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Generalization of Planck Radiation Law: Possible Testing of the Nature of Space-Time

by

C. Tsallis[1] and *F.C. Sá Barreto*

//[1]Centro Brasileiro de Pesquisas Físicas – CPBF/CNPq
Rua Dr. Xavier Sigaud, 150
22290-180 – Rio de Janeiro, RJ – Brazil

[2]Departamento de Física
Universidade Federal de Minas Gerais
C.P. 702
31270-970 – Belo Horizonte – MG, Brazil

Abstract

Within the framework of the recently introduced nonextensive Statistical Mechanics, we generalize the Planck law for the black-body radiation. Quantitative criteria are established that could provide astrophysical observational evidence for a discontinuous (or continuous but not differentiable) nature for space-time.

Key-words: Black-body radiation; Nonextensive Statistical Mechanics and Thermodynamics; Space-time; Quantum Gravity.

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The deep nature of space-time no doubt is one of the most challenging questions in contemporary Physics. The possibility for space-time not having the 4-dimensional *continuous and differentiable* structure we are accustomed to (in Classical and Quantum Mechanics, Maxwell equations, Special and General Relativity) has been tackled in many occasions. For example, the possible physical interpretation of Planck's length ($L_P \equiv \sqrt{\hbar G/c^3} \simeq 1.6 \times 10^{-33} cm$) and time ($T_P \equiv L_P/c \simeq 5.4 \times 10^{-44} s$) has been lengthily discussed ([1] and references therein). At extremely high energies (those corresponding to just after a Bing-Bang-like explosion or, equivalently, those at which all elementary forces, including the gravitational one, would be unified), relevant phenomena would presumably occur at a scale comparable to L_P and T_P. At this extremely microscopic level (or, perhaps, even before), the usual differential-equation Physics would become unsatisfactory approximations and should have to be replaced by finite-difference equations (see, for instance, [2, 3] and references therein). Consistently, the possibility of *non-conventional* (discontinuous, or continuous but not differentiable) space-time is considered in [4] (fractal-like) and also in [5] (lattice-like). The latter is discussed in the framework of Quantum Groups ([6–8] and references therein), where it becomes explicitly related to *nonextensive* (or *nonadditive*) Mechanics, the *conventional* (continuous and differentiable) space-time being recovered in the *extensive* particular case. Moroever, the need for nonextensive thermodynamics is well known in Cosmology, Gravitation and Astrophysics (e.g., black holes, superstrings, 3-dimensional gravitational N-body problem; see [9–11] and references therein); it might even be a common feature whenever the (linear) size of the system is smaller than or comparable to the range of the relevant interactions between the elements of the system [12]. Very recently, formal nonextensive Thermodynamics and Statistical Mechanics have become available ([13, 14]; see also [15–19]). It is within this (generalized) framework that we intend to discuss the black-body radiation.

The formalism starts by postulating [13] a generalized form for the entropy, namely

$$S_q \equiv k \frac{1 - \sum_s p_s^q}{q-1} \qquad (q \in \Re) \tag{1}$$

where $\{p_s\}$ are the probabilities of the microscopic states and k is a conventional positive constant. In the $q \to 1$ limit, S_q recovers the well known Shannon form $-k_B \sum_s p_s \ln p_s$, from which standard Statistical Mechanics and Thermodynamics follow. S_q satisfies a variety of (*generalized*) basic properties such as positivity, concavity, H-theorem [15], Ehrenfest theorem [17], von Neumann equation [18], Shannon theorem [14], fluctuation-dissipation theorem, Langevin and Fokker-Planck equations, Bogolyubov inequality (see [12] for a review), among many others. Also, it has been shown [16] that it overcomes, for $q \neq 1$, the quite known Boltzmann-Gibbs inability to provide *finite* mass for astrophysical systems within the polytropic model (as studied by Chandrasekhar and others).

Let us now reproduce here a property (*pseudo-additivity*), which is relevant in the present context. If Σ and Σ' are two *independent* systems (in the sense that $p_{s,s'}^{\Sigma U \Sigma'} = p_s^{\Sigma} p_{s'}^{\Sigma'}$) then

$$\frac{S_q^{\Sigma U \Sigma'}}{k} = \frac{S_q^{\Sigma}}{k} + \frac{S_q^{\Sigma'}}{k} + (1-q) \frac{S_q^{\Sigma}}{k} \frac{S_q^{\Sigma'}}{k} \tag{2}$$

In other words, $(1-q)$ is a *measure of the lack of extensivity* of the system (and is herein thought as being related to the *lack of continuity, or of differentiability, of space time*).

