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ANTIPODAL UNIVERSES IN THE TOPOLOGY S^3xR AND H^3xR

by

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ABSTRACT

On the Lie groups $S^3 xR$ and $H^3 xR$ a two-parameter family of left-invariant metrics g_L and right-invariant metrics g_R is introduced. The resulting space-times admit a five-parameter group of motions; in the family with topology $H^3 xR$ the case occurs in which $g_L = g_R$, and this particular space-time admits a seven-parameter group of motions, with the 3-dimensional section H^3 maximally symmetric. Global causality problems are easily characterized and it is shown how they can be avoided by the introduction of a line of singularities (strings) in the space-times. A coordinate system is defined where the left-invariant metrics g_L and right-invariant metrics g_R differ by a coordinate inversion - equivalently g_L and g_R correspond to rotating universes with opposite matter vorticity - and it is discussed how weak interaction processes could allow distinguishing the universes physically (this is the subject of a companion paper, where we introduce neutrinos as test particles in these universes). Coordinate transformations are also presented where both metrics g_L and g_R can be recast in the form of a Gödel-type metric.

Key-words: Antipodal universes; Left and right-invariant space-times; Rotating universes; Causality and strings.

1. INTRODUCTION

In the geometrical study of cosmological models some powerful methods were developed and extensively used, to construct invariant Lorentzian metrics over space-time manifolds which are Lie groups [1-5]. As well known the action of the Lie group on itself can be divided into two independent subgroups, namely the left (L) and right (R) action of the group on itself [1,6,7]. In this context two sets of invariant Lorentzian metrics can be introduced, denoted L-invariant and R-invariant geometries in which the bilinear metric form is constructed with vector fields/forms which are invariant under the left, and right action of the Lie group, respectively [6]. Our purpose in the present paper is two-fold. First we examine global properties of a two-parameter family of solutions of Einstein equations which are L- and R-invariant Lorentzian metrics over the Lie groups S^3xR and H^3xR. The metrics are constructed according to the above prescription, after properly deforming the associated Lie algebra of the L- and R-invariant 1-forms and vector fields of the group manifolds (the deformation depending on two parameters). These 1-forms and fields are obtained by standard procedures using the algebra of quaternions to characterize the semi-simple Lie groups S^3 and H^3 [6,8]. By coordinate transformations the resulting metrics can assume the form of a Gödel-type geometry which has been examined in the literature mainly from the point of view of local properties (see Ref. [9] for a review; see also [10,11]). The methods used here also clarify from the global point of view some of the results of the existing literature. Second we examine the geometrical and physical relation between the L-invariant and R-invariant geometries. We exhibit a system of coordinates in which the L-and R-invariant vector fields, forms and the bilinear metric forms are related by an improper transformation of coordinates. A possible *physical* distinction between L- and R-invariant universes is discussed. This in fact is the subject of the next following paper where we introduce neutrinos as test particles in these universes.

We organize the paper as follows. In Section II we characterize globally the space-time manifolds S^3xR and H^3xR, and introduce on these manifolds the two parameter family of invariant geometries. This construction gives us directly the isometries of the space-times as shown in Section III. Also we discuss the causality problems from a global aspect and some possible modifications in the topology to circumvent these pathologies. In Section IV cylindrical coordinate transformations are exhibited which recast the metrics in the form of a Gödel-type geometry. Possible sources of curvature for the space-times are also discussed.

2. THE SPACE-TIME MANIFOLDS S^3xR and H^3xR, AND THE CONSTRUCTION OF INVARIANT GEOMETRIES

The methods used in this Section are adapted from Oszváth and Schüking [1] and are presented here concisely. Calculations are not given in detail, but they can be checked without difficulty.

Let E_4 be the four dimensional Euclidean space with Cartesian coordinates $a = (a^0, a^1, a^2, a^3)$. We define the surface M as the set of points of E_4 which satisfy

$$(a^2)^2 + (a^1)^2 - (\varepsilon a^2)^2 - (\varepsilon a^3)^2 = 1\,, \tag{2.1}$$

where $\varepsilon = i, 1$ whether M is S^3 or H^3, respectively. For every $a = (a^0, a^1, a^2, a^3)$, $b = (b^0, b^1, b^2, b^3) \in M$ we define the multiplication law [12]

$$\begin{aligned} ab = (&a^0b^0 - a^1b^1 + (\varepsilon)^2a^2b^2 + (\varepsilon)^2a^3b^3, \\ &a^0b^1 + a^1b^0 - (\varepsilon)^2a^2b^3 + (\varepsilon)^2a^3b^2, \\ &a^0b^2 + a^2b^0 + a^3b^1 - a^1b^3, \\ &a^0b^3 + a^3b^0 + a^1b^2 - a^2b^1) \end{aligned} \tag{2.2}$$

Under (2.2) M becomes a group, acting on itself by left multiplication; namely, for a given $v \in M$, a left motion of M into itself is expressed as

$$a' = v\,a \tag{2.3}$$

and we have $a' \in M$ for all $a \in M$. M is simply transitive since for each $a \neq 0$ there exists only one left motion v from a to a given $a' \in M$.

