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2D Feynman's propagator for a time variable magnetic field: Exact solution

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Abstract

We evaluate exactly the Feynman's propagator of a quantum mechanical two dimensional particle in a time variable magnetic field.

An important topic in the mathematical physics of Quantum mechanics is the study of the direct solubility of the Schrödinger equation in the presence of adequate potentials ([1]).

In this note we solve exactly the Green function for initial conditions of a two-dimensional particle in the presence of a time-variable magnetic field (the Feynman quantum mechanical propagator) by means of a suitable space time Ray-Reid coordinate transformation as proposed in ref. [2].

Let us start our note by considering the Hamiltonian of our system

$$H = \frac{1}{2m}\left[(P_x + \frac{e}{c}\ A_x(x,y,t))^2 + (P_y + \frac{e}{c}\ A_y(x,y,t))^2\right] \tag{1}$$

where the generic time varying magnetic field $B(t)$ is applied along the z-axis and the vector potential $\vec{A} = (A_x(x,y,t), A_j(x,y,t))$ is written in the suitable form

$$\vec{A} = -\frac{1}{2}(yB(t))\vec{i} + \frac{1}{2}(xB(t))\vec{j} \tag{2}$$

The associated Schrödinger equation reads

$$i\hbar\ \frac{\partial}{\partial t}\psi(x,y,t) = \\ = \left[-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - \frac{\hbar}{i}w_c(t)\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) + \frac{1}{8}m\ w_c^2(t)(x^2+y^2)\right]\psi(x,y,t) \tag{3}$$

with

$$\psi(x,y,t\to 0^+) = \delta(x-\vec{x}')\delta(y-y') \tag{4}$$

and $w_c(t) = \dfrac{e}{mc}B(t)$ is the cyclotron frequency.

In order to integrate exactly eq. (4) we make the following space-time, coordinate change ([2])

$$\begin{aligned} x &= s_1(\tau)\bar{x} + s_2(\tau)\bar{y} \\ y &= -s_2(\tau)\bar{x} + s_1(\tau)\bar{y} \\ t &= \int_0^\tau \mu^{-1}(\sigma)d\sigma \\ \psi(x,y,t) &= \bar{\psi}(\bar{x},\bar{y},\tau) \end{aligned} \tag{5}$$

with the correspondent inverse

$$\bar{x} = \frac{s_1(\tau)}{(s_1^2+s_2^2)(\tau)}\ x - \frac{s_2(\tau)}{(s_1^2+s_2^2)(\tau)}\ y$$

$$\bar{y} = \frac{s_2(\tau)}{(s_1^2+s_2^2)(\tau)}\ x - \frac{s_1(\tau)}{(s_1^2+s_2^2)(\tau)}\ y$$
$$\tau = \tau(t) = \int_0^t \mu(\sigma) d\sigma \tag{6}$$

Let us introduce the notation:

$$A = \frac{1}{(s_1^2+s_2^2)(\tau)}$$
$$a = s_1(\tau)s_1'(\tau) + s_2(\tau)s_2'(\tau) \tag{7}$$
$$b = s_1'(\tau)s_2(\tau) - s_2'(\tau)s_1(\tau)$$

so that we have the following results:

$$i\hbar\ \frac{\partial\psi}{\partial t}(x,y,t) = i\hbar\ \mu(\tau)\left\{\frac{\partial}{\partial\tau} + A(-a\bar{x}+b\bar{y})\frac{\partial}{\partial\bar{x}} - A(b\bar{x}+a\bar{y})\frac{\partial}{\partial\bar{y}}\right\}\bar{\phi}(\bar{x},\bar{y},\tau) \tag{8.a}$$

with

$$\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\psi(x,y,t) = \left(\bar{y}\frac{\partial}{\partial\bar{x}} - \bar{x}\frac{\partial}{\partial\bar{y}}\right)\bar{\psi}(\bar{x},\bar{y},\tau) \tag{8.b}$$

and

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi(x,y,\tau) = A\left(\frac{\partial^2}{\partial\bar{x}^2} + \frac{\partial^2}{\partial\bar{y}^2}\right)\psi(\bar{x},\bar{y},\bar{\tau}) \tag{8.c}$$

