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# Construction of a non-standard quantum field theory through a generalized Heisenberg algebra

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**Abstract**

We construct a Heisenberg-like algebra for the one dimensional quantum free Klein-Gordon equation defined on the interval of the real line of length $L$. Using the realization of the ladder operators of this type Heisenberg algebra in terms of physical operators we build a $3+1$ dimensional free quantum field theory based on this algebra. We introduce fields written in terms of the ladder operators of this type Heisenberg algebra and a free quantum Hamiltonian in terms of these fields. The mass spectrum of the physical excitations of this quantum field theory are given by $\sqrt{n^2\pi^2/L^2+m_q^2}$, where $n=1,2,\cdots$ denotes the level of the particle with mass $m_q$ in an infinite square-well potential of width $L$.

**Keywords:** quantum field theory; Heisenberg algebra; hadrons.

# 1 Introduction

There is a vast range of energy from the present accelerators energies ($\approx 10^3$Gev) to the Planck energy ($10^{19}$ Gev) where it is believed that there is room for surprises [1]. It is also believed that field theories based on deformed algebras could play an important role to describe physics in this vast energy range [2]. These algebras have parameters, known as deformation parameters, that it is expected to regularize the ultraviolet divergences in deformed field theories [3].

This paper is the first step towards an analysis of possible consequences in quantum field theories (QFTs) of a class of generalized Heisenberg algebras we have recently constructed [4]. The generators of each algebra in this class are the Hamiltonian of the one-dimensional quantum system in consideration and the ladder operators. The physical systems that are described by these type Heisenberg algebras are characterized by those one-dimensional quantum systems having an spectrum where the successive energy levels are related by $\epsilon_{n+1} = f(\epsilon_n)$ [5].

Within this class of algebras we find deformed and also non-deformed type Heisenberg algebras [4] and as a first step of this program we explore the consequences in QFT of a non-deformed type Heisenberg algebra belonging to the large above mentioned class. We hope that the QFT we obtain in this paper can have an intrinsic interest as an alternative phenomenological QFT for hadronic interactions. Moreover, the whole procedure could be seen as a prototype since it seems possible to implement the approach developed here to a deformed Heisenberg algebra belonging to the above mentioned class of algebras in order to construct a deformed QFT that could be appropriate for very high energy ($10^3$Gev$< E < 10^{19}$Gev) physics.

Our approach extends an standard aspect of QFTs. In the very well-known theory of quantum spin-0 particles [6], scalar particle states appear in QFT as vacuum excitations through the application of the Heisenberg algebra creation operator to the vacuum. Roughly speaking, what we do here is to construct a free QFT where the Hamiltonian eigenvectors are obtained by the successive application to the vacuum of the creation operator of another physical system instead the ordinary harmonic oscillator.

We construct here the Heisenberg-like algebra of a relativistic particle in an infinite square-well potential: the quantum one dimensional free Klein-Gordon equation defined on the interval $0 \leq x \leq L$ with special boundary conditions. Within this picture, the creation operator of the algebra when applied to the vacuum of the theory creates particle states with mass spectrum $\sqrt{n^2\pi^2/L^2 + m_q^2}$ where $n = 1, 2, \cdots$ gives the level of the particle with mass $m_q$ in an infinite square-well potential of width $L$. Afterwards, we use the ladder operators of this physical system to construct fields and a free QFT Hamiltonian.

In section 2, we briefly review a class of generalized Heisenberg algebras having $q$-oscillators as a particular example of this class. Within the general class of Heisenberg algebras presented in section 2 there is also the Heisenberg algebra of a relativistic particle in an one-dimensional infinite square-well potential (a non-deformed type Heisenberg algebra). This relativistic square-well algebra is presented in section 3 where it is also given the realization of the ladder operators in terms of the physical operators of the model. In section 4, we construct the first steps towards a quantum field theory based on the square-well Heisenberg algebra. We introduce in this section fields written in terms of the generators of the algebra and a Hamiltonian written in terms of these fields, describing a system with an infinite number of degrees of freedom with mass spectrum excitations given by $\sqrt{n^2\pi^2/L^2 + m_q^2}$, where $n = 1, 2, \cdots$ denotes the level of the particle with mass $m_q$ in an infinite square-well potential of width $L$. In section 5 we present our final comments where we conjecture a possible application of this formalism as an alternative phenomenological quantum field approach to hadronic interactions.