M acting on itself by left multiplication (2.3) is a Lie group with the three independent left-invariant vector fields

[13] on M

$$e^{\mu}_{(1)} = (-a^1, a^0, a^3, -a^2) ,$$

$$e^{\mu}_{(2)} = ((\varepsilon)^2 a^2, (\varepsilon)^2 a^3, a^0, a^1) , \qquad (2.4)$$

$$e^{\mu}_{(3)} = ((\varepsilon)^2 a^3, -(\varepsilon)^2 a^2, -a^1, a^0) .$$

They are obtained by an arbitrary left motion a of the three independent unit vectors (0,1,0,0), (0,0,1,0), (0,0,0,1) which define the infinitesimal tangent space of M at the identity (1,0,0,0).

We have the analogous picture for right motions of the Lie group M into itself, namely

$$a' = a\,v , \qquad (2.5)$$

(cf. eq. (2.3)) with the corresponding independent right - invariant vector fields on M,

$$d^{\mu}_{(1)} = (-a^1, a^0, -a^3, a^2) ,$$

$$d^{\mu}_{(2)} = ((\varepsilon)^2 a^2, -(\varepsilon)^2 a^3, a^0, -a^1) , \qquad (2.6)$$

$$d^{\mu}_{(3)} = ((\varepsilon)^2 a^3, (\varepsilon)^2 a^2, a^1, a^0) .$$

We obviously have [14]

$$[e_{(i)}, d_{(j)}] = 0 , \quad i,j = 1,2,3 . \qquad (2.7)$$

We introduce on M the coordinate system (χ,η,r) by the transformations

$$a^0 = \cosh(\tfrac{\varepsilon r}{2})\cos\chi ,$$

$$a^1 = \cosh(\tfrac{\varepsilon r}{2})\sin\chi , \qquad (2.8)$$

$$a^2 = \frac{1}{\varepsilon}\sinh(\tfrac{\varepsilon r}{2})\cos\eta ,$$

$$a^3 = \frac{1}{\varepsilon}\sinh(\tfrac{\varepsilon r}{2})\sin\eta .$$

For $\varepsilon = i$ we obtain a chart on S^3, the coordinates being Euler angles with range $0 \le \chi,\eta < 2\pi$, $0 \le r/2 \le \pi$. For $\varepsilon = 1$ a chart is defined on H^3 with $0 \le \chi,\eta < 2\pi$, $-\infty < r < \infty$. The invariant fields (2.4) and (2.6) expressed as $E_i = e^{\mu}_{(i)} \frac{\partial}{\partial a^\mu}$ and $D_i = d^{(\mu)}_{i} \frac{\partial}{\partial a^\mu}$ have the form

$$E_1 = \frac{\partial}{\partial\chi} - \frac{\partial}{\partial\eta} ,$$

$$E_2 = 2\cos(\chi-\eta)\frac{\partial}{\partial r} + \frac{2\varepsilon\sin(\chi-\eta)}{\sin(\varepsilon r)}\left[-\sin(\tfrac{\varepsilon r}{2})\frac{\partial}{\partial\chi} + \cosh^2(\tfrac{\varepsilon r}{2})\frac{\partial}{\partial\eta}\right], \qquad (2.9)$$

$$E_3 = -2\sin(\chi-\eta)\frac{\partial}{\partial r} + \frac{2\varepsilon\cos(\chi-\eta)}{\sinh(\varepsilon r)}\left[-\sinh^2(\tfrac{\varepsilon r}{2})\frac{\partial}{\partial\chi} + \cosh^2(\tfrac{\varepsilon r}{2})\frac{\partial}{\partial\eta}\right].$$

and

$$D_1 = \frac{\partial}{\partial\chi} + \frac{\partial}{\partial\eta} ,$$

$$D_2 = -2\cos(\chi+\eta)\frac{\partial}{\partial r} + \frac{2\varepsilon\sin(\chi+\eta)}{\sinh\varepsilon r}\left[\sinh^2(\frac{\varepsilon r}{2})\frac{\partial}{\partial\chi} + \cosh^2(\frac{\varepsilon r}{2})\frac{\partial}{\partial\eta}\right] ,$$