Eq. (1) yields for the canonical ensemble, the following equilibrium density operator [13, 14],

$$\hat{\rho} = [\hat{1} - (1-q)\beta\hat{\mathcal{H}}]^{\frac{1}{1-q}}/Z_q \tag{3}$$

where $\beta \equiv 1/kT$ is the Lagrange parameter associated with the thermostat, $\hat{\mathcal{H}}$ is the Hamiltonian, and the generalized partition function is given by

$$Z_q \equiv Tr[\hat{1} - (1-q)\beta\hat{\mathcal{H}}]^{\frac{1}{1-q}} \tag{4}$$

In the $q \to 1$ limit we recover Boltzmann-Gibbs distribution $\hat{\rho} \propto \exp(-\beta\hat{\mathcal{H}})$. It can be shown [14] that, $\forall q$, $1/T = \partial S_q/\partial U_q$, $U_q \equiv Tr\hat{\rho}^q\hat{\mathcal{H}} = -\frac{\partial}{\partial\beta}\frac{Z_q^{1-q}-1}{1-q}$ and $F_q \equiv U_q - TS_q = -\beta\frac{Z_q^{1-q}-1}{1-q}$. The relevant mean value associated with any observable $\hat{O}$ is given [14, 17, 18] by

$$\langle\hat{O}\rangle_q \equiv Tr\hat{\rho}^q\hat{O} = \langle\hat{\rho}^{q-1}\hat{O}\rangle_1 \tag{5}$$

In the $\beta(1-q) \to 0$ limit (which we focus from now on), Eq. (4) *asymptotically* becomes

$$\begin{aligned} Z_q &= Tr\exp\left\{\frac{1}{1-q}\ln\,[\hat{1} - (1-q)\beta\hat{\mathcal{H}}]\right\} \\ &\sim Z_{BG}\left\{1 - \frac{1}{2}(1-q)\beta^2\langle\hat{\mathcal{H}}^2\rangle_{BG}\right\} \end{aligned} \tag{6}$$

where BG stands for Boltzmann-Gibbs and where we have retained only the $(1-q)$-correction to the leading term. Also, by using Eq. (3), we can consistently rewrite Eq. (5) as follows

$$\begin{aligned} \langle\hat{O}\rangle_q &= Z_q^{1-q}\left\langle\frac{\hat{O}}{\hat{1} - (1-q)\beta\hat{\mathcal{H}}}\right\rangle_1 \\ &\sim Z_q^{1-q}\langle\hat{O}\rangle_{BG}\left\{1 + (1-q)\beta\left[\frac{\langle\hat{O}\hat{\mathcal{H}}\rangle_{BG}}{\langle\hat{O}\rangle_{BG}} + \frac{\beta}{2}\left(\langle\hat{\mathcal{H}}^2\rangle_{BG} - \frac{\langle\hat{O}\hat{\mathcal{H}}^2\rangle_{BG}}{\langle\hat{O}\rangle_{BG}}\right)\right]\right\} \\ &\sim Z_{BG}^{1-q}\langle\hat{O}\rangle_{BG}\left\{1 + (1-q)\beta\left[\frac{\langle\hat{O}\hat{\mathcal{H}}\rangle_{BG}}{\langle\hat{O}\rangle_{BG}} + \frac{\beta}{2}\left(\langle\hat{\mathcal{H}}^2\rangle_{BG} - \frac{\langle\hat{O}\hat{\mathcal{H}}^2\rangle_{BG}}{\langle\hat{O}\rangle_{BG}}\right)\right]\right\} \end{aligned} \tag{7}$$

where we used Eq. (6) in the last step.