## 2 Generalized Heisenberg algebras

Let us consider an algebra generated by $J_0$, $A$ and $A^\dagger$ described by the relations [4]

$$J_0\, A^\dagger = A^\dagger\, f(J_0), \tag{1}$$

$$A\, J_0 = f(J_0)\, A, \tag{2}$$

$$\left[A^\dagger, A\right] = J_0 - f(J_0), \tag{3}$$

where $^\dagger$ is the Hermitian conjugate and, by hypothesis, $J_0^\dagger = J_0$ and $f(J_0)$ is a general analytic function of $J_0$. It is easy to see that the Jacobi identity is trivially satisfied for general $f$. The case where $f(J_0) = r\, J_0\,(1 - J_0)$ was analyzed in ref. [7].

Using the algebraic relations in eqs. (1-3) we see that the operator

$$C = A^\dagger A - J_0 = A\, A^\dagger - f(J_0) \tag{4}$$

satisfies

$$[C, J_0] = [C, A] = \left[C, A^\dagger\right] = 0, \tag{5}$$

being thus a Casimir operator of the algebra.

We analyze now the representation theory of the algebra when the function $f(J_0)$ is a general analytic function of $J_0$. We assume we have an $n$-dimensional irreducible representation of the algebra given in eqs. (1-3). Consider the state $|0\rangle$ with the lowest eigenvalue of the Hermitian operator $J_0$

$$J_0\,|0\rangle = \alpha_0\,|0\rangle. \tag{6}$$

For each value of $\alpha_0$ and the parameters of the algebra we have a different vacuum that for simplicity will be denoted by $|0\rangle$.

Let $|m\rangle$ be a normalized eigenstate of $J_0$,

$$J_0|m\rangle = \alpha_m|m\rangle\,. \tag{7}$$

Applying eq. (1) to $|m\rangle$ we have

$$J_0(A^\dagger|m\rangle) = A^\dagger f(J_0)|m\rangle = f(\alpha_m)(A^\dagger|m\rangle)\,. \tag{8}$$

Thus, we see that $A^\dagger|m\rangle$ is a $J_0$ eigenvector with eigenvalue $f(\alpha_m)$. Starting from $|0\rangle$ and applying successively $A^\dagger$ to $|0\rangle$ we create different states with $J_0$ eigenvalue given by

$$J_0\left((A^\dagger)^m|0\rangle\right) = f^m(\alpha_0)\left((A^\dagger)^m|0\rangle\right)\,, \tag{9}$$

where $f^m(\alpha_0)$ denotes the $m$-th iterate of $f$. Since the application of $A^\dagger$ creates a new vector, whose respective $J_0$ eigenvalue has iterations of $\alpha_0$ through $f$ augmented by one unit, it is convenient to define the new vectors $(A^\dagger)^m|0\rangle$ as proportional to $|m\rangle$ and we then call $A^\dagger$ a raising operator. Note that

$$\alpha_m = f^m(\alpha_0) = f(\alpha_{m-1})\,, \tag{10}$$

where $m$ denotes the number of iterations of $\alpha_0$ through $f$.

Following the same procedure for $A$, applying eq. (2) to $|m+1\rangle$, we have

$$A\,J_0|m+1\rangle = f(J_0)\,(A|m+1\rangle) = \alpha_{m+1}\,(A|m+1\rangle)\ , \tag{11}$$

showing that $A\,|m+1\rangle$ is also a $J_0$ eigenvector with eigenvalue $\alpha_m$. Then, $A\,|m+1\rangle$ is proportional to $|m\rangle$ being $A$ a lowering operator.

Since we consider $\alpha_0$ the lowest $J_0$ eigenvalue, we require

$$A\,|0\rangle = 0. \tag{12}$$

As was shown in [7], depending on the function $f$ and its initial value $\alpha_0$, it may happen that the $J_0$ eigenvalue of state $|m+1\rangle$ is lower than the one of state $|m\rangle$. Then, as shown in [4], given an arbitrary analytical function $f$ (and its associated algebra in eqs. (1-3)) in order to satisfy eq. (12), the allowed values of $\alpha_0$ are chosen in such a way that the iterations $f^m(\alpha_0)$ $(m \geq 1)$ are always bigger than $\alpha_0$.

As was proven in [4] in general we obtain

$$\begin{aligned}
J_0\,|m\rangle &= f^m(\alpha_0)\,|m\rangle, \quad m = 0, 1, 2, \cdots\ , && (13)\\
A^\dagger\,|m-1\rangle &= N_{m-1}\,|m\rangle, && (14)\\
A\,|m\rangle &= N_{m-1}\,|m-1\rangle, && (15)
\end{aligned}$$

where $N_{m-1}^2 = f^m(\alpha_0) - \alpha_0$.