$$D_3 = 2\sin(\chi+\eta)\frac{\partial}{\partial r} + \frac{2\varepsilon\cos(\chi+\eta)}{\sinh\varepsilon r}\left[\sinh^2(\frac{\varepsilon r}{2})\frac{\partial}{\partial\chi} + \cosh^2(\frac{\varepsilon r}{r})\frac{\partial}{\partial\eta}\right] , \tag{2.10}$$

in the coordinate system defined by (2.8). They satisfy the algebra

$$[E_1 , E_2] = 2E_3, [E_3 , E_1] = 2E_2, [E_2 , E_3] = -2\varepsilon^2 E_1 ,$$

$$[D_1 , D_2] = -2D_3, [D_3 , D_1] = -2D_2, [D_2 , D_3] = 2\varepsilon^2 D_1 . \tag{2.11}$$

Taking on the 1-dimensional manifold R the coordinate z $(-\infty < z < \infty)$ with vector field $\partial/\partial z$, the Lie group $M \times R$ can be characterized by the invariant vector fields $\{E_1, E_2, E_3, \partial/\partial z\}$ and $\{D_1, D_2, D_3, \partial/\partial z\}$, which satisfy the algebra (2.11) and

$$[E_i, \partial/\partial z] = 0 \quad , \quad [D_i, \partial/\partial z] = 0 \ , \ i = 1,2,3 \tag{2.12}$$

and which constitute bases for the vector fields on $M \times R$. Next we deform the algebra (2.11) by rescaling the vector fields as

$$X_0 = \frac{1}{\alpha} E_1 \ , \ X_1 = \frac{1}{\beta} E_2 \ , \ X_2 = \frac{1}{\beta} E_3 \ , \ X_3 = \frac{\partial}{\partial z}$$

$$Y_0 = \frac{1}{\alpha} D_1 \ , \ Y_1 = \frac{1}{\beta} D_2 \ , \ Y_2 = \frac{1}{\beta} D_3 \ , \ Y_3 = \frac{\partial}{\partial z} \ , \tag{2.13}$$

where α,β are constant parameters. From (2.7) and (2.12) we have

$$[X_A, y_B] = 0 \quad , \quad A,B = 0,1,2,3 . \tag{2.14}$$

The deformation of the algebra defined in (2.13) corresponds geometrically to a deformation of the 3-sphere ($\epsilon=i$) or the 3-hyperboloid ($\epsilon=1$) into an ellipsoidal or a deformed hyperboloidal surface immersed in E_4. Since the deformation does not alter the topology of the groups we still refer to them as S^3xR and H^3xR.

We now define invariant Lorentzian metrics on MxR. We make a particular choice, by prescribing the following scalar product rules for the invariant vector fields X_A and y_A,

$$g_L(X_A,X_B) = \eta_{AB} \quad , \quad g_R(y_A, y_B) = \eta_{AB} , \tag{2.15}$$

where $\eta_{AB} = \mathrm{diag}(+1,-1,-1,-1)$. We shall refer to g_L and g_R as left- and right-invariant metrics, respectively. From (2.9), (2.10) and (2.13), equations (2.15) can be equivalently expressed as

$$g(e) = A(r)d\chi^2 + B(r)d\eta^2 + \frac{1}{2}\, e(\alpha^2 - \frac{\beta^2}{\epsilon^2})\sinh^2(\epsilon r)d\chi d\eta + \\ - \frac{\beta^2}{4}\, dr^2 - dz^2 , \tag{2.16}$$

where

$$A(r) = \frac{\alpha^2}{4}\sinh^2 2r(\coth^2\frac{\epsilon r}{2} - \frac{\beta^2}{\epsilon^2\alpha^2}) , \tag{2.17}$$

$$B(r) = \frac{\alpha^2}{4} \sinh^2 \varepsilon r (\mathrm{th}^2 \frac{\varepsilon r}{2} - \frac{\beta^2}{\varepsilon^2 \alpha^2})$$

and $e = \pm 1$ for g_L or g_R, respectively. It is to be noted that the left- and right-invariant metrics expressed in the coordinate system introduced in (2.8), differ by the sign of the cross term $d\chi d\eta$ only; in other words, they are connected by a coordinate inversion $\chi \to -\chi$ or $\eta \to -\eta$. We have denoted these space-times antipodal. In the realm of pure gravitational interaction the geometries are indistinguishable if the covariance group of the theory includes improper transformations. The possibility of the physical distinction between the two universes is discussed in a companion paper where the nature of these transformations is analysed when we include neutrinos as test particles. We show also that improper transformations are no longer symmetries of the system universe-plus-neutrinos of a given helicity. An intuitive argument can be given in this direction. To the *passive* transformation $\eta \to -\eta$ there corresponds the *active* transformation $g(e) \to g(-e)$, or equivalently, the change of the sign of the vorticity associated to the velocity field of the matter content of the universe (cf. Appendix). On the other hand neutrinos can be used as an absolute standard for the sign of the rotation of the universe because, as prescribed by weak interactions processes, a massless neutrino is an absolute left-handed screw. Therefore the *active* transformation $g(e) \to g(-e)$ or, equivalently, the change of the sign of the universe rotation, is no longer a symmetry of the system universe-plus-neutrino because - to preserve the symmetry - left-handed neutrinos should then be transformed into right-handed neutrinos, which is a forbidden configuration.

