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LIGHT-FRONT QUANTIZED FIELD THEORY: *SOME NEW RESULTS* *

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Abstract

A review is made on some recent studies which support the point of view that the relativistic field theory quantized on the light-front (LF), as proposed by Dirac, seems to be more transparent compared to the conventional equal-time quantized one. Some ideas following from these studies may be of some relevance in the context of the quantization of gravitation theory.

It is argued on general grounds that the LF quantization is *equally appropriate* as the conventional equal-time one and that the two should lead, assuming the microcausality principle, to the same physical content. This is shown to be true by considering several model field theories. The description on the LF of the spontaneous symmetry breaking (SSB), (tree level) Higgs mechanism, of the *condensate* or θ-vacua in the Schwinger model (SM), of the absence of such vacua in the Chiral SM (CSM), and of the BRS-BFT quantization of the *front form* CSM are among the topics discussed.

The LF phase space is strongly constrained and is different from the one in the conventional theory. The removal of the constraints by following the Dirac procedure results in a substantially reduced number of independent operators. The discussion of the physical Hilbert space and the vacuum becomes more tractable.

Some comments on the irrelevance, in the quantized field theory, of the fact that the hyperplanes $x^{\pm}=0$ constitute characteristic surfaces of the hyperbolic partial differential equation are also made. The LF theory quantized on, say, the $x^{+}=const.$ hyperplanes seems to contain in it the information on the equal-x^{-} commutators as well.

A theoretical reaffirmation of the universally accepted notion that the experimental data is to be confronted with the predictions of a classical theory model only after it has been upgraded through its quantization seems to emerge. The LF quantization promises to be a powerful tool, complementary to the functional integral method, for handling the nonperturbative calculations.

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1- Introduction

Dirac [1], in his paper, in 1949, discussed the unification, in a relativistic theory, of the principles of the quantization and the special relativity theory. The Light-Front (LF) quantization which studies the relativistic quantum dynamics of physical system on the hyperplanes : $x^0 + x^3 \equiv \sqrt{2}x^+ = const.$, called the *front form* theory, was also proposed there. The *instant form* or the conventional equal-time theory on the contrary uses the $x^0 = const.$ hyperplanes. The LF coordinates x^μ : $(x^+, x^-, x^\perp)$, where $x^\pm = (x^0 \pm x^3)/\sqrt{2} = x_\mp$ and $x^\perp = (x^1, x^2)$, are convenient to use in the *front form* theory. They are *not related by a Lorentz transformation* to the coordinates $(x^0 \equiv t, x^1, x^2, x^3)$ usually employed in the *instant form* theory and as such the descriptions of the same physical content of a dynamical theory on the LF may come out to be different from that given in the conventional treatment. The LF quantized field theory may hence be of some relevance in the understanding of the unification of the principles of the quantization with that of the general covariance[1].

We will make the *convention* to regard[2] $x^+ \equiv \tau$ as the LF-time coordinate while $x^- \equiv x$ as the *longitudinal spatial* coordinate. The temporal evolution in x^0 or x^+ of the system is generated by Hamiltonians which are different in the two *forms* of the theory. The LF components, with $\mu = +, -, 1, 2$, of any tensor are defined likewise.

Consider [2] the invariant distance between two spacetime points : $(x - y)^2 = (x^0 - y^0)^2 - (\vec{x} - \vec{y})^2 = 2(x^+ - y^+)(x^- - y^-) - (x^\perp - y^\perp)^2$. On an equal $x^0 = y^0 = const.$ hyperplane the points have spacelike separation except for if they are *coincident* when it becomes lightlike one. On the LF with $x^+ = y^+ = const.$ the distance becomes *independent of* $(x^- - y^-)$ and the seperation is again spacelike; it becomes lightlike one when $x^\perp = y^\perp$ but with the difference that now the points need *not* necessarily be coincident along the longitudinal direction. The LF field theory hence need not necessarily be local in x^-, even if the corresponding *instant form* theory is formulated as a local one. For example, the commutator $[A(x^+, x^-, x^\perp), B(0, 0, 0^\perp)]_{x^+=0}$ of two scalar observables would vanish on the grounds of microcausality principle if $x^\perp \neq 0$ when $x^2|_{x^+=0}$ is spacelike. Its value would hence be proportional to $\delta^2(x^\perp)$ and a finite number of its derivatives, implying locality only in $x^\perp$ but not necessarily so in x^-. Similar arguments in the *instant form* theory lead to the locality in all the three spatial coordinates. In view of the microcausality both of the commutators $[A(x), B(0)]_{x^+=0}$ and $[A(x), B(0)]_{x^0=0}$ are nonvanishing only on the light-cone.

We remark that in the LF quantization we time order with respect to τ rather than t. The microcausality principle, however, ensures that the retarded commutators $[A(x), B(0)]\theta(x^0)$ and $[A(x), B(0)]\theta(x^+)$, which appear [3] in the S-matrix elements of relativistic field theory, do not lead to disagreements in the two formulations. In the regions $x^0 > 0, x^+ < 0$ and $x^0 < 0, x^+ > 0$, where the commutators seem different the x^2 is spacelike and both of them vanish. Hence, admitting the *microcausality principle* to hold, the LF hyperplane seems *equally appropriate* as the conventional one of the *instant form*

[1]We recall the Kruskal-Szekers coordinates which threw a new light on the problem of the Schwarzshild singularity.