When the functional $f(J_0)$ is linear in $J_0$, i.e., $f(J_0) = q^2 J_0 + s$, it was shown in [4] that the algebra in eqs. (1-3) is a generalization of $q$-oscillators, reducing to $q$-oscillators for $\alpha_0 = 0$. Moreover, it was shown in [4], where the representation theory was constructed in detail for the linear and quadratic functions $f(x)$, that the essential tool in order to construct representations of the algebra in (1-3) for a general analytic function $f(x)$ is the analysis of the stability of the fixed points of $f(x)$ and their composed functions.

It was shown in [4] and [5] that there is a class of one-dimensional quantum systems that are described by these generalized Heisenberg algebras. This class is characterized by those quantum systems having energy eigenvalues that can be written as $\epsilon_{n+1} = f(\epsilon_n)$, where $\epsilon_{n+1}$ and $\epsilon_n$ are successive energy levels and $f(x)$ a different function for each physical system. This function $f(x)$ is exactly the same function that appears in the construction of the algebra in eqs. (1-3). In this algebraic description of the class of quantum systems, $J_0$ is the Hamiltonian operator of the system, $A^\dagger$ and $A$ are the creation and annihilation operators that are related as in eq. (4) where $C$ is the Casimir operator of the representation associated to the quantum system.

# 3 Relativistic square-well algebra

We are going to construct in this section an algebraic formalism, similar to the harmonic oscillator algebra, for the infinite one-dimensional square-well potential in relativistic quantum mechanics: the quantum one dimensional free Klein-Gordon equation defined on the interval $0 \leq x \leq L$ with special boundary conditions. This Heisenberg-like algebra, that we call relativistic square-well algebra, is an example in the large class of generalized Heisenberg algebras described in the previous section for a specific functional $f(x)$ that we shall determine. In [5] it was constructed the Heisenberg-like algebra of a non-relativistic particle in a square-well potential, here in this section we present the relativistic generalization of this algebra.

We briefly review the formalism of non-commutative differential and integral calculus on a one-dimensional lattice developed in [8] and [9]. Let us consider an one dimensional lattice in a momentum space where the momenta are allowed only to take discrete values, say $p_0$, $p_0 + a$, $p_0 + 2a$, $p_0 + 3a$ etc, with $a > 0$.

The non-commutative differential calculus is based on the expression[8],[9]

$$[p, dp] = dp\, a \,, \tag{16}$$

implying that

$$f(p)\, dg(p) = dg(p)\, f(p+a) \,, \tag{17}$$

for all functions $f$ and $g$. We introduce partial derivatives by

$$d\, f(p) = dp\, (\partial_p\, f)\, (p) = (\bar{\partial}_p\, f)\, (p)\, dp\,, \tag{18}$$

where the left and right discrete derivatives are given by

$$(\partial_p\, f)\, (p) \;=\; \frac{1}{a}\, [f(p+a) - f(p)]\,, \tag{19}$$

$$(\bar{\partial}_p\, f)\, (p) \;=\; \frac{1}{a}\, [f(p) - f(p-a)]\,, \tag{20}$$

that are the two possible definitions of derivatives on a lattice. The Leibniz rule for the left discrete derivative can be written as,

$$(\partial_p\, fg)\, (p) = (\partial_p f)\, (p) g\, (p) + f(p+a)(\partial_p g)\, (p)\,, \tag{21}$$

with a similar formula for the right derivative[8].

Let us now introduce the momentum shift operators

$$T \;=\; 1 + a\, \partial_p \tag{22}$$

$$\bar{T} \;=\; 1 - a\, \bar{\partial}_p\,, \tag{23}$$

which increases (decreases) the value of the momentum by $a$

$$(T f)\, (p) \;=\; f(p+a) \tag{24}$$

$$(\bar{T} f)\, (p) \;=\; f(p-a) \tag{25}$$

and satisfies

$$T\, \bar{T} = \bar{T} T = \hat{1}\,, \tag{26}$$

where $\hat{1}$ means the identity on the algebra of functions of $p$.