3. ISOMETRIES AND CAUSALITY IN SECTIONS

From (2.14) and (2.15) it follows

$$\mathcal{L}_{y_A} g_L = 0 \quad , \quad \mathcal{L}_{x_A} g_R = 0 \quad , \quad A = 0,1,2,3 \tag{3.1}$$

This means that by construction, the left (right) invariant geometry $g_L(g_R)$ has the four right (left)-invariant vector fields $\{y_A\}(\{x_A\})$ as Killing vectors. A direct inspection of the geometries (2.16) shows that $\partial/\partial\eta$ is an additional independent Killing vector. Summing up, we have in general

(a) Five Independent Killing vectors associated to g_L

$$\{y_A, \partial/\partial\eta\} \quad , \tag{3.2}$$

(b) Five Independent Killing vectors associated to g_R

$$\{x_A, \partial/\partial\eta\} \ . \tag{3.3}$$

In another terminology the space-times with metrics (2.16) are endowed with a G_5 group of isometries acting transitively [15] on the space-time manilfolds. These space-times are said to be *homogeneous* in the sense that G_5 contains a subgroup of isometric transformations which acts *simply transitively* on MxR, by construction. It is however important to remark that the space-times are not *spatially homogeneous* since there is no subgroup G_3 of G_5 whose orbit is a 3-dimensional space-like surface.

For the hyperbolic family ($\varepsilon = 1$) an exceptional case occurs when $\alpha^2 = \beta^2$ - inspection of formulae (2.15) and (2.17) yields immediately $g_L = g_R$. It then follows from a trivial counting in (3.2)-(3.3) that this particular geometry has seven independent Killing vectors, for instance $(X_o, X_1, X_2, Y_o, Y_1, Y_2, Y_3)$. In other words the geometry has a seven-parameter group of motions, and the sections z = const. are maximally symmetric with a G_6 group of motions generated by $(X_o, X_1, X_2, Y_o, Y_1, Y_2)$. [10].

The class of left- and right-invariant geometries (2.16) for the hyperbolic case has the structure of the reflection group of hyperbolae through their asymptotes [16]. In fact the constant function (cf. (2.16)) $y = e(\alpha^2 - \beta^2)$, defined in the plane of metric parameters (α, β), describes a congruence of hyperbolae which are reflected through the asymptotes y = 0, when we change the sign of e, namely, when we go from g_L to g_R and vice versa. The asymptotes y = 0 correspond to the exceptional geometry discussed above with a G_7 isometry group.

Causality and Topological Defects in Sections

The space-times introduced here - the invariant geometries (2.16) over the Lie groups S^3xR and H^3xR - present some pathological properties like the existence of time-like or null-like closed curves. As we shall see this is connected to the fact that the sections z = const. have, by construction, the structure of S^3 or H^s, and the restriction of the invariant geometry (2.16) to S^3 or H^3 has signature (+--). In some cases, by a legitimate alteration of the topology, we can avoid these a-causal curves, but they are in general inevitably present.

We examine separately the two cases

(i) $H^3 xR$.

The sections z = const. with the topology of H^3 are described, in terms Cartesian coordinates of the embedding Euclidean space E_4, by (2.1) with $\varepsilon = 1$. Let us consider the 2-dimensional sections of H^3, which we shall describe in the coordinate system (r,χ,η) defined in (2.8), with $-\infty < r < \infty$, $0 \leq \chi,\eta < 2\pi$. Taking firstly the sections η = const. we choose for convenience $\eta = 0$ and obtain from (2.8)

$$
\begin{aligned}
a^0 &= \cosh\left(\frac{r}{2}\right)\cos\chi \quad , \\
a^1 &= \cosh\left(\frac{r}{2}\right)\sin\chi \quad , \\
a^2 &= \sinh\left(\frac{r}{2}\right) \quad , \\
a^3 &= 0 \quad ,
\end{aligned}
\tag{3.4}
$$

which represent the points of the one-leaf hyperboloid of Fig. 1.