[2]The coordinates x^+ and x^- appear in a symmetric fashion and we note that $[x^+, \frac{1}{i}\partial^-] = [x^-, \frac{1}{i}\partial^+] = i$ where $\partial^\pm = \partial_\mp = (\partial^0 \pm \partial^3)/\sqrt{2}$ etc..

theory for the canonical quantization.

The structure of the *LF phase space*, however, is different from that of the one in the conventional theory. Consequently, we may require on the LF a different description of to the same physical content as found in the conventional treatment. For example, the SSB needs a different description [2] or mechanism on the LF when compared with the conventional one. The broken continuous symmetry is now inferred from the study of the residual unbroken symmetry of the LF Hamiltonian operator while the symmetry of the LF vacuum remains intact. The expression which counts the number of Goldstone bosons present in the theory, a physical content, comes out to be the same as that found in the equal-time quantized theory. A new proof of the Coleman's theorem [4] on the absence of the Goldstone bosons in two dimensional theory also emerges [2] easily on the LF. The LF vacuum is generally found to be simpler [5, 6] and in many cases the interacting theory vacuum is seen to coincide[3].

An important advantge pointed out by Dirac of *front form* theory is that here *seven* out of the ten Poincaré generators are *kinematical*, e.g., they leave the plane $x^+ = 0$ invariant [1]. They are P^+, P^1, P^2, $M^{12} = -J_3$, $M^{+-} = M^{03} = -K_3$, $M^{1+} = (K_1 + J_2)/\sqrt{2}$ and $M^{+2} = (K_2 - J_1)/\sqrt{2}$. In the conventional theory only six such ones, *viz.*, $\vec{P}$ and $M^{ij} = -M^{ij}$, leave the hyperplane $x^0 = 0$ invariant. In fact, in the standard notation $K_i = -M^{0i}, J_i = -(1/2)\epsilon_{ijk}M^{kl}, i, j, k = 1, 2, 3$ and the generator K_3 is dynamical one in the *instant form* theory. It is in contrast kinematical in the *front form* theory where it generates the scale transformations of the LF components of x^μ, P^μ and $M^{\mu\nu}$, with $\mu, \nu = +, -, 1, 2$. It is also worth remarking that the + component of the Pauli-Lubanski pseudo-vector W^μ is special in that it contains only the LF kinematical generators. This suggests us to define the *LF Spin operator* by $\mathcal{J}_3 = -W^+/P^+$. The other two components of $\vec{\mathcal{J}}$ are shown to be $\mathcal{J}_a = -(\mathcal{J}_3 P^a + W^a)/\sqrt{P^\mu P_\mu}$, $a = 1, 2$, which, however, do carry [4] in them also the LF *dynamical generators* P^-, M^{1-}, M^{2-}.

Another distinguishing feature of the *front form* theory is that it gives rise generally to a (strongly) constrained dynamical system [7] which leads to an appreciable reduction in the number of independent operators on the phase space. The vacuum structure, for example, then becomes more tractable and the computation of physical quantities simpler. This is verified in the recent study of the LF quantized SM [8] and the Chiral SM (CSM) discussed below where we are led directly to the *physical Hilbert space*, once

[3]In fact, in many cases the interacting theory vacuum may coincide with the perturbation theory one. This results from the fact that momentum four-vector is now given by $(k^-, k^+, k^\perp)$ where $k^\pm = (k^0 \pm k^3)/\sqrt{2}$. Here k^- is the LF energy while $k^\perp$ and k^+ indicate the transverse and the *longitudinal* components of the momentum respectively. For a free massive particle on the mass shell we have $2k^-k^+ = ({k^\perp}^2 + m^2) > 0$ so that $k^\pm$ are both positive definite when $k^0 > 0$. The conservation of the total longitudinal momentum does not permit the excitation of massive quanta by the LF vacuum. We require $k^+ \to 0$ for each particle (and antiparticle) entering the ground state, which has vanishing total momentum. Such configurations constitute a point with zero measure in the LF phase space and may [5] not be of relevance in many cases. However, it should be noted that when dealing with the momentum space integrals, say, the loop integrals, a significant contribution may arise precisely from such a corresponding configuration in the integrand; the reason being that we have to deal with the products of several distributions. On the other hand, (k^1, k^2, k^3) may take positive or negative values and we may construct in the conventional theory eigenstates of zero momentum with an arbitrary number of particles (and antiparticles) which may mix with the vacuum state, with no particles, to form the ground state.

[4]See refs. [2] and [5].

the constraints are taken into account by following the Dirac procedure [7].