Let us now consider the system $\hat{\mathcal{H}} = h\nu\hat{n}$, ν being the (photon) frequency, and $\hat{n}$ being the bosonic particle-number operator. By choosing $\hat{O} \equiv \hat{n}$, Eq. (7) becomes

$$\langle\hat{n}\rangle_q \sim \langle\hat{n}\rangle_{BG} Z_{BG}^{q-1}\left\{1 + (1-q)x\left[\frac{\langle\hat{n}^2\rangle_{BG}}{\langle\hat{n}\rangle_{BG}} + x\left(\langle\hat{n}^2\rangle_{BG} - \frac{\langle\hat{n}^3\rangle_{BG}}{\langle\hat{n}\rangle_{BG}}\right)\right]\right\} \tag{8}$$

with $x \equiv \beta h\nu$ and

$$Z_{BG} = \sum_{n=0}^{\infty} e^{-\beta x} = (1 - e^{-x})^{-1} \tag{9}$$

$$\langle\hat{n}\rangle_{BG} = \frac{e^{-x}}{1 - e^{-x}} = \frac{1}{e^x - 1} \tag{10}$$

$$\langle \hat{n}^2 \rangle_{BG} = \frac{e^{-x} + e^{-2x}}{(1-e^{-x})^2} \tag{11}$$

$$\langle \hat{n}^3 \rangle_{BG} = \frac{e^{-x} + 4e^{-2x} + e^{-3x}}{(1-e^{-x})^3} \tag{12}$$

Let us now calculate the $q = 1$ black-body radiation law assuming a $d+1$ space-time and $(d-1)$ transverse modes for the electromagnetic field (with spectrum given by $2\pi\nu = c|\vec{k}|$, where $\vec{k}$ is the d-dimensional wave vector). By following along the standard lines we obtain for the *photon energy density per unit volume*

$$D_1(\nu) = \frac{\pi^{d/2}(d-1)dh\nu^d}{\Gamma\left(\frac{d}{2}+1\right)c^d(e^{h\nu/k_BT}-1)} \tag{13}$$

For the $3+1$ space-time this expression recovers Planck law

$$D_1(\nu) = \frac{8\pi h\nu^3}{c^3(e^{h\nu/k_BT}-1)} \equiv D_{Planck}(\nu) \tag{14}$$

Although everything that follows could be done for a $d+1$ space-time, we shall from now on focus the standard $d = 3$ case. So, if we maintain the one-photon approach, Eq. (8) implies, for $q \simeq 1$,

$$D_q(\nu) \sim D_{Planck}(\nu)(1-e^{-x})^{q-1}\left\{1 + (1-q)x\left[\frac{1+e^{-x}}{1-e^{-x}} - \frac{x}{2}\frac{1+3e^{-x}}{(1-e^{-x})^2}\right]\right\} \tag{15}$$

hence

$$\frac{D_q(\nu)h^2c^3}{8\pi(k_BT)^3} \sim \frac{x^3}{e^x-1}(1-e^{-x})^{q-1}\left\{1 + (1-q)x\left[\frac{1+e^{-x}}{1-e^{-x}} - \frac{x}{2}\frac{1+3e^{-x}}{(1-e^{-x})^2}\right]\right\} \tag{16}$$

which generalizes *Planck law* and is illustrated in the figure. In the $h\nu << k_BT$ region we have

$$D_q(\nu) \sim \frac{8\pi(k_BT)^3}{h^2c^3}\left(\frac{h\nu}{k_BT}\right)^{1+q} \tag{17}$$

which generalizes *Rayleigh-Jeans law.* In the $h\nu >> k_BT$, the $(1-q)$-correction diverges, hence the present expansion is not valid; in other words, the high-frequency tail should be calculated by its own. Nevertheless, the natural variable indeed is $h\nu/k_BT$, consequently the *total emitted power per unit surface* $P_q \propto \int_0^\infty d\nu D_q(\nu)$ is given by

$$P_q \sim \sigma_q T^4 \tag{18}$$

Therefore, the *Stefan-Boltzmann law* remains the usual one, but with a q-dependent prefactor ($\sigma_1 = \pi^2 k_B^4/60\hbar^3c^2 \simeq 5.67\times10^{-12}\,watt/cm^2K^4$ is Stefan constant; $(d\sigma_q/dq)_{q=1} > 0$). A similar type of behavior was obtained within the Quantum Groups formalism (q_G-black-body [20]), where we introduced q_G to avoid confusion with q). A second immediate consequence of $h\nu/k_BT$ being the natural variable for all values of q is that the *Wien shift law* also preserves its form with a q-dependent prefactor. More precisely, a straightforward expansion of $D_q(\nu)$ (Eq. (16)) in the neighborhood of ν_M (location of the maximum of $D_q(\nu)$) yields