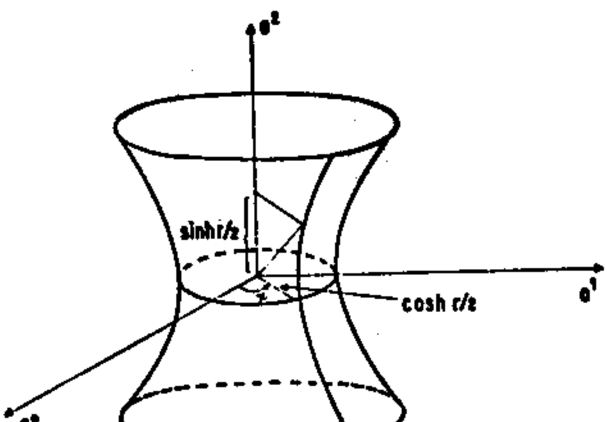

Fig. 1 - One-leaf hyperpoloid embedded in E_4 corresponding to the section η,z = const. of the manifoid $H^3 xR$.

For the sections χ = const, choosing for simplicity $\chi = 0$ we have

$$a^0 = \cosh(\tfrac{r}{2}) ,$$

$$a^1 = 0 ,$$

$$a^2 = \sinh(\tfrac{r}{2})\cos\eta , \qquad (3.5)$$

$$a^3 = \sinh(\tfrac{r}{2})\sin\eta .$$

Equations (3.5) describes in E_4 the points of the two-leaf hyperboloid of Fig. 2.

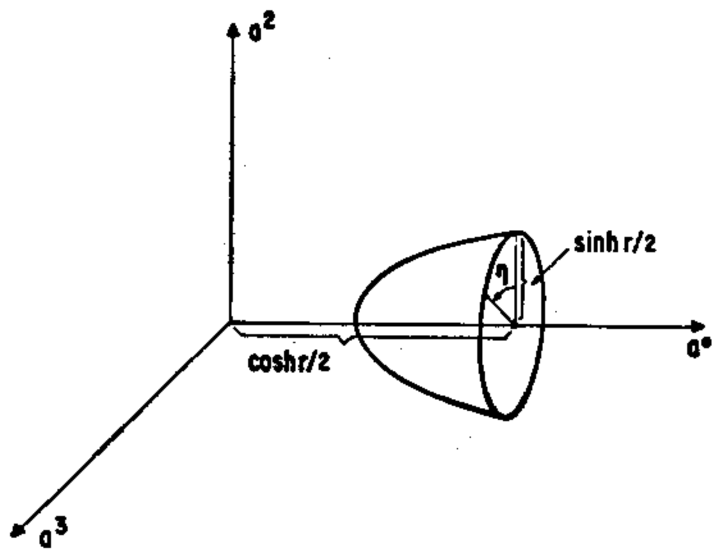

Fig. 2 - Two-leaf hyperboloid corresponding to the sections χ, z = const. of the manifold H^3xR.

We are now ready to examine how the topology of the sections is related to the causality problem in these space-times. To this end let us consider the invariant Lorentzian geometry (2.16) with $\varepsilon = 1$ defined on H^3xR. We distinguish

(a) Case $\alpha^2 > \beta^2$: in the sections η, z = const. we have $A(r) > 0$ for all r. Therefore the integral curves of the vector field $\partial/\partial\chi$ are time-like closed curves (cf. (2.16)) on the hyperboloid of Fig. 1. This violation of causality can however be circumvented - in fact the hyperboloid of Fig. 1 is homeomorphic to the cylinder and can be continuously developed onto the plane. The curves defined by $\partial/\partial\chi$ can thus be open into infinite lines and the causality problem avoided. This procedure corresponds actually to modifying the connectivity-in-the-large properties [17, 18] of the manifold - the connectivity in the large is changed by identification of certain point sets, namely the manifold H^3 differs from the present one by identification of the points $(\chi + 2n\pi, \eta, r, z)$, n = integer [19]

In the sections χ, z = const. the space-like, time-like or null-like character of the vector field $\partial/\partial\eta$ depends on the sign of the function B(r) (cf. (2.17) for $\epsilon = 1$). The *closed* curves defined by the field $\partial/\partial\eta$ on the two-leaf hyperboloid of Fig. 2 are time-like or null-like for values of r such that $th^2 \frac{r}{2} \gtrsim \frac{\beta^2}{\alpha^2}$, respectively. Contrary to the case of the sections η, z = const. however, the presence of closed time-like lines cannot be circumvented by modifying the topology *without* introducing singularities in the space-time manifold. This is the case since a branch of the two-leaf hyperboloid of Fig. 2 is homeomorphic to the cylinder only if one point of the hyperboloid is extracted. In other words, to eliminate the causality problem in these sections - by developing the closed coordinate lines η into open infinite lines - we must introduce topological defects (a line of singularities) in the space-time manifold. A possible way to implement the extraction of points of the space-time is the introduction of a string in the sense of Refs. [20,21,22]: with the geometry (2.16) expanded about the point $r=0$, for instance, the point $r=0$ is transformed in a conical singularity. The neighborhood about $r=0$ of the hyperboloid of Fig. 2 is approximated by a flat conical space, and the conical singularity at $r=0$ contributes to the curvature tensor of the space-time with a term proportional to a δ-type function which vanishes outside the singular line defined by $r=0$ [20,23,24].

