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Non-Linear Ohm's Law

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Abstract

We deduce from the second Newton's law the medium constitute Ohm's law for an electrical flow in a medium with square power law for the condutor resistivity.

The elementary discussions of the electrical conductivity in mediums always have been made in a context where the effective medium resistence has an effective behavior of a linear damping term proportional to the flow's velocity ([1],[2]). In this comment we present an elementary approach on electrical conductivities in a medium with a non-linear quadratic polinomial damping term on the electron flow velocities [3].

Let us start our analysis from the second newton's law applied to a one-dimensional electron flow through a conductor with a square power law for the conductor resistence under the presence of a steady electric field from the initial conduction (electron mass $m_e = 1$)

$$\frac{dV}{d\tau} = eE - vV^2 \tag{1}$$

$$V(0) = 0 \tag{2}$$

here $V(\tau)$ is the electron velocity and $I(\tau) = \rho A V(\tau)$ is the current flow where A is the normal cross section of the conduction and ρ is the electron charge density.

The solution of the Riccati equation (1) can be obtained through the substitution

$$V(t) = \frac{1}{v}\ \frac{dy(\tau)/d\tau}{y(\tau)} \tag{3}$$

with $y(t)$ satisfying the linear equation

$$y\ \frac{d^2 y(\tau)}{dt^2} - (v^2 eE) y(\tau) = 0$$

$$y(0) = y_0\ ;\ y'(0) = 0 \tag{4}$$

the solution of eq. (4) is given by

$$y(\tau) = c_1\ e^{\sqrt{evE}\tau} + c_2 e^{-\sqrt{evE}\tau} \tag{5}$$

performing the inverse transformation in eq. (3) we obtain the physical electron flow velocity

$$V(\tau) = \sqrt{\frac{eE}{v}} \left(\frac{1 - e^{-2\sqrt{evE}\tau}}{1 + e^{-2\sqrt{evE}\tau}} \right) \tag{6}$$

In a steady-state processes, we obtain

$$V_s = \sqrt{\frac{eE}{v}} \tag{7}$$

and leading, thus, to the non-linear Ohm's law

$$I = \rho A \sqrt{\frac{e}{v}\frac{V}{d}}$$

or (8)

$$V = I^2 \bar{R}$$

with the effective electrical resistance

$$\bar{R} = \frac{vd}{\rho^2 A^2 e} \tag{9}$$

At this point let us analyze the more general case with the microscopic friction law $\alpha V + \eta V^2$ (α and η positive constants):

$$\frac{dV(\tau)}{d\tau} = eE - \alpha V - vV^2 \tag{10}$$

$$V(0) = 0 \tag{11}$$

Performing the Riccati transformation (3) in eq. (9) we obtain

$$\frac{d^2y(\tau)}{d\tau^2} + \alpha\ \frac{dy(\tau)}{d\tau^2} - (eEv)y(\tau) = 0 \tag{12}$$

with $y(0) = y_0$ and $y'(0) = 0$.

as a consequence we have the result

$$V(\tau) = \frac{\alpha}{2v}(\gamma - 1)\left[\frac{1 - Ae^{-\alpha\gamma\tau}}{1 + Ae^{-\alpha\gamma\tau}}\right] \tag{13}$$

Here $\gamma = (1 + \frac{4evE}{\alpha^2})^{1/2}$ and A is a constant the steady solution is given by the $t \to \infty$ lim in the above equation

$$V_c = \frac{\alpha}{2v}(\gamma - 1) = \frac{1}{v}\left[-\frac{\alpha}{2} + \frac{\alpha}{2}\sqrt{1 + \frac{4evE}{\alpha^2}}\right] \tag{14}$$

and thus, giving the associated non-linear Ohm's law

$$\bar{I} = \frac{\rho A}{\eta}\frac{\alpha}{2}\left[-1 + \frac{4e\eta eV}{\alpha^2 d}\right] \tag{15}$$

or

$$V = \left(\frac{\alpha d}{\rho A e}\right)\bar{I} + \left(\frac{vd}{\rho^2 A^2 e}\right)\bar{I}^2 \tag{16}$$

It is worth remark that eq. (15) reduces to the usual Ohm's law for $\upsilon \to 0$. Is remarkable that eq. (13) can not be obtained from the usual procedure of considering $\frac{dV}{d\tau} = 0$.

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References

[1] Luiz C.L. Botelho, "Ohm's Law: a Path-Integral Study at Classical and Quantum Level", Notas de Física.

[2] M.A.L. Omar, "Elementary Solid State Physics", World Student Series Edition, 1975.