We recall that the LF field theory was rediscovered in 1966 by Weinberg [9] in his Feynman rules adapted for infinite momentum frame. It was demonstrated [10] latter that these rules, in fact, correspond to the *front form* quantized theory. It was also employed successfully in the *nonabelian bosonization* of the field theory of N free Majorana fermions, where the corresponding LF current algebra was compared [11] with the one in the bosonized theory described by the WZNW action at the critical point. The interest in LF quantization has been revived [5, 6] also due to the difficulties encountered in the computation, in the conventional framework, of the nonperturbative effects in the context of QCD and the problem of the relativistic bound states of light fermions [6, 5] in the presence of the complicated vacuum. Studies show that the application of Light-front Tamm-Dancoff method may be feasible here. The technique of the regularization on the lattice has been quite successful for some problems but it cannot handle, for example, the light (chiral) fermions and has not been able yet to demonstrate, for example, the confinenment of quarks. The problem of reconciling the standard constituent quark model and the QCD to describe the hadrons is also not satisfactorily resolved. In the former we employ few valence quarks while in the latter the QCD vacuum state itself contains, in the conventional theory, an infinite sea of constituent quarks and gluons (partons) with the density of low momentum constituents getting very large in view of the infrared slavery. The *front form* dynamics may serve as a complementary tool to study such probelms since we have a simple vacuum here while the complexity of the problem is now transferred to the LF Hamiltonian. In the case of the scalar field theory, for example, the corresponding LF Hamiltonian is, in fact, found [2] to be nonlocal due to the appearence of *constraint equations* on the *LF phase space*.

We discuss here only some of the interesting conclusions reached from the detailed study of some model relativistic theories on the LF and where the standard Dirac procedure for constrained dynamical systems is followed in order to build the self-consistent Hamiltonian formulation. Among some of the results obtained we find that

- The LF hyperplane is *equally appropriate* as the conventional equal-time one for the field theory quantization.

- The hyperplanes $x^{\pm} = 0$ define the characteristic surfaces of a hyperbolic partial differential equation. From the mathematical theory of classical partial differential equations [12] it is known that the Cauchy initial value problem would require us to specify the data on both the hyperplanes. ¿From our studies we conclude that it is sufficient in the *front form* theory to choose, as proposed by Dirac [1], one of the two LF hyperplanes for canonically quantizing the theory.

  - In the quantized theory the equal-τ commutators of the field operators, at a fixed initial LF-time, form now a part of the initial data instead and we deal with operator differential equations.

  - The studies show that the information on the commutators on the other characteristic hyperplane are already contained [8] in the quantized theory and need not thus be specified separately.

- The inherent symmetry with regard to equal-$x^{\pm}$ commtators along with the reduced number of independent field operators which survive after the constraints are taken care of seem to be responsible for the very transparent discussion on the LF.

- The physical content following from the *front form* theory is the same, even though arrived at through different description on the LF, when compared with the one in the *instant form* case.
- In the conventional treatment we sometimes are required to introduce external constraints in the theory based on physical considerations, say, when describing the spontaneous symmetry breaking. Many of the analogous constraints may be shown to be already icorporated in the quantized theory considered on the LF.
- A theoretical demonstration of the well accepted notion that a classical model field theory must be upgraded first through its quantization before we confront it with the experimental data, seems to emerge.
- The recently proposed BRS-BFT [13] quantization procedure is extended straightforwardly on the LF as well as is illustrated below in the context of CSM (Appendix D).
- Topological considerations often employed in the context of the functional integral techniques, where the Euclidean theory action is ususally employed, also seem to have their counterpart, though now interpreted differently. This is suggested, for example, from the studies of the LF quantized SM and CSM.

For illustration purposes we discuss in the following Sections the description on the LF of the spontaneous symmetry breaking (SSB) and of the structure of the vacuum state in the CSM while some other topics related to the *front form* theory are collected in the Appendices.

2- Spontaneous Symmetry Breaking Mechanism on the LF

On the canonical quantization of the *instant form* scalar field Lagrangian theory we obtain as well known the Hamiltonian and the commutation relations among the field operators. The description of, say, the tree level SSB emerges when we require also (e.g., introduce *external* constraints), based on physical considerations, that the $\phi_{classical} \equiv \omega$ corresponds to the minimum of the Hamiltonian functional. The *front form* of the same theory describes [2] a constrained dynamical system and the canonical Hamiltonian framework, which may be quantized by the correspondence principle, is shown to contain in it a new ingredient in the form of the *constraint equations*, in addition to the Hamiltonian and commmutators among the field operators. The constraint equations may also be derived from the Lagrange equations of motion [5]. The new ingredient permits us to

[5]That the constraint equations can be derived simply by integrating the Lagrange equations of motion over the longitudinal spatial coordinate x^- was noted also in: P.P. Srivastava, *On spontaneous symmetry breaking mechanism on the light-front quantized field theory,* Ohio-State preprint 91-0481, Slac database PPF-9148 (see also 92-0012, PPF-9202), November '91; available as *scanned* copies. In fact, Dirac in his paper does consider some examples where the constraints on the form of the potential are required if we would like to unify in the dynamical theory the principles of the quantization and the relativistic invariance. It is interesting to note that soon after in 1950-52 he formulated also the systematic method (Dirac procedure) for constructing Hamiltonian formulation for a constrained dynamical system.

describe [2] SSB on the LF without requiring us to deal with the difficult task of introducing constraints on the LF based on physical considerations. Some of them may be shown to be already incorporated [2] in the formulation itself due to the requirement of the selfconsistency [7].