(b) Case $\alpha^2<\beta^2$: in the sections η, z = const. the integral curves of $\partial/\partial\chi$) on the hyperboloid of Fig. 1 are time-like or null like whether $\text{th}^2 \frac{r}{2} \geq \frac{\alpha^2}{\beta^2}$, respectively. As in case (a) here the curves defined by $\partial/\partial\chi$ can be open into infinite lines by an appropriate modification of the topology, and the causality problem avoided, without introducing singularities in the space-time. In the sections χ, z = const. we have $B(r)<0$ for all r so that no acausal curves occur in these sections.

In this sense the space-times of Case (b): $\alpha^2<\beta^2$, are said to be causal.

(c) Case $\alpha^2=\beta^2$: this can be considered a limiting case of (b), and we have here $A(r)>0$ and $B(r)<0$ for all r. The only time-like closed curves present are defined by the vector field $\partial/\partial\chi$ on the one-leaf hyperboloid of Fig. 1, and can be eliminated as in (b).

(ii) $S^3 \times R$

The geometry here is given by (2.6), for $\varepsilon=1$. On the sections z = const. with the topology of S^3 we introduce the coordinate system (r,χ,η) defined through the Cartesian coordinates of the embedding Euclidean space by formulae (2.8) for $\varepsilon = i$, where $0 < r/2 \leq \pi$, $0 \leq \eta, \chi < 2\pi$. The two-dimensional sections of S^3 corresponding to y = const. or χ = const. are spheroidal surfaces.

In the sections η = const [χ = const] of S^3 the character of the vector field $\partial/\partial\chi$ [$\partial/\partial\eta$] depends on the sign of the function A(r) [B(r)]. For values of r such that $\frac{1+\cos}{1-\cos r}\left[\frac{1-\cos r}{1+\cos r}\right]$ is greater than, smaller than on equal to β^2/α^2, the coordinate lines defined by $\partial/\partial\chi$ [$\partial/\partial\eta$] are closed curves of time-like, space-time or null-like character respectively. The causality problem in these sections cannot be circumvented *unless* we extract two points from each two-dimensional spheroidal surface z,η = const. [z,χ = const.] to produce a topological cylinder. This can be done by a procedure analogous to the one discussed in case (i.a), and, in this sense, the space-times discussed here with topology S^3xR can be made free from the presence of the above mentioned closed time-like curves only by the introduction of at least two strings in the manifold.

4. THE CYLINDRICAL COORDINATE SYSTEM AND SOURCES OF CURVATURE

The class of invariant geometries (2.16)-(2.17) introduced on S^3xR and H^3xR can be cast in a simpler form by the use of cylindrical coordinates defined below. Let us consider the right invariant geometries g_R and a new coordinate system $(t,\phi,\bar{r},\bar{z})$ given by the transformation equations

$$r = \frac{m}{\varepsilon}\,\bar{r} \quad , \qquad \eta = \frac{m^2}{4\Omega}\,t - \phi \ , \tag{4.1}$$

$$\chi = \frac{m^2}{4\Omega}\,t \quad , \qquad z = \bar{z}$$

where

$$\Omega = \varepsilon^2\alpha/\beta^2 \quad , \quad m = 2\varepsilon/\beta \ . \tag{4.2}$$

In this coordinate system g_R is written

$$g_R = (dt + H(\bar{r})d\phi)^2 - D^2(\bar{r})d\phi^2 - d\bar{r}^2 - d\bar{z}^2 \ , \tag{4.3}$$

where

$$H(\bar{r}) = \frac{4\Omega}{m^2}\sinh^2(\frac{m\bar{r}}{2}) \quad , \quad D(\bar{r}) = \frac{1}{m}\sinh m\bar{r} \ . \tag{4.4}$$

In terms of the new parameters (m,Ω) the family of geometries g_R over S^3xR or H^3xR are obtained by taking $m^2 < 0$ or $m^2 > 0$ respectively. Although the geometry g_L does not assume a simple form in the coordinate system defined by (4.1), an analogous cylindrical coordinate system can be introduced where g_L takes the form (4.3)-(4.4). The transformation equations are obtained from (4.1) by the substitution $\eta \to -\eta$ or $\chi \to -\chi$, for instance

$$r = \frac{m}{\varepsilon}\bar{r} \ , \qquad \eta = \frac{m^2}{4\Omega}t - \phi \ ,$$

$$\chi = -\frac{m^2}{4\Omega}t \quad , \quad z = \bar{z} \ , \tag{4.5}$$

where Ω and m are defined through (4.2). The difference here lies in that (4.5) is a discontinuous transformation involving an inversion operation.