Introducing the momentum operator $P$[8]

$$(P f)\, (p) = p\, f(p)\,, \tag{27}$$

we have

$$T P \;=\; (P+a) T \tag{28}$$

$$\bar{T} P \;=\; (P-a) \bar{T}\,. \tag{29}$$

Integrals can also be defined in this formalism. It is shown in ref. [8] that the property of an indefinite integral

$$\int df = f + \text{periodic function in } a \ , \tag{30}$$

suffices to calculate the indefinite integral of an arbitrary one form. It can be shown that[8] for an arbitrary function $f$

$$\int d\bar{p}\, f(\bar{p}) = \begin{cases} a \sum_{k=1}^{[p/a]} f(p-ka)\,, & \text{if } p \geq a \\ 0\ , & \text{if } 0 \leq p < a \\ -a \sum_{k=0}^{-[p/a]-1} f(p+ka)\,, & \text{if } p < 0 \end{cases} \tag{31}$$

where $[p/a]$ is by definition the highest integer $\leq p/a$.

All equalities involving indefinite integrals are understood modulo the addition of an arbitrary function periodic in $a$. The corresponding definite integral is well-defined when the length of the interval is multiple of $a$. Consider the integral of a function $f$ from $p_d$ to $p_u$ ($p_u = p_d + Ma$, where $M$ is a positive integer) as

$$\int_{p_d}^{p_u} dp f(p) = a \sum_{k=0}^{M} f(p_d + ka). \tag{32}$$

Using eq. (32), an inner product of two (complex) functions $f$ and $g$ can be defined as

$$\langle f\,,\, g\rangle = \int_{p_d}^{p_u} dp\, f(p)^*\, g(p)\,, \tag{33}$$

where $^*$ indicates the complex conjugation of the function $f$. The norm $\langle f\,,\, f\rangle \geq 0$ is zero only when $f$ is identically null. The set of equivalence classes [1] of normalizable functions $f$ ($\langle f\,,\, f\rangle$ is finite) is a Hilbert space. It can be shown that[8]

$$\langle f, Tg\rangle = \langle \bar{T} f, g\rangle\,, \tag{34}$$

so that

$$\bar{T} = T^\dagger\ , \tag{35}$$

where $T^\dagger$ is the adjoint operator of $T$. Eqs. (26) and (35) show that $T$ is a unitary operator. Moreover, it is easy to see that $P$ defined in eq. (27) is an Hermitian operator and from (35) one has

$$(i\partial_p)^\dagger = i\bar{\partial}_p\ . \tag{36}$$

[1]Two functions are in the same equivalence class if their values coincide on all lattice sites.

Now, we go back to address our main problem of this section, i.e. to construct an algebraic formalism, similar to the harmonic oscillator algebra, for the infinite one-dimensional square-well potential in relativistic quantum mechanics. Let us assume we have the one dimensional quantum free Klein-Gordon equation defined on the interval $0 \leq x \leq L$ with $\phi(x=0,t) = \phi(x=L,t) = 0$ where $\phi(x,t)$ is the one dimensional Klein-Gordon field. It can easily be checked that the solution of this equation is very similar to the solution of the Heisenberg equation for the square-well potential. The stationary part of the solution can be interpreted as being the spectrum of the Hamiltonian $H = \sqrt{P^2 + m_q^2}$, $\hbar = c = 1$, where the momentum is quantized with eigenvalues $n\pi/L$ for $n = 1, 2, 3, \ldots$. Therefore, the momentum space is an one-dimensional periodic lattice with constant spacing $a = \pi/L$, clearly a candidate to apply the non-commutative differential calculus sketched before. We then take the momentum operator, $P$, in the Hamiltonian $H = \sqrt{P^2 + m_q^2}$ as defined in eq. (27).

We can rewrite the Hamiltonian's eigenvalue associated with the (n+1)-th level as

$$e_{n+1}^2 = (\sqrt{e_n^2 - m_q^2} + a)^2 + m_q^2 \,, \tag{37}$$

where $e_n$ is the Hamiltonian's eigenvalue associated with the $n$-th level and $a = \pi/L$ is the lattice spacing.

As $J_0$ is related to the Hamiltonian[7] and their eigenvalues are the iterations given by a function $f$ , we see that if we choose this function as

$$f(x) = \sqrt{\left(\sqrt{x^2 - m_q^2} + a\right)^2 + m_q^2} \,, \tag{38}$$

the $J_0$ in eqs. (13-15) has eigenvalues equal to the energy eigenvalues of the square-well potential. Note that $\alpha_0 = \sqrt{a^2 + m_q^2}$. Eqs. (1-3) can then be rewritten for this case as

$$\begin{aligned} J_0 A^\dagger &= A^\dagger \sqrt{\left(\sqrt{J_0^2 - m_q^2} + a\right)^2 + m_q^2} \,, & (39) \\ A J_0 &= \sqrt{\left(\sqrt{J_0^2 - m_q^2} + a\right)^2 + m_q^2} \quad A \,, & (40) \\ \left[A^\dagger, A\right] &= J_0 - \sqrt{\left(\sqrt{J_0^2 - m_q^2} + a\right)^2 + m_q^2} \,. & (41) \end{aligned}$$

As $J_0$ is a Hermitian operator it can be diagonalized and, as we are considering only the representations where the eigenvalues of $J_0$ are positive greater or equal to $m_q$, the square roots in eqs. (39-41) are well defined.