$$h\nu_M/k_BT \sim a_1 + a_2(q-1) \tag{19}$$

with $a_1 \simeq 2.821439$ and $a_2 \simeq 5.171965$. The comparison of this result with the corresponding one in Quantum Groups [20] shows an important difference: at fixed temperature, $\nu_M(q>1) > \nu_M(q=1) > \nu_M(q<1)$, whereas $\nu_M(q_G) = \nu_M(1/q_G)$ and $\nu_M(q_G<1) < \nu_M(q_G=1)$.

We verify that, for $\nu \simeq \nu_M$,

$$\frac{D_q(\nu)}{D_q(\nu_M)} \sim 1 - B\left(\frac{\nu}{\nu_M} - 1\right)^2 \tag{20}$$

where

$$B \sim b_1 - b_2(q-1) \tag{21}$$

with $b_1 \simeq 1.232159$ and $b_2 \simeq 2.690571$, and

$$D_q(\nu_M)\frac{c^3h^2}{8\pi(k_BT)^3} \sim d_1 + d_2(q-1) \tag{22}$$

with $d_1 \simeq 1.421435$ and $d_2 \simeq 2.933276$.

If we have (in arbitrary units) the experimental photon energy density $D^{\exp}(\nu)$ of a black-body radiation, we can, in the neighborhood of its maximum, fit it with

$$D^{\exp}(\nu) \sim D_M^{\exp}\left[1 - B^{\exp}\left(\frac{\nu}{\nu_M^{\exp}} - 1\right)^2\right] \tag{23}$$

thus obtaining $D_M^{\exp}$, $B^{\exp}$ and $\nu_M^{\exp}$. From Eq. (21) we have

$$q - 1 \simeq \frac{b_1 - B^{\exp}}{b_2} \tag{24}$$

If, within the experimental error, we can guarantee that this quantity is different from zero (i.e., if $B^{exp} \neq 1.232159$), we can say that the observation is consistent with a non-conventional (discontinuous, or continuous but not differentiable) structure for the space-time where the black-body is located! Its temperature can be obtained by replacing Eq. (24) into Eq. (19), which yields

$$T \simeq \frac{h\nu_M^{\exp}b_2}{k_B(b_2a_1 + a_2b_1 - a_2B^{\exp})} \tag{25}$$

Summarizing, we have extended to $q \simeq 1$, i.e., to slightly nonextensive thermodynamics (and, consistently, to non-conventional space-time) Planck law (hence Rayleigh-Jeans, Stefan-Boltzmann and Wien shift laws) for the radiation of a black-body. The main results are: (i) In the low frequency region ($h\nu << k_BT$), the photon energy density D_q is proportional to ν^{1+q}, hence, the slope in a $\log D_q$ *vs.* $\log \nu$ plot provides q; (ii) The location (ν_M) of the maximum of $D_q(\nu)$ and the associated curvature provide (through Eqs. (24) and (25)) q and T; (iii) The behavior of $D_q(\nu)$ for fixed $h\nu/k_BT$ is very different from that obtained within Quantum Groups [20]: for increasing q, D_q *monotonically increases* (except for $h\nu/k_BT$ below roughly unity, where it very slightly decreases) and crosses Planck law for $q = 1$, whereas, for increasing q_G, Dq_G becomes *maximal* (Planck law) at $q_G = 1$.

As candidates for the present test one naturally thinks of the Universe background radiation, light coming from very dense massive stars (with a relatively well defined temperature), black holes. A *positive* result for the present test would possibly guide our understanding of the intimate structure of space-time. If so, remains great open the question on why the black-body emits, for $q > 1$ and $q < 1$, respectively more and less light than the Planck one, as if something was respectively enhancing and trapping the exit of the photons.