The space-times characterized by a line element of the form (4.3) without specification of the functions $H(\bar{r})$ and $D(\bar{r})$ are denoted in the literature as Gödel-type space-times. When $H(\bar{r})$ and $D(\bar{r})$ are given by (4.4) the space-times are denoted Gödel-type homogeneous. We have obtained the functions (4.4) by construction, starting from the global Lie group structure of the space-times plus the choice (2.15). The functions could be equivalently derived if space-time homogeneity is assumed, that is, if the geometry (4.3) is restricted to admit a simply transitive isometry group. This was the point of view taken in Ref. [9]. The Gödel solution [8] can be recognized in the cylindrical coordinate system to correspond to the particular case $m^2 = 2\Omega^2$.

We must finally comment on the possible physical sources of curvature compatible with geometries (2.16) via the field equations. We start with the model proposed by Gödel in 1949, the geometry of which is obtained taking $\alpha^2 = 2\beta^2$, $\varepsilon = 1$ in (2.16) and (2.17) (or $m^2 = 2\Omega^2$, as mentioned above). The metric is a solution of Einstein field equations with the cosmological constant term, and incoherent matter. In this respect Oszváth [25] proved a theorem stated without proof by Gödel [8], saying that the metric of the Einstein static universe and the Gödel metric are the only space-time homogeneous solutions of Einstein equations with incoherent matter and rigid rotation. Later Bampi and Zordan [26] showed that - for Gödel-type metrics (4.3), without specification of the functions $H(\bar{r})$ and $D(\bar{r})$, and for the energy-momentum tensor of a perfect fluid - *all* resulting solutions of Einstein equations are isometric to the

Gödel solution.

Let us consider again the homogeneous space-time characterized by the metric (2.16) (or (4.3)). For $\alpha^2 \geq 2\beta^2$, $\epsilon = 1$ (or $0 \leq m^2 \leq 2\Omega^2$ in cylindrical coordinates), and for $\epsilon = i$ and arbitrary α, β (or $-\infty < m^2 < 0$), the corresponding classes of metrics are solutions of Einstein and Einstein-Maxwell equations with charged dust [27,3], or neutral dust plus a free electromagnetic field [28,3]. The admissible range of parameters can be extended by adding to the energy-momentum tensor of dust and free electromagnetic fields, the energy momentum tensor of a free scalar field [9]. For $\alpha^2 = \beta^2$ (or $m^2 = 4\Omega^2$) only a pure massless scalar field is allowable. The spectrum of Gödel type homogeneous solutions was further extended [11] to $\beta^2 > \alpha^2, \epsilon = 1$ (or $m^2 > 4\Omega^2$), in the context of Einstein-Cartan theory [28]. These models have as source a perfect fluid with spin in rigid rotation, the spin distribution being uniform and parallel to the rotation axis.

In the above examples where a perfect fluid distribution is present, the kinematical parameters [5,30] characterizing the models can be unambiguously defined [31], associated to the four-velocity field of the fluid. The velocity field of the fluid has zero acceleration, expansion and shear, but has a non-null vorticity

$$\omega = (e\ \epsilon^2\ \frac{\alpha}{\beta^2})\ \frac{\partial}{\partial z} \qquad , \qquad (4.6)$$

relative to the local compass of inertia (cf. Appendix).
The infinitesimal elements of the perfect fluid in these models are therefore in geodesic motion with constant rigid rotation.

5. FINAL CONCLUSIONS

In this paper we have constructed space-times with the topology of the Lie groups S^3xR and H^3xR, and studied their global properties. Left-invariant metrics g_L and right-invariant metrics g_R depending on two parameters are introduced on the group manifolds of the space-times. This global construction allows us to conclude directly that the space-times admit a G_5 group of motions. In the family of space-times with structure H^3xR there occurs a particular case when $g_L = g_R$ which admits a G_7 group of motions, with the 3-dimensional sections corresponding to H^3 maximally symmetric.

