $\kappa q3$ is related to the stiffness to which a given mechanical system is submitted [25].

In terms of the classical Wigner function we find the equation

$$\imath\ell\frac{\partial\chi}{\partial t}+\frac{\ell^2}{2m}\left[\frac{\partial^2\chi}{\partial q_1^2}-\frac{\partial^2\chi}{\partial q_2^2}\right]+\frac{\imath\ell\beta}{2}(\,q_1-q_2)\left(\frac{\partial\chi}{\partial q_1}-\frac{\partial\chi}{\partial q_2}\right)-\kappa\left(\frac{q_1+q_2}{2}\right)3\,(\,q_1-q_2)\,\chi-\mathcal{A}\chi=0 \tag{38}$$

($\mathcal{A}$ is given by (27)) the quantization of which is

$$\imath\hbar\frac{\partial\rho}{\partial t}+\frac{\hbar^2}{2m}\left[\frac{\partial^2\rho}{\partial q_1^2}-\frac{\partial^2\rho}{\partial q_2^2}\right]+\frac{\imath\hbar\beta}{2}(\,q_1-q_2)\left(\frac{\partial\rho}{\partial q_1}-\frac{\partial\rho}{\partial q_2}\right)-\kappa\left(\frac{q_1+q_2}{2}\right)3\,(\,q_1-q_2)\,\rho-\mathcal{B}\rho=0, \tag{39}$$

with $\mathcal{B}\rho$ given by (34).

In order to show that the quantum equations of motion above obtained are not a mere artifact of the dynamic quantization method, in the next section we will evaluate their classical limit.

III. CLASSICAL LIMIT OF QUANTUM DYNAMICS

Inspired on the purely formal works by Hermann [26] we shall motivate the definition of a classical limiting process starting with the one-dimensional Schrödinger equation for a particle with mass m subjected to an external scalar potential $V = V(q,t)$

$$\Theta_\hbar \psi_\hbar(q,t) = 0, \tag{40}$$

where $\Theta_\hbar = (-\hbar/2m)(\partial^2/\partial q^2) + V - \imath\hbar\partial/\partial t$. Taking the limit $\hbar \to 0$ directly about Eq.(40) is just senseless. Our main idea then is to perform the following unitary transformation

$$\psi'_\hbar = e^{-\imath\xi/\hbar}\psi_\hbar \quad ; \quad \Theta'_\hbar = e^{-\imath\xi/\hbar}\Theta_\hbar e^{\imath\xi/\hbar}, \tag{41}$$

so that Eq.(40) becomes

$$\left\{ \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial q^2} - \frac{\imath\hbar}{2m}\left[\frac{\partial^2 \xi}{\partial q^2} + 2\frac{\partial \xi}{\partial q}\frac{\partial}{\partial q} + 2m\frac{\partial}{\partial t}\right] + \left[\frac{1}{2m}\left(\frac{\partial \xi}{\partial q}\right)^2 + V + \frac{\partial \xi}{\partial t}\right]\right\}\psi'_\hbar = 0. \tag{42}$$

Now we can take $\hbar \to 0$ about Eq.(42) yielding (with $\xi = S(q,t)$)

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + V + \frac{\partial S}{\partial t} = 0 \tag{43}$$

which in turn may be put in the form $-\partial S/\partial t = H(\partial S/\partial q, q, t)$, because $\lim_{\hbar\to 0}\left\{\hat{p}' = e^{-\imath S/\hbar}\left(-\imath\hbar\partial/\partial q\right)e^{\imath S/\hbar}\right\} = \partial S/\partial q \equiv p$. Note that once obtained the classical equation (43), we can also evaluate the classical limit of quantum kinematics, e.g., for the Heisenberg relations $[\hat{p},\hat{q}] = \hat{p}\hat{q} - \hat{q}\hat{p} = -\imath\hbar$ we find that $\lim_{\hbar\to 0}\left\{e^{-\imath S/\hbar}[\hat{p},\hat{q}]e^{\imath S/\hbar}\right\} = \lim_{\hbar\to 0}\left\{e^{-\imath S/\hbar}(-\imath\hbar)e^{\imath S/\hbar}\right\}$ yields $pq = qp$.

Equation (43) is exactly the well-known Hamilton-Jacobi partial differential equation of classical mechanics. Therefore, the classical limit of the Schrödinger equation (40) is the Hamilton-Jacobi equation (43). An important point to be emphasized is that the limit $\hbar \to 0$ about (42) only exists if the following *asymptotic* conditions are obeyed:

$$\lim_{\hbar\to 0}\psi'_\hbar \sim \psi''_\hbar \neq 0 \tag{44}$$

$$\lim_{\hbar\to 0}\hbar\psi'_\hbar \sim 0 \tag{45}$$

$$\lim_{\hbar\to 0} \hbar \frac{\partial \psi'_\hbar}{\partial x} \sim 0, (x = q, t) \tag{46}$$

$$\lim_{\hbar\to 0} \hbar^2 \frac{\partial^2 \psi'_\hbar}{\partial q^2} \sim 0. \tag{47}$$