We then have an algebra given in eqs. (39-41) where the eigenvalues of $J_0$ , $e_n$, are the energy eigenvalues of the relativistic one dimensional infinite square-well potential and $A^\dagger(A)$ act as ladder operators. From the previous discussion and from eqs. (13-15), taking $\alpha_0 = \sqrt{a^2 + m_q^2}$, the eigenvalues of $J_0$ when applied to the states $|m\rangle$ give $f^m(m_q)$ that are, for $f$ given in eq. (38), the energy eigenvalues of the relativistic one dimensional infinite square-well potential. Moreover, as said before and as seen from eqs. (13-15), $A^\dagger(A)$ act as ladder operators.

In order to have a complete description, similar to the case of the one-dimensional harmonic oscillator, we must realize the operators $J_0$, $A^\dagger$ and $A$ in terms of the physical operators of the system. The solution to this problem is

$$A^\dagger = S\,\bar{T} \tag{42}$$

$$A = T\,S \tag{43}$$

$$J_0 = \sqrt{P^2 + m_q^2}\,. \tag{44}$$

where $T$ and $\bar{T}$ are given by eqs. (22-23), $P$ by eq. (27) and $S = (J_0 + C)^{1/2}$, with $C$ the Casimir of the algebra shown in eq. (4) having eigenvalue $-\sqrt{a^2 + m_q^2}$. Using eqs. (28-29) it is straightforward to check that the operators given in eqs. (42-44) indeed satisfy eqs. (39-41). Moreover, using eq. (26) we have

$$A^\dagger A = S^2 = \sqrt{P^2 + m_q^2} + C\,. \tag{45}$$

In summary, we have constructed an algebraic formalism, similar to the harmonic oscillator algebra, for the relativistic one-dimensional square-well potential. This Heisenberg-like algebra is an example in the recently constructed class of generalized Heisenberg algebras[4]. This class of algebras contains also $q$-oscillators as an special case. Moreover, it is also interesting to stress that the ladder operators of the relativistic square-well algebra are realized in terms of the physical operators of the system.

# 4 Square-well quantum field theory

We are going to construct in this section a free quantum field theory based on the relativistic square-well algebra presented in the last section, i.e., the vacuum excitations of this quantum field are particles confined by the square-well potential.

In the momentum space appropriated to the construction of the square-well algebra, as presented in the previous section, besides the operator $P$ defined in eq. (27) one can define two type-coordinate self-adjoint operators as

$$\begin{aligned} \chi &= i(\bar{\partial}_p + \partial_p) \ , & (46) \\ Q &= \bar{\partial}_p - \partial_p \ , & (47) \end{aligned}$$

where $\partial_p$ and $\bar{\partial}_p$ are the left and right discrete derivatives defined in eqs. (19, 20). Of course, in the continuous limit the operator $Q$ is identically null since $\partial_p$ and $\bar{\partial}_p$ represent the same derivative in this limit.

It can be checked that the operators $P$, $\chi$ and $Q$ generate an algebra on the momentum lattice with lattice spacing $a = \pi/L$[8]

$$\begin{aligned} [\chi, P] &= 2i\left(1 - \frac{a}{2}Q\right), & (48) \\ [P, Q] &= -ia\chi, & (49) \\ [\chi, Q] &= 0. & (50) \end{aligned}$$

Note that, in the continuous limit $a \to 0$ we recover the standard Heisenberg algebra, $[x, p] = i$.

With the help of eqs. (22-23 and 42-44) we can rewrite $\chi$ and $Q$ as

$$\begin{aligned} \chi &= \frac{i}{a}\left(S^{-1}A^\dagger - AS^{-1}\right), & (51) \\ Q &= \frac{1}{a}\left(-2 + S^{-1}A^\dagger + A\,S^{-1}\right), & (52) \end{aligned}$$

where $S = (J_0 + C)^{1/2} = \left(\sqrt{P^2 + m_q^2} + C\right)^{1/2}$ with $C$ the Casimir operator shown in eq. (4) having eigenvalue $-\sqrt{a^2 + m_q^2}$ for the representation of interest. We stress that $A^\dagger$ and $A$ are the creation and annihilation operators respectively, of a relativistic particle with mass $m_q$ in an infinite square-well potential as explained in the previous section.