If we consider the general decomposition of the wave function in a "complex" polar form ([1])

$$\bar{\psi}(\bar{x},\bar{y},\tau) = \chi(\bar{x},\bar{y},\tau)\ e^{\frac{i}{\hbar}F(\bar{x},\bar{y},\tau)} \tag{9}$$

we obtain the Schrödinger equation in the new coordinate system eq. (6)

$$\begin{aligned}
O = &-\frac{\hbar^2}{2m}A\ \left(\frac{\partial^2}{\partial\bar{x}^2} + \frac{\partial^2}{\partial\bar{y}^2}\right)\chi(\bar{x},\bar{y},\tau) + i\hbar\mu(\tau)\frac{\partial}{\partial\tau}\ \chi(\bar{x},\bar{y},\tau) + \\
&+\frac{\partial\chi}{\partial\bar{x}}(\bar{x},\bar{y},\tau)\left[\frac{i\hbar A}{m}\frac{\partial F}{\partial\bar{x}} + i\hbar\mu A(-a\bar{x}+b\bar{y}) - i\frac{\hbar w_c\bar{y}}{2}\right] \\
&+\frac{\partial\chi}{\partial\bar{y}}(\bar{x},\bar{y},\tau)\left[\frac{i\hbar A}{m}\frac{\partial F}{\partial\bar{y}} - i\hbar\mu A(b\bar{x}+a\bar{y}) + i\frac{\hbar w_c}{2}\bar{x}\right] \\
&+\chi\left(\frac{i\hbar A}{m}\left(\frac{\partial^2 F}{\partial\bar{x}^2} + \frac{\partial^2 F}{\partial\bar{y}^2}\right) - \frac{A}{2\mu}\left(\left(\frac{\partial F}{\partial\bar{x}}\right)^2 + \frac{\partial F}{\partial\bar{y}}\right)^2\right) - \\
&-\frac{1}{8}\frac{mw_c^2}{A}(\bar{x}^2+\bar{y}^2)) - \mu\ \frac{\partial F}{\partial\tau} - \mu\ A(-a\bar{x}+b\bar{y})\ \frac{\partial F}{\partial\bar{x}} \\
&+\mu A(b\bar{x}+a\bar{y})\frac{\partial F}{\partial\bar{y}} + \frac{w_c}{2}(\tau)\left(\bar{y}\frac{\partial F}{\partial\bar{x}} - \bar{x}\frac{\partial F}{\partial\bar{y}}\right)
\end{aligned} \tag{10}$$

At this point we determine the exact functional form of our space-time variable change by imposing the vanishing of the first order derivative terms in eq. (10)

$$F = \frac{m\mu a}{2}(\bar{x}^2 + \bar{y}^2) + g(\tau) \tag{11.a}$$

$$\mu(\tau) = \frac{w_c(t(\tau))}{2(bA)(\tau)} \tag{11.b}$$

By imposing either the vanishing of the potential like term in our transferred Schrödinger equation we get the equation for function $g(\tau)$ in eq. (11.a)

$$\frac{dg}{d\tau} = i\hbar(Aa)(\tau) \tag{12}$$

The situation now is the following:

$$\frac{\hbar^2}{2m}A(\tau)\left(\frac{\partial^2}{\partial\bar{x}^2} + \frac{\partial^2}{\partial\bar{y}^2}\right)\chi(\bar{x},\bar{y},\tau) = -i\hbar\ \mu(\tau)\ \frac{\partial\chi}{\partial\tau}(\bar{x},\bar{y},\tau) \tag{13}$$

which is the free-particle equation with unitary mass if we choose the $\mu(\tau)$ function in the form

$$\frac{A(\tau)}{m(\tau)\mu(\tau)} = 1 \tag{14}$$

with the solution

$$\chi(\bar{x},\bar{y},\tau) = \int d\bar{x}'d\bar{y}'\ K_0(\bar{x},\bar{y},\tau|(\bar{x}',\bar{y}',\tau'))\chi(\bar{x}',\bar{y}',\tau') \tag{15}$$

where the Free-particle Feynman propagator given by

$$K_0\ (\bar{x},\bar{y},\tau|(\bar{x}',\bar{y}',\tau'))\ = \frac{1}{(2\pi i\hbar(\tau-\tau'))}\ exp\left\{\frac{i}{2\hbar}\left[\frac{(\bar{x}-\bar{x}')^2 + (\bar{y}-\bar{y}')^2}{\tau-\tau'}\right]\right\} \tag{16}$$

which due to our "complex" polar form eq. (5) leads to the complete coordinate transformed propagator

$$\begin{aligned} &K(\bar{x},\bar{y},\tau|(\bar{x}',\bar{y}',\tau')) \\ &= exp\left(\frac{i}{\hbar}F(\bar{x},\bar{y},\tau)\right)K_0(\bar{x},\bar{y},\tau|(\bar{x}',\bar{y}',\tau'))exp\left(\frac{i}{\hbar}F(\bar{x}',\bar{y}',\tau')\right) \end{aligned} \tag{17}$$