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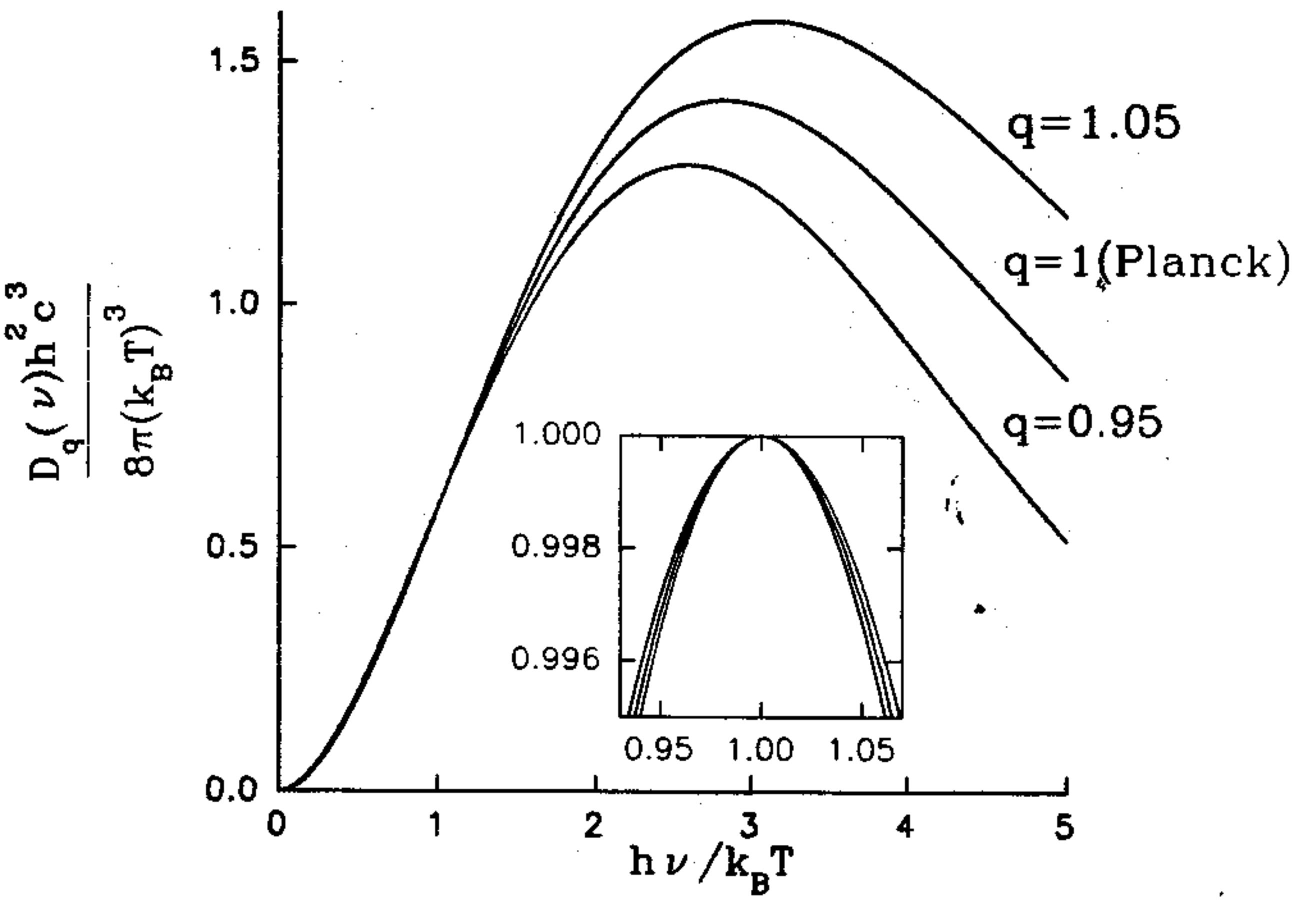
Caption for Figure

Black-body photon energy density per unit volume (D_q) within extensive ($q = 1$; Planck law) and slightly nonextensive ($q = 0.95$ and $q = 1.05$) statistical mechanics. The inset presents $Dq(\nu)/Dq(\nu_M)$ *vs.* ν/ν_M (the top, middle and bottom curves respectively correspond to $q = 1.05$, $q = 1$ and $q = 0.95$) where the curvature effect is exhibited.


q=1.05
q=1(Planck)
q=0.95
1.5
1.0
0.5
0.0
0
1
2
3
4
5
$\frac{D_q(\nu)h^2c^3}{8\pi(k_BT)^3}$
$h\nu/k_BT$
1.000
0.998
0.996
0.95
1.00
1.05

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