The existence of the continuum limit of the Discretized Light Cone Quantized (DLCQ) [14] theory, the nonlocal nature of the LF Hamiltonian, and the description of the SSB on the LF were clarified [2, 15] only recently.

Consider first the two dimensional case of single real scalar theory with the Lagrangian $\mathcal{L} = [\dot{\phi}\phi' - V(\phi)]$. Here $\tau \equiv x^+ = (x^0 + x^1)/\sqrt{2}$, $x \equiv x^- = (x^0 - x^1)/\sqrt{2}$, $\partial_\tau \phi = \dot{\phi}$, $\partial_x \phi = \phi'$, and $d^2x = d\tau dx$. It is the simplest example of a constrained field theory. The eq. of motion, $\dot{\phi}' = (-1/2)\delta V(\phi)/\delta\phi$, shows that $\phi = const.$ is a possible solution. We propose to make the following *separation*[6] $\phi(\tau, x) = \omega(\tau) + \varphi(\tau, x)$ where the $\omega(\tau)$ is the *dynamical variable* representing the *bosonic condensate* and $\varphi(\tau, x)$ describes (quantum) *fluctuations* above it. We set $\int dx^- \varphi(\tau, x) = 0$ so that the fluctuation field carries no zero momentum mode in it. Subsequently, we apply the standard Dirac procedure in order to construct a selfconsistent Hamiltonian formulation which may be quantized canonically.

We are led [2] to

$$[\varphi(x,\tau), \varphi(y,\tau)] \;=\; -\frac{i}{4}\epsilon(x-y), \tag{1}$$

$$[\omega(\tau), \varphi(x,\tau)] \;=\; 0, \tag{2}$$

and

$$H^{lf} \equiv P^- = \int dx \left[\omega(\lambda\omega^2 - m^2)\varphi + \frac{1}{2}(3\lambda\omega^2 - m^2)\varphi^2 + \lambda\omega\varphi^3 + \frac{\lambda}{4}\varphi^4\right], \tag{3}$$

along with the (second class) constraint equation

$$\begin{aligned} lim_{R\to\infty}\frac{1}{R}\int_{-R/2}^{R/2} dx\, V'(\phi) \;\equiv\; & \omega(\lambda\omega^2 - m^2) + lim_{R\to\infty} \\ & \frac{1}{R}\int_{-R/2}^{R/2} dx \Big[\,(3\lambda\omega^2 - m^2)\varphi + \lambda(3\omega\varphi^2 + \varphi^3)\Big] = 0 \end{aligned} \tag{4}$$

where we have assumed $V(\phi) = (\lambda/4)(\phi^2 - m^2/\lambda)^2$, $\lambda \geq 0$, $m \neq 0$.

Eliminating ω would lead to a nonlinear and nonlocal Hamiltonian in the *front form* theory even when the scalar theory is written above a local one in the conventional *instant form* formulation.

At the tree or classical level φ are bounded ordinary functions in x^- and only the first term survives in the constraint equation leading to $V'(\omega) = 0$, which is the same as found in the conventional theory. There it is essentially added to the theory, on the physical considerations which require the energy functional to attain is minimum (extremam) value. The stability property, say, of a particular constant solution may be inferred as

[6] It was first proposed in the ref. cited in the previous footnote; in $3+1$ dimensions the separation is: $\phi(\tau, x^-, x^\perp) = \omega(\tau, x^\perp) + \varphi(\tau, x^-, x^\perp)$. See papers contribuited to *XXVI Intl. Conference on High energy Physics, Dallas, Texas*, August '92, *AIP Conf. Proc.*, 272 (1993) 2125, Ed. J.R. Sanford.

usual from the classical partial differential equation of motion. For example, $\omega = 0$ is shown to be an unstable solution for the potential V considered above while the other two solutions with $\omega \neq 0$ give rise to the stable phases[7].

The construction of the Hamiltonian formulation using the Dirac method [7] is a straightforward exercise. We may use [2] the continuum formulation directly or proceed from the DLCQ [14], and take its infinitie volume limit [15] to obtain the same results. The canonical quantization is performed via the correspondence which relates the final Dirac brackets with the commutators (or anticommutators). We note that the Dirac procedure when applied to the scalar theory written in the continuum shows that the variable ω is a c-number or a background field; in the theory described in finite volume, however, its commutator with φ is nonvanishing [2] and as such it is a q-number operator. We stress that in our discussion the condensate variable is introduced as a dynamical variable and we let the Dirac procedure decide if it comes out as a c- or q-number. In the SM it comes out to be an operator rather than a background field.