In summary, if a given physical system is described by the Schrödinger equation, only for those solutions $\psi'_\hbar$ obeying the above asymptotic relations as $\hbar \to 0$, it can be classically described by the Hamilton-Jacobi equation. For instance, let us consider $\psi_\hbar = e^{\imath A/\hbar}$ as a solution of Eq.(40). Expanding the function $A(q,t)$ in powers of $\hbar/\imath$ [27]: $A = S_0 + (\hbar/\imath)S_1 + (\hbar/\imath)^2 S_2 + ...$, it follows that $\psi'_\hbar = e^{-\imath\xi/\hbar} e^{\imath[S_0 + (\hbar/\imath)S_1 + (\hbar/\imath)^2 S_2 + ...]/\hbar}$. Using the validity conditions of the WKB method [27] in order to neglect terms of order $\hbar^3$ and so on, we obtain the asymptotics

$$\lim_{\hbar\to 0} \psi'_\hbar \sim \psi''_\hbar = e^{S_1} e^{\hbar S_2/\imath},$$

where we have used the fact that ξ and S_0 obey the same equation of motion, hence $\xi \equiv S_0 = S$. It is worth noticing that for the superposition of WKB functions $\psi_\hbar = e^{\imath A/\hbar} + e^{\imath B/\hbar}$, with $B = S_0 - (\hbar/\imath)S_1 - (\hbar/\imath)^2 S_2 - ...$, our conditions (44-47) are also satisfied. This means that the validity conditions of the WKB approximation are not necessary to obtain the classical limit of the Schrödinger equation.

Based on above procedure we propose the following general classical limiting method of quantum-mechanical equations of motion:

DEFINITION. Let a quantum-mechanical differential equation be given by

$$D_\hbar \Psi_\hbar = 0. \tag{48}$$

By performing the transformation

$$\Psi'_\hbar = e^{-\alpha\xi/\hbar} \Psi_\hbar, \tag{49}$$

α being a free parameter, Eq.(48) becomes

$$D'_\hbar \Psi'_\hbar = 0, \tag{50}$$

with $D'_\hbar = e^{-\alpha\xi/\hbar} D_\hbar e^{\alpha\xi/\hbar}$*. By taking the classical limit* $\hbar \to 0$ *in Eq.(50), we arrive at the classical evolution equation for the function* ξ *(independent of* $\hbar$*):* $D^\xi = 0$*, since asymptotic conditions are imposed on the behaviour of the functions* $\Psi'_\hbar, \hbar\Psi'_\hbar$*, etc., and their derivatives.* These asymptotics are valid in the semiclassical domain of quantum mechanics.

In order to apply our above definition, let us evaluate the classical limit of the time evolution equation for the Wigner function $W(p,q,t)$

$$\left\{ \hbar \frac{\partial}{\partial t} + \hbar \frac{p}{m}\frac{\partial}{\partial q} - \hbar \sum_{k=0}^{\infty} \frac{(-)^k (\hbar/2)^{2k}}{(2k+1)!} \frac{\partial^{2k+1} V}{\partial q^{2k+1}} \frac{\partial^{2k+1}}{\partial p^{2k+1}} \right\} W = 0, \tag{51}$$

obtained from the von Neumann equation for the density matrix $\rho(q + \hbar\eta/2, q - \hbar\eta/2) = \psi^*(q + \hbar\eta/2)\psi(q - \hbar\eta/2)$ and making use of the Wigner transform $W(p,q,t) = (1/2\pi)\int_{-\infty}^{+\infty} \rho(q+\hbar\eta/2, q-\hbar\eta/2) e^{\imath p\eta} d\eta$ [28]. In the literature [29] one still believes that it is enough making $\hbar \to 0$ directly about Eq.(51) (divided by $\hbar$) to obtain immediately the classical Liouville equation. This procedure is not generally correct, simply because W is just a quantal object and does not possess a limit when $\hbar \to 0$ [30]. Furthermore, to make the Wigner function W propagate classically does not mean to obtain the classical limit of Eq.(51). However, it is straightforward to show that by means of the transformation (49) for $\Psi_\hbar \equiv W$, and assuming the parameter α infinitesimal, i.e., $\alpha^2 \approx 0$, we obtain the classical Liouville equation for the probability distribution $\xi \equiv F(p,q,t) \geq 0$:

$$\frac{\partial F}{\partial t} = -\frac{p}{m}\frac{\partial F}{\partial q} + \frac{\partial V}{\partial q}\frac{\partial F}{\partial p}, \tag{52}$$

since there exist the following asymptotics

$$\lim_{\hbar \to 0} W' \sim W'' \neq 0 \tag{53}$$

$$\lim_{\hbar \to 0} \hbar^n W' \sim 0, (n = 2, 4, 6, ..., \infty) \tag{54}$$

$$\lim_{\hbar \to 0} \hbar \frac{\partial W'}{\partial x} \sim 0, (x = q, t) \tag{55}$$

$$\lim_{\hbar \to 0} \hbar^j \frac{\partial^n W'}{\partial p^n} \sim 0, (j, n = 1, 2, 3, ..., \infty). \tag{56}$$

In expression (56) $n \leq j$ for j even and $n = j$ for j odd.