Let us now introduce a three-dimensional discrete $\vec{k}$-space,

$$k_i = \frac{2\pi l_i}{L_i}, \quad i = 1, 2, 3 \quad , \tag{53}$$

with $l_i = 0, \pm 1, \pm 2, \cdots$ and $L_i$, the lengths of the three sides of a rectangular box $\Omega$. For each point of this $\vec{k}$-space we associate an independent copy of the one-dimensional momentum lattice defined in the previous section such that $P^\dagger_{\vec{k}} = P_{\vec{k}}$ and $T_{\vec{k}}$ and $\bar{T}_{\vec{k}}$ are defined by means of the previous definitions, eqs. (22-23), through the substitution $P \to P_{\vec{k}}$ and $S_{\vec{k}}$ is given by

$$S_{\vec{k}} \equiv (J_0(\vec{k}) + C(\vec{k}))^{1/2} = \left(\sqrt{P^2_{\vec{k}} + m_q^2 + \vec{k}^2} + C(\vec{k})\right)^{1/2} \quad , \tag{54}$$

where $C(\vec{k})$ has for this representation eigenvalue $-\sqrt{a^2 + m_q^2 + \vec{k}^2}$.

We now introduce for each point of this $\vec{k}$-space independent operators $A^\dagger_{\vec{k}}$, $A_{\vec{k}}$ and $J_0(\vec{k})$ that commute for any two different point of this $\vec{k}$-space and for the same point we have

$$\begin{aligned}
J_0(\vec{k})A^\dagger_{\vec{k}} &= A^\dagger_{\vec{k}}\sqrt{\left(\sqrt{J_0^2(\vec{k}) - m_k^2} + a\right)^2 + m_k^2}\,, && (55)\\
A_{\vec{k}}J_0(\vec{k}) &= \sqrt{\left(\sqrt{J_0^2(\vec{k}) - m_k^2} + a\right)^2 + m_k^2}\quad A_{\vec{k}}\,, && (56)\\
\left[A^\dagger_{\vec{k}}, A_{\vec{k}}\right] &= J_0(\vec{k}) - \sqrt{\left(\sqrt{J_0(\vec{k})^2 - m_k^2} + a\right)^2 + m_k^2}\,, && (57)
\end{aligned}$$

where $m_k = \sqrt{m_q^2 + \vec{k}^2}$. Since $J_0(\vec{k})$ is Hermitian it can be diagonalized, moreover because the square roots the above algebra, eqs. (55-57), are well-defined for representations where $J_0(\vec{k})$ eigenvalues are greater or equal to $m_k$.

It is not difficult to see that the realization of the generators in terms of the physical operators is given as

$$\begin{aligned}
A^\dagger_{\vec{k}} &= S_{\vec{k}}\bar{T}_{\vec{k}}\,, && (58)\\
A_{\vec{k}} &= T_{\vec{k}}S_{\vec{k}}\,, && (59)\\
J_0(\vec{k}) &= \sqrt{P^2_{\vec{k}} + m_k^2}\,, && (60)
\end{aligned}$$

where $T_{\vec{k}}$ and $\bar{T}_{\vec{k}}$ are given by eqs. (22-23) for $P \to P_{\vec{k}}$.

Now, we define the type-coordinate operators for each point of the three-dimensional lattice as

$$\chi_{\vec{k}} = i(\bar{\partial}_{p_{\vec{k}}} + \partial_{p_{-\vec{k}}}) , \quad (61)$$

$$Q_{\vec{k}} = \bar{\partial}_{p_{\vec{k}}} - \partial_{p_{-\vec{k}}} , \quad (62)$$

such that $\chi^\dagger_{\vec{k}} = \chi_{-\vec{k}}$ and $Q^\dagger_{\vec{k}} = Q_{-\vec{k}}$, exactly as it happens in the construction of a spin-0 field for the spin-0 quantum field theory [6]. With the previous definitions, eqs. (58-59 and 61-62), we can rewrite the type-coordinate operators in terms of the ladder operators of the square-well Heisenberg algebra

$$\chi_{\vec{k}} = \frac{i}{a}\left(-S^{-1}_{-\vec{k}}A^\dagger_{-\vec{k}} + A_{\vec{k}}S^{-1}_{\vec{k}}\right) , \quad (63)$$

$$Q_{\vec{k}} = \frac{1}{a}\left(-2 + S^{-1}_{-\vec{k}}A^\dagger_{-\vec{k}} + A_{\vec{k}}\, S^{-1}_{\vec{k}}\right) . \quad (64)$$