In the quantized theory the constraint equation above shows that the value of ω would be altered from its tree level value due to the quantum corrections arising from the other terms. It is, in fact, straightforward to renormalize the theory, say, up to one-loop order by employing the Dyson-Wick expansion. We do not need to solve the constraint first which would give rise to a very complicated LF Hamiltonian. It is more convenient to derive [2] the renormalized constraint eqn. which together with the expression of mass renormalization condition give us two eqns. which may be used to study [2] the phase transition as conjectured by Simon and Griffiths [16].

In view of the LF commutator above the scalar field has the LF momentum space expansion : $\varphi(x,\tau) = (1/\sqrt{2\pi}) \int dk\ \theta(k)\ [a(k,\tau)\, e^{-ikx} + a^\dagger(k,\tau) e^{ikx}]/(\sqrt{2k})$, were $a(k,\tau)$ and $a^\dagger(k,\tau)$ satisfy the canonical equal-τ commutation relations, $[a(k,\tau), a(k',\tau)^\dagger] = \delta(k-k')$ etc.. The vacuum state is defined by $a(k,\tau)|vac\rangle = 0$, $k > 0$ and the tree level description of the SSB is given as follows. The values of $\omega = \langle|\phi|\rangle_{vac}$ obtained from $V'(\omega) = 0$ characterize the different vacua in the theory. Distinct Fock spaces corresponding to different values of ω are built as usual by applying the creation operators on the corresponding vacuum state. The $\omega = 0$ corresponds to a *symmetric phase* since the Hamiltonian operator is then symmetric under $\varphi \to -\varphi$. For $\omega \neq 0$ this symmetry is violated and the system is in a *broken or asymmetric phase.*

The extension to 3+1 dimensions and to the global continuous symmetry is straightforward[8]. Consider real scalar fields $\phi_a (a = 1, 2, ..N)$ which form an isovector of global internal symmetry group $O(N)$. We now write[9] $\phi_a(x, x^\perp, \tau) = \omega_a + \varphi_a(x, x^\perp, \tau)$ and the Lagrangian density is $\mathcal{L} = [\dot{\varphi}_a \varphi'_a - (1/2)(\partial_i\varphi_a)(\partial_i\varphi_a) - V(\phi)]$, where $i = 1, 2$ indicate the transverse space directions. The Taylor series expansion of the constraint equations $\beta_a = 0$ gives a set of coupled equations $R\, V'_a(\omega) + V''_{ab}(\omega) \int dx\varphi_b + V'''_{abc}(\omega) \int dx\varphi_b\varphi_c/2 + ... = 0$. Its discussion at the tree level leads to the conventional theory results. The LF symmetry generators are found to be $G_\alpha(\tau) = -i \int d^2x^\perp dx \varphi'_c (t_\alpha)_{cd}\varphi_d = \int d^2k^\perp\, dk\, \theta(k)\, a_c(k, k^\perp)^\dagger (t_\alpha)_{cd} a_d(k, k^\perp)$

[7] A similar analysis of the corresponding *partial* differential equations in the conventional treatmnet can also be made; the Fourier transform theory is convenient to use.

[8] See Nuovo Cimento A107 (1994) 549 and ref. [2].

[9] In general $\phi_a(x^-, x^\perp, \tau) = \omega_a(x^\perp, \tau) + \varphi_a(x^-, x^\perp, \tau)$ and the $x^\perp$ dependent tree level configurations (e.g. kinks etc.) are determined from $\left[V'_a(\omega) - \partial^\perp.\partial^\perp\omega_a\right] = 0$.

where $\alpha, \beta = 1, 2, .., N(N-1)/2$, are the group indices, t_α are hermitian and antisymmetric generators of $O(N)$, and $a_c(k, k^\perp)^\dagger$ ($a_c(k, k^\perp)$) is creation (destruction) operator contained in the momentum space expansion of φ_c. These are to be contrasted with the generators in the equal-time theory, $Q_\alpha(x^0) = \int d^3x\, J^0 = -i \int d^3x (\partial_0\varphi_a)(t_\alpha)_{ab}\varphi_b - i(t_\alpha\omega)_a \int d^3x (d\varphi_a/dx_0)$. All the symmetry generators thus annihilate the LF vacuum and the SSB is now seen in the broken symmetry of the quantized theory Hamiltonian. The criteria for the counting of the number of Goldstone bosons on the LF is found to be the same as in the conventional theory. In contrast, the first term on the right hand side of $Q_\alpha(x^0)$ does annihilate the conventional theory vacuum but the second term gives now non-vanishing contributions for some of the (broken) generators. The symmetry of the conventional theory vacuum is thereby broken while the quantum Hamiltonian remains invariant. The physical content of SSB in the *instant form* and the *front form*, however, is the same though achieved by different descriptions. Alternative proof on the LF, in two dimensions, can be given of the Coleman's theorem related to the absence of Goldstone bosons; we are unable [2] to implement the second class constraints over the phase space. We remark that the simplicity of the LF vacuum is in a sense compensated by the involved nonlocal Hamiltonian. The latter, however, may be treatable using advance computational techniques. Also in connection with renormalization it may not be necessary[10] first to solve all the constraint equations.