In order to write this Feynman propagator in the original variables (x, y, t) we consider the polar form for the functions $s_1(\tau)$ and $s_2(\tau)$ as in refs. [2], [3]

$$\begin{aligned} s_1(\tau) &= \rho(\tau)cos\theta(\tau) \\ s_2(\tau) &= \rho(\tau)sen\theta(\tau) \end{aligned} \tag{18}$$

so that

$$a(\tau) = \rho(\tau)\ \frac{d\rho}{d\tau}(\tau) \tag{19.a}$$

$$b(\tau) = -\rho^2(\tau)\ \frac{d\theta}{d\tau}(\tau) \tag{19.b}$$

$$A(\tau) = \frac{1}{\rho^2(\tau)} \tag{19.c}$$

which by its turn leads to the following explicit results (see eq. (1) and eq. ((2)) and

$$g(\tau) = i\hbar \ln \rho(\tau), \tag{20}$$

$$F(\bar{x},\bar{y},\tau) = \mu(\tau)\rho(\tau)\frac{d\rho}{d\tau}(\tau)\ \frac{(\bar{x}^2+\bar{y}^2)}{2} + i\hbar\ \ln\ \rho(\tau) \tag{21}$$

By writing eq. (20) in the original coordinate system we get the exact expression for our complex phase of our propagator

$$F(x,y,t) = \frac{1}{2}\ \frac{\frac{d}{dt}\rho(t)}{\rho(t)}(x^2+y^2) + i\hbar\ \ln\rho(t) \tag{22}$$

Let us note that

$$\theta(t) = -\frac{e}{2mc}\int_0^t\ B(\sigma)d\sigma \tag{23}$$

with

$$\mu(t) = \frac{1}{\rho^2(t)} \tag{24}$$

and

$$\frac{d^2\rho(t)}{dt^2} + \frac{e^2}{8c^2m}\ B^2(t)\rho(t) = 0 \tag{25}$$

with suitable Sturn-Liouville boundary conditions ([1]).

By grouping together the above results we obtain the exact expression for our Feynman propagator (see eqs. (3)-(4))

$$K(x,y,t,x',y',t') = \frac{\rho(t')}{\rho(t)}\left(\frac{1}{2\pi i\hbar\int_{t'}^t\mu(\sigma)d\sigma}\right)^{\frac{1}{2}}$$
$$exp\left\{\frac{i}{\hbar}\frac{m}{2}\frac{d\rho(t)}{dt}/\rho(t)(x^2+y^2)\right\}\exp\left\{-\frac{i}{\hbar}\frac{m}{2}\frac{d\rho(t')}{dt'}/\rho(t')(x'^2+y'^2)\right\}$$
$$exp\left\{\frac{i}{2\hbar}\left[\frac{Re\left[\left(\frac{e^{i\theta(t)}}{\rho(t)}x-\frac{e^{i\theta(t')}}{\rho(t')}x'\right)^2+\left(\frac{e^{i\theta(t)}}{\rho(t)}y-\frac{e^{i\theta(t')}}{\rho(t')}y'\right)^2\right]+Im\left[\frac{xy'+yx'}{\rho(t)\rho(t')}e^{i(\theta(t)-\theta(t'))}\right]}{\int_{t'}^t\mu(\sigma)d\sigma}\right]\right\} \tag{26}$$

This is our main and new result of our note.

One very important result which can be obtained from eq. (26) is that one get by considering the quantum mechanical probability density of our system ([4])

$$|K(x,y,t;\bar{x},\bar{y},t')|^2 = \left(\frac{\rho(t')}{\rho(t)}\right)^2 \cdot \frac{1}{2\pi\hbar\int_{t'}^{t}\mu(\sigma)d\sigma} \tag{27}$$

At this point we note that if $B(t)$ is periodic with period T, one get on the basis of eq. (23) and eq. (25) that the probability density eq. (27) will be periodic too by the same period T ([4]).

Finally let us comment that extensive applications of the exact Feynman propagator in random magnetic fields $B(t)$ ([5]) will be presented elsewhere.

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