Also the knowledge of the topology of the space-times allows us to identify immediately the nature of the coordinates used, and global causality problems - associated to the existence of closed time-like lines of coordinates - are easily characterized. The basic result is that in the family with topology H^3xR some of the acausal curves can be avoided simply by the operation of developing one-leaf hyperboloids onto the plane while other acausal curves can only be avoided by extracting points of the space-time. The latter procedure corresponds to introducing a topological defect or a string (a line of delta-type singularities in the curvature) in the space-times. In the family with the topology S^3xR the modification of the topology involves the introduction of at least two strings (lines of singularities) to avoid the acausal curves.

A coordinate system is introduced where the left-invariant metrics g_L and right-invariant metrics g_R defined on the group manifolds differ by a coordinate inversion only, and we discuss how weak interaction processes could allow distinguishing these universes physically. This is the subject of the following paper where the symmetries of the system *universe plus neutrinos* (neutrinos considered here as test particles) are examined; there it is shown that the physical transformations on the system universe plus neutrinos, corresponding to the passive operation of coordinate inversion, are not a symmetry of the system if neutrinos have one type of helicity only.

APPENDIX

We assume now that the metrics g_L and g_R are associated to distinct space-times, both defined on the same manifold. We shall prove in this Appendix that the material content (for the cases it can be present as source of curvature) of these space-times have opposite rotation, relative to the compass of inertia [1,32]. In the coordinate system defined by (2.8), let us consider for our calculations the vector field basis (cf. (2.13) and (2.15))

$$X_o = \frac{1}{\alpha}\left(\frac{\partial}{\partial\chi} - e\,\frac{\partial}{\partial\eta}\right) ,$$

$$X_1 = \frac{1}{\beta}\left[2\cos(\chi-e\eta)\frac{\partial}{\partial r} + \frac{2\varepsilon\sin(\chi-e\eta)}{\sinh\varepsilon r}\left(-\sinh^2\frac{\varepsilon r}{2}\frac{\partial}{\partial\chi} + e\cosh^2\frac{\varepsilon r}{2}\frac{\partial}{\partial\eta}\right)\right] ,$$

$$X_2 = \frac{1}{\beta}\left[-2\sin(\chi-e\eta)\frac{\partial}{\partial r} + \frac{2\varepsilon\cos(\chi-e\eta)}{\sinh\varepsilon r}\left(-\sinh^2\frac{\varepsilon r}{2}\frac{\partial}{\partial\chi} + e\cosh^2\frac{\varepsilon r}{2}\frac{\partial}{\partial\eta}\right)\right] , \qquad (A.1)$$

$$X_3 = \frac{\partial}{\partial z} ,$$

which define local Lorentz frames in the space-times (2.16).

The four-velocity field of matter [33] relative to this basis is given by

$$V = V^A X_A = X_o . \qquad (A.2)$$

We can then describe the motion of the matter with respect to the frame $\{X_A\}$ by means of the equation

$$\mathcal{L}_{X_o}\eta = 0 \quad , \tag{A.3}$$

where $\eta=\sum_{a=1}^{3}\eta^a X_a$ is a vector orthogonal to X_o (cf. (2.15) and connecting two neighboring fluid particles, one of them located at the origin of the Lorentz frame determined by (A.1). Denotating $\dot{\eta}^a = X_o\eta^a$ we obtain from (A.3), (2.11) and (2.13)

$$\dot{\eta}^1 = \frac{2e}{\alpha}\eta^2 \quad , \quad \dot{\eta}^2 = -\frac{2e}{\alpha}\eta^1 \quad , \quad \dot{\eta}^3 = 0 \; . \tag{A.4}$$

The motion of the frame (A.1) along a material world- line determined by X_o can be calculated by

$$\dot{X}_A = \nabla_{X_o} X_A$$

and we obtain

$$\dot{X}_1 = e(\varepsilon^2\frac{\alpha}{\beta^2} - \frac{2}{\alpha})X_2 \quad , \quad \dot{X}_2 = -e(\varepsilon^2\frac{\alpha}{\beta^2} - \frac{2}{\alpha})X_1, \tag{A.5}$$

$$\dot{X}_3 = 0 = X_o \; ,$$

that is, the plane 1-2 of the frame $\{X_A\}$ rotates with respect to the local compass of inertia with a circular frequency $e(\frac{\varepsilon^2\alpha}{\beta^2} - \frac{2}{\alpha})$ (the axes of the local compass of inertia being determined, for instance, by gyroscopes). Since the shear of the matter velocity field (A.2) is zero, the rotation of matter relative to the compass of inertia is given by the angular velocity

$$\omega^A = (0,0,0,e\,\varepsilon^2\frac{\alpha}{\beta^2}) \qquad (A.6)$$

which changes sign as $e \to -e$, that is, as we change g_L into g_R and vice versa.

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