To summarize, the simple procedure of separating first the condensate variable ω in the scalar field before applying the Dirac procedure is found to be successful also in describing [2] the phase transition in two dimensional scalar theory, the SSB of continuous symmetry, a new proof of the Coleman's theorem and the tree level Higgs mechanism. It is again found successful in showing [8] the emergence on the LF of the θ-vacua along with their continuum normalization in the bosonized SM while explaining at the same time their absence in the CSM. The condensate variable, we remind, is introduced as a dynamical variable and we let the Dirac procedure decide if it comes out to be a c-number (background field) or a q-number operator in the quantized field theory. It is shown [2] to be c-number in the scalar theory studied in the continuum while it is an operator in the SM whose eigenvalues characterize the θ-vacua. In the next Sec. we discuss in some detail the vacuum structure in the CSM which illustrates the remarkable transparency attained in the discussion on the LF.

It is worth remarking that the LF formulation is inherently symmetrical with respect to x^+ and x^- and it is a matter of convention that we take the plus component as the LF *time* while the other as a spatial coordinate. The theory quantized at $x^+ = const.$ hyperplanes seems already to incorporate in it the information on the equal-x^- commutation relations. We need to quantize the theory, as suggested by Dirac, only on one of the LF hyperplanes. Consider, for example, the free scalar theory for which

$$\varphi(x^+, x^-) = \frac{1}{\sqrt{2\pi}} \int_{k^+>0}^{\infty} \frac{dk^+}{\sqrt{2k^+}} \Big[a(k^+) e^{-i(k^+x^- + k^-x^+)} + a^\dagger(k^+) e^{i(k^+x^- + k^-x^+)} \Big]$$

with $\left[a(k^+), a(l^+)^\dagger \right] = \delta(k^+ - l^+)$ etc. and $2k^+k^- = m^2$. We find easily

[10] See Nuovo Cimento A108(1995) 35.

$$\left[\varphi(x^+,x^-),\varphi(y^+,x^-)\right]=\frac{1}{2\pi}\int_{k^+>0}^{\infty}\frac{dk^+}{2k^+}\left[e^{ik^-(y^+-x^+)}-e^{-ik^-(y^+-x^+)}\right].$$

We may change the integration variable to k^- by making use of $k^-dk^+ + k^+dk^- = 0$ and employ the integral representation $\epsilon(x)=(i/\pi)\mathcal{P}\int_{-\infty}^{\infty}(d\lambda/\lambda)\,e^{(-i\lambda x)}$ to arrive at the equal-x^- commutator

$$\left[\varphi(x^+,x^-),\varphi(y^+,x^-)\right]=-\frac{i}{4}\epsilon(x^+-y^+)$$

The above field expansion on the LF, in contrast to the equal-time case, does not involve the mass parameter m and the same result follows in the massless case also if we assume that $k^+=l^+$ implies $k^-=l^-$. Defining the right and the left movers by $\varphi(0,x^-)\equiv\varphi^R(x^-)$, and $\varphi(x^+,0)\equiv\varphi^L(x^+)$ we obtain $\left[\varphi^R(x^-),\varphi^R(y^-)\right]=(-i/4)\epsilon(x^--y^-)$ while $\left[\varphi^L(x^+),\varphi^L(y^+)\right]=(-i/4)\epsilon(x^+-y^+)$. The symmetry under discussion is responsible for an appreciable simplification found in the recent study of the gauge theory SM and the CSM on the LF discussed in the next Sec..

3- Bosonized CSM on the LF. Absence of θ-vacua [19]

The Lagrangian density of the chiral QED_2 or CSM model is

$$\mathcal{L}=-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\bar{\psi}_R\,i\gamma^\mu\partial_\mu\psi_R+\bar{\psi}_L\,\gamma^\mu(i\partial_\mu+2e\sqrt{\pi}A_\mu)\psi_L, \tag{5}$$

where[11]. $\psi=\psi_R+\psi_L$ is a two-component spinor field and A_μ is the abelian gauge field, $\gamma_5\psi_L=-\psi_L$, and $\gamma_5\psi_R=\psi_R$. The classical Lagrangian is invariant under the local $U(1)$ gauge transformations $A_\mu\rightarrow A_\mu+\partial_\mu\alpha/(2\sqrt{\pi}e)$, $\psi\rightarrow[P_R+e^{i\alpha}P_L]\psi$ and under the global $U(1)_5$ chiral transformations $\psi\rightarrow exp(i\gamma_5\alpha)\,\psi$.

The model under study can be solved completely using the technique of bosonization. The latter consists in the replacement of a known system of fermions with a theory of bosons which has a completely equivalent physical content, including, for example, identical spectra, the same current commutation relations and the energy-momentum tensor when expressed in terms of the currents. The bosonized version is convenient to study the vacuum structure and it was shown [17] to be given by

$$S=\int d^2x\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}\partial_\mu\phi\partial^\mu\phi+eA_\nu(\eta^{\mu\nu}-\epsilon^{\mu\nu})\partial_\mu\phi+\frac{1}{2}ae^2A_\mu A^\mu\right] \tag{6}$$

Here the explicit mass term for the gauge field parametrized by the constant parameter a represents a regularization ambiguity and the breakdown of $U(1)$ gauge symmetry. The action may be derived by the functional integral method or by the canonical quantization.