Once applied our classical limiting process to conservative equations of motion, we want now to apply it to the dissipative equations (28), (33) and (39). To evaluate the classical limit of Eq.(28) we use the quantum Wigner function $W(q, p, t)$, so that

$$\frac{\partial W}{\partial t} + \frac{p}{m}\frac{\partial W}{\partial q} - \left[\ \beta p + \frac{\partial V}{\partial q}\right]\frac{\partial W}{\partial p} + GW = \beta W, \tag{57}$$

with

$$GW = -\frac{2}{\imath 3!}\left(\frac{-\hbar}{2\imath}\right)^3 \frac{\partial^3 V}{\partial q^3}\frac{\partial^3 W}{\partial p^3} - \frac{2}{\imath 5!}\left(\frac{-\hbar}{2\imath}\right)^5 \frac{\partial^5 V}{\partial q^5}\frac{\partial^5 W}{\partial p^5} - \ldots \quad . \tag{58}$$

We perform the transformation

$$W' = e^{-\epsilon\xi/\hbar} W; (\epsilon 2 \approx 0), \tag{59}$$

we take $\hbar \to 0$ and get Eq.(25) for $\xi \equiv F$, since

$$\lim_{\hbar \to 0} W' \sim W'' \neq 0 \tag{60}$$

$$\lim_{\hbar \to 0} \frac{\hbar W'}{\epsilon} \sim \xi W' \tag{61}$$

$$\lim_{\hbar \to 0} \hbar^n W' \sim 0, (n = 2, 4, 6, \ldots, \infty) \tag{62}$$

$$\lim_{\hbar \to 0} \hbar \frac{\partial W'}{\partial x} \sim 0, (x = q, t) \tag{63}$$

$$\lim_{\hbar \to 0} \hbar^j \frac{\partial^n W'}{\partial p^n} \sim 0, (j, n = 1, 2, 3, \ldots, \infty), n \leq j(\text{par}), n = j(\text{ímpar}). \tag{64}$$

In quantum phase space Eq.(33) has the form

$$\frac{\partial W}{\partial t} + \frac{p}{m}\frac{\partial W}{\partial q} + \left[\ \beta p(1 - q2) - \frac{\partial V}{\partial q}\right]\frac{\partial W}{\partial p} + GW = -\beta(1 - q2)W. \tag{65}$$

Using (49) in the form (59) we obtain in the classical limit the generalized Liouville equation (31) with the same asymptotic conditions (60-64).

Now expressing the quantum Duffing system (39) in phase space as

$$\frac{\partial W}{\partial t}+\frac{p}{m}\frac{\partial W}{\partial q}-\left(\beta p+\kappa q^{3}\frac{\partial V}{\partial q}\right)\frac{\partial W}{\partial p}+GW=\beta W, \tag{66}$$

it follows easily that the classical limit of this equation, once performed the transformation (59), is Eq.(37).

Here we restrict ourselves only to investigate the *logical consistence* of the dynamic quantization process as applied to a linearly damped particle and the van der Pol and the Duffing systems. The main result is that a description of nonconservative systems in terms of wave function is secondary and, in general, impossible. Such systems can be quantally described through the von Neumann function or equivalently by the Wigner function in the quantum phase space. A study of the solutions of these quantized systems will appear in a future paper.

IV. FINAL REMARKS

In this paper was presented a mathematical and objective criterion to classify a deterministic dynamical system as conservative or nonconservative. In particular we saw that the Hamiltonian formalism is not unique concerning the nonconservative Newtonian systems. This implies that the (canonical) quantization of these systems is totally ambiguous. In order to overcome this difficult, recently [18-21] it has been proposed and investigated a quantization process starting from the *equations of motion* within a probabilistic framework. Using such *dynamic* quantization we were able to quantize a particle with linear friction and the van der Pol and the Duffing systems. The logical consistence of these nonconservative equations of motion was verified through the definition of a novel classical limiting process $\hbar \to 0$ about the quantum dynamics.

One can argue [31] that a correct quantization of nonconservative systems should take into account explicitly the physics of the heat bath in order that the usual quantization methods, based on Lagrangians or Hamiltonians, can be employied. However, we have shown in Ref.[20] that the explicit treatment of the thermal reservoir is not necessary to quantize a particle realizing a Brownian-type motion. The important is the stochastic dynamical system or the correspondent Fokker-Planck equation, and not the Hamiltonian model.

In conclusion, the dynamic quantization reveals that nonconservative

systems are not described in terms of wave function. In the case of conservative systems the important result also is valid [21]: The wave function ψ is *derived* from the density matrix or von Neumann function ρ. The mathematical object fundamental, to quantum mechanics, is the von Neumann function, and not the function ψ.

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