By means of the type-coordinate operators, $\chi_{\vec{k}}S_{\vec{k}}$ and $Q_{\vec{k}}S_{\vec{k}}$, we can define two fields $\phi_1(\vec{r},t)$ and $\phi_2(\vec{r},t)$ as

$$\phi_1(\vec{r},t) = \frac{1}{2}\sum_{\vec{k}} \frac{i}{\sqrt{\Omega}}\frac{1}{\omega(\vec{k})}\left(-S^{-1}_{\vec{k}}A^\dagger_{\vec{k}}S_{-\vec{k}}\, e^{-i\vec{k}.\vec{r}} + A_{\vec{k}}\, e^{i\vec{k}.\vec{r}}\right), \quad (65)$$

$$\phi_2(\vec{r},t) = \frac{1}{2}\sum_{\vec{k}} \frac{1}{\sqrt{\Omega}}\frac{1}{\omega(\vec{k})}\left(-2S_{\vec{k}}\, e^{i\vec{k}.\vec{r}} + S^{-1}_{\vec{k}}A^\dagger_{\vec{k}}\, S_{-\vec{k}}\, e^{-i\vec{k}.\vec{r}} + A_{\vec{k}}\, e^{i\vec{k}.\vec{r}}\right), \quad (66)$$

where $\omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}$, $m$ a real parameter and $\Omega$ is the volume of a rectangular box.

What we have done so far is similar to the construction of spin- 0 fields in relativistic quantum field theory in terms of the creation and annihilation operators of the harmonic oscillator algebra with the difference that now, we have the ladder operators of a relativistic particle in a square-well potential. Moreover, if we define type-momentum fields as

$$\Pi(\vec{r},t) = \sum_{\vec{k}} \frac{\beta_1}{\sqrt{\Omega}}\ S_{\vec{k}}\, e^{i\vec{k}.\vec{r}}, \quad (67)$$

$$\wp(\vec{r},t) = \frac{1}{2\beta_1}\sum_{\vec{k}} \frac{1}{\sqrt{\Omega}}\left(-S_{\vec{k}}\, e^{i\vec{k}.\vec{r}} + S^{-1}_{\vec{k}}A^\dagger_{\vec{k}}\, S_{-\vec{k}}\, e^{-i\vec{k}.\vec{r}} + A_{\vec{k}}\, e^{i\vec{k}.\vec{r}}\right), \quad (68)$$

where $\beta_1$ is an arbitrary real number we can show that the Hamiltonian

$$H = \int\ d^3r \left(\Pi(\vec{r},t)^\dagger\ \wp(\vec{r},t) + \wp(\vec{r},t)^\dagger\ \Pi(\vec{r},t) + \phi_1(\vec{r},t)^\dagger(-\vec{\nabla}^2 + m^2)\phi_1(\vec{r},t) + \phi_2(\vec{r},t)^\dagger(-\vec{\nabla}^2 + m^2)\phi_2(\vec{r},t)\right) , \quad (69)$$

can be rewritten as

$$H = \sum_{\vec{k}} A^\dagger_{\vec{k}} A_{\vec{k}} = \sum_{\vec{k}} S^2_{\vec{k}} = \sum_{\vec{k}} \sqrt{P^2_{\vec{k}} + m_q^2 + \vec{k}^2} + C(\vec{k}) \ , \tag{70}$$

where $P_{\vec{k}}$, for each $\vec{k}$, is the momentum operator for a particle with mass $m_k$ in a square-well potential. The Casimir operator, $C(\vec{k})$, in the representation in consideration has eigenvalue $-\sqrt{a^2 + m_q^2 + \vec{k}^2}$.

The eigenvectors of $H$ form a complete set and span the Hilbert space of this system. The eigenvectors are

$$|1\rangle, \ A^\dagger_{\vec{k}}|1\rangle, \ A^\dagger_{\vec{k}} A^\dagger_{\vec{k}'}|1\rangle \ \text{for} \ \vec{k} \neq \vec{k}', \ (A^\dagger_{\vec{k}})^2|1\rangle, \ \cdots \tag{71}$$

Note that the lowest energy state is the one of a particle in the lowest level in the relativistic square-well potential which is represented by $|1\rangle$. This Hilbert space has a different interpretation with respect to the standard spin-0 quantum field theory based on the harmonic oscillator. In the standard case the analog of the formula (70) is

$$H = \sum_{\vec{k}} N_{\vec{k}} \sqrt{\vec{k}^2 + m^2} \ , \tag{72}$$

where $N$ is the number operator. While in the standard quantum field theory the creation operator creates one particle of mass $m$ each time it is applied to the vacuum, in the square-well quantum field theory one reads from eq. (70) that the creation operator in this case creates excited states of particles with mass spectrum given by $\sqrt{a^2\, n^2 + m_q^2}$ where $n = 1, 2, \cdots$ is the level of the particle with mass $m_q$ in the square-well confining potential. In other words, for instance, the state $(A^\dagger_{\vec{k}})^n\,|1\rangle$ represents not a $n$-particle state but a state of one particle with mass $\sqrt{a^2\,(n+1)^2 + m_q^2}$.

The time evolution of the fields can be studied by means of Heisenberg's equation for $A^\dagger_{\vec{k}}$ and $A_{\vec{k}}$. Taking into account eqs. (39-41, 70) we have

$$\frac{d}{dt} A^\dagger_{\vec{k}}(t) \equiv \dot{A}^\dagger_{\vec{k}}(t) = i A^\dagger_{\vec{k}}(t) \ \Delta H_{\vec{k}} \ , \tag{73}$$

with

$$\Delta H_{\vec{k}} = \sqrt{(P_{\vec{k}} + a)^2 + m_q^2 + \vec{k}^2} - \sqrt{P^2_{\vec{k}} + m_q^2 + \vec{k}^2} \ . \tag{74}$$

Eq. (73) has as solution

$$A^\dagger_{\vec{k}}(t) = A^\dagger_{\vec{k}}(0)\, \exp\left(i\Delta H_{\vec{k}} t\right) \ , \tag{75}$$

with a similar expression for $A_{\vec{k}}(t)$.

Even for the free theory we have just constructed there are some additional points that deserve to be further investigated. For example, it would be interesting to construct the Lorentz algebra operators of the system. We hope to understand this and other points in a near future.

# 5 Final comments

This paper was the first step towards an analysis of possible consequences in QFT of a class of generalized Heisenberg algebras we have recently constructed [4]. We hope that the QFT we developed here can be used as an alternative phenomenological approach to hadronic interactions and also that the entire treatment developed here could be applied to a deformed Heisenberg algebra in order to construct a deformed QFT for very high energy physics ($10^3$Gev$< E < 10^{19}$Gev).

We constructed here a free quantum field theory based on the Heisenberg algebra of a relativistic particle in an infinite square-well potential: the quantum one dimensional free Klein-Gordon equation defined on the interval $0 \leq x \leq L$ with special boundary conditions. It is interesting to stress that this formalism shows that it is possible to construct a different class of quantum field theories based on an algebraic structure, having ladder operators, different from the harmonic oscillator algebra. Moreover, note that even if the Heisenberg relativistic square-well algebra shown in eqs. (39-41) has a non-simple form, the realization of the ladder operators in terms of the physical operators of the model as seen in eqs. (22-23 and 42-44) is really very simple.

In the previous section, we introduced fields written in terms of the generators of the square-well algebra and a Hamiltonian written in terms of these fields describing a system with an infinite number of degrees of freedom and mass spectrum excitations given by $\sqrt{n^2\pi^2/L^2 + m_q^2}$, where $n = 1, 2, \cdots$ denotes the level of the particle with mass $m_q$ in an infinite square-well potential of width $L$. As we have already noted, even for this free quantum field theory there are certainly some points, not analyzed in this paper, as the mentioned Lorentz algebra, that deserve to be investigated. It would be also interesting to construct the interacting theory of this quantum field theory, for the excitations produced by the square-well creation operator as well as for their interaction with an electromagnetic field.

Finally, since the mass spectrum of this quantum field theory are excited states of a confined particle it is tempting to use it to try to address at least some aspects of hadronic interactions. In other words, we conjecture a possible application of this formalism as a phenomenological quantum field approach to hadronic interactions. Of course, as the "potential" (or the mass spectrum) used in this quantum field theory is the simple square-well potential it is hard to imagine that the results given by the interacting quantum field theory can be very precise. Thus, we also think it would be interesting to extend the formalism developed in this paper to a more realistic mass spectrum.

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