We make the *separation*: $\phi(\tau,x^-)=\omega(\tau)+\varphi(\tau,x^-)$ and follow it through the application of the Dirac method as done with the SSB above. On the other hand in the bosonized SM on the LF we recall [8] that $\omega(\tau)$ turned out to be q-number operator and its eigenvalues were shown [8] to characterize the *condensate* or θ-vacua [18], which

[11] Here $\gamma^0=\sigma_1$, $\gamma^1=i\sigma_2$, $\gamma_5=-\sigma_3$, $x^\mu:(x^+\equiv\tau,x^-\equiv x)$ with $\sqrt{2}x^\pm=\sqrt{2}x_\mp=(x^0\pm x^1)$, $A^\pm=A_\mp=(A^0\pm A^1)/\sqrt{2}$, $\psi_{L,R}=P_{L,R}\,\psi$, $P_L=(1-\gamma_5)/2$, $P_R=(1+\gamma_5)/2$, $\bar{\psi}=\psi^\dagger\gamma^0$.

were shown also to emerge naturally with a continuum normalization in contrast to what found in the conventional equal-time treatment. We remind [8] also that the *chiral transformation* is defined by: $\omega \to \omega + const.$, $\varphi \to \varphi$, and $A_\mu \to A_\mu$. This ensures that the *boundary conditions* on the φ are kept unaltered under such transformations and our mathematical framework may be considered *well posed*, before we proceed to build the canonical Hamiltonian framework.

Written explicitly the action takes the following form on the LF

$$S = \int d^2x \left[\dot{\varphi}\varphi' + \frac{1}{2}(\dot{A}_- - A'_+)^2 + ae^2[A_+ + \frac{2}{ae}(\dot{\omega} + \dot{\varphi})]A_-\right] \tag{7}$$

where an overdot (a prime) indicates the partial derivative with respect to τ (x). In order to suppress the finite volume effects we work in the *continuum formulation* and require, based on physical considerations, that the fields satisfy the boundary conditions needed for the existence of their Fourier transforms in the spatial variable x^-.

We note now that A_+ appears in the action as an *auxiliary* field, without a kinetic term. It is clear that the condensate variable may thus be subtracted out from the theory using the frequently adopted procedure of *field redefinition* [20] on it: $A_+ \to A_+ - 2\dot{\omega}/(ae)$, obtaining thereby

$$\mathcal{L}_{CSM} = \dot{\varphi}\varphi' + \frac{1}{2}(\dot{A}_- - A'_+)^2 + 2e\dot{\varphi}A_- + ae^2A_+A_-, \tag{8}$$

which signals the emergence of a *different structure* of the Hilbert space compared to that of the SM where we had instead [8]

$$L = \int dx^- \left[\dot{\varphi}\varphi' + \frac{1}{2}(\dot{A}_- - A'_+)^2 - (\frac{e}{\sqrt{\pi}})(A_+\varphi' - A_-\dot{\varphi})\right] + (\frac{e}{\sqrt{\pi}})\dot{\omega}h(\tau) \tag{9}$$

with $h(\tau) = \int dx^- A_-(\tau, x^-)$, the zero mode associated with the gauge field A_-. We recall that the *condensate* or θ-vacua in SM emerged due to the presence in the theory of *three* linearly independent operators: the condensate ω, its canonically conjugate $h(\tau)$ and φ with the vanishing commutator with the other two while the H^{lf} contained in it only the field φ. The Hilbert space could be described in two fashions. Selecting φ abd h as forming the complete set of operators led to the chiral vacua while φ together with ω led to the description in terms of the condensate or θ-vacua.

The *Lagrange equations in the CSM* follow to be

$$\begin{aligned} \partial_+\partial_-\varphi &= -e\partial_+A_-, \\ \partial_+\partial_+A_- - \partial_+\partial_-A_+ &= ae^2A_+ + 2e\partial_+\varphi, \\ \partial_-\partial_-A_+ - \partial_+\partial_-A_- &= ae^2A_-. \end{aligned} \tag{10}$$

and for $a \neq 1$ they lead to:

$$\begin{aligned} \Box G(\tau, x) &= 0 \\ \left[\Box + \frac{e^2a^2}{(a-1)}\right] E(\tau, x) &= 0, \end{aligned} \tag{11}$$

where $E = (\partial_+ A_- - \partial_- A_+)$ and $G = (E - ae\varphi)$. Both the massive and massless scalar excitations are present in the theory and the tachyons would be absent in the specrtum if $a > 1$; the case considered in this paper. We will confirm in the Hamiltonian framework below that the E and G represent, in fact, the two independent field operators on the LF phase space.

The *Dirac procedure* [7] as applied to the very simple action of the CSM is straightforward. The canonical momenta are $\pi^+ \approx 0, \pi^- \equiv E = \dot{A}_- - A'_+, \pi_\varphi = \varphi' + 2eA_-$ which result in two primary weak constraints $\pi^+ \approx 0$ and $\Omega_1 \equiv (\pi_\varphi - \varphi' - 2eA_-) \approx 0$. A secondary constraint $\Omega_2 \equiv \partial_- E + ae^2 A_- \approx 0$ is shown to emerge when we require the τ independence (persistency) of $\pi^+ \approx 0$ employing the preliminary Hamiltonian

$$H' = H_c{}^{lf} + \int dx\ u_+ \pi^+ + \int dx\ u_1 \Omega_1, \tag{12}$$

where u_+ and u_1 are the Lagrange multiplier fields and $H_c{}^{lf}$ is the canonical Hamiltonian

$$H_c{}^{lf} = \int\ dx\ \left[\frac{1}{2}E^2 + EA'_+ - ae^2 A_+ A_-\right]. \tag{13}$$

and we assume initially the standard equal-τ Poisson brackets : $\{E^\mu(\tau, x^-), A_\nu(\tau, y^-)\} = -\delta^\mu_\nu \delta(x^- - y^-)$, $\{\pi_\varphi(\tau, x^-), \varphi(\tau, y^-)\} = -\delta(x^- - y^-)$ etc.. The persistency requirement for Ω_1 results in an equation for determining u_1. The procedure is repeated with the following extended Hamiltonian which includes in it also the secondary constraint

$$H_e{}^{lf} = H_c{}^{lf} + \int dx\ u_+ \pi^+ + \int dx\ u_1 \Omega_1 + \int dx\ u_2 \Omega_2. \tag{14}$$

No more secondary constraints are seen to arise; we are left with the persistency conditions which determine the multiplier fields u_1 and u_2 while u_+ remains undetermined. We also find[12] $(C)_{ij} = \{\Omega_i, \Omega_j\} = D_{ij}\ \ (-2\partial_x \delta(x-y))$ where $i, j = 1, 2$ and $D_{11} = 1,\ D_{22} = ae^2,\ D_{12} = D_{21} = -e$ and that π^+ has vanishing brackets with $\Omega_{1,2}$. The $\pi^+ \approx 0$ is first class weak constraint while Ω_1 and Ω_2, which does not depend on A_+ or π^+, are second class ones.

We go over from the Poisson bracket to the Dirac bracket $\{,\}_D$ constructed in relation to the pair, $\Omega_1 \approx 0$ and $\Omega_2 \approx 0$

$$\{f(x), g(y)\}_D = \{f(x), g(y)\} - \int\int dudv\ \{f(x), \Omega_i(u)\}(C^{-1}(u,v))_{ij}\{\Omega_j(v), g(y)\}. \tag{15}$$

Here C^{-1} is the inverse of C and we find $(C^{-1}(x,y))_{ij} = B_{ij}\ K(x,y)$ with $B_{11} = a/(a-1)$, $B_{22} = 1/[(a-1)e^2]$, $B_{12} = B_{21} = 1/[(a-1)e]$, and $K(x,y) = -\epsilon(x-y)/4$. Some of the Dirac brackets are $\{\varphi, \varphi\}_D = B_{11}\ K(x,y)$; $\{\varphi, E\}_D = eB_{11}\ K(x,y)$; $\{E, E\}_D = ae^2 B_{11}\ K(x,y)$; $\{\varphi, A_-\}_D = -B_{12}\ \delta(x-y)/2$; $\{A_-, E\}_D = B_{11}\ \delta(x-y)/2$; $\{A_-, A_-\}_D = B_{12}\partial_x\ \delta(x-y)/2$ and the only nonvanishing one involving A_+ or π^+ is $\{A_+, \pi^+\}_D = \delta(x-y)$.

[12] We make the convention that the first variable in an equal- τ bracket refers to the longitudinal coordinate $x^- \equiv x$ while the second one to $y^- \equiv y$ while τ is suppressed.

The eqns. of motion employ now the Dirac brackets and inside them, in view of their very construction [7], we may set $\Omega_1 = 0$ and $\Omega_2 = 0$ as strong relations. The Hamiltonian is therefore effectively given by H_e with the terms involving the multipliers u_1 and u_2 dropped. The multiplier u_+ is not determined since the constraint $\pi^+ \approx 0$ continues to be first class even when the above Dirac bracket is employed. The variables π_φ and A_- are then removed from the theory leaving behind φ, E, A_+, and π^+ as the remaining independent variables. The canonical Hamiltonian density reduces to $\mathcal{H}_c^{lf} = E^2/2 + \partial_-(A_+E)$ while $\dot{A}_+ = \{A_+, H_e^{lf}\}_D = u_+$. The surface term in the canonical LF Hamiltonian may be ignored if, say, $E(= F_{+-})$ vanishes at infinity. The variables π^+ and A_+ are then seen to describe a decoupled (from φ and E) free theory and we may hence drop these variables as well. The effective LF Hamiltonian thus takes the simple form

$$H_{CSM}^{lf} = \frac{1}{2}\int dx\ E^2, \tag{16}$$

which is to be contrasted with the one found in the conventional treatment [21, 22]. E and G (or E and φ) are now the independent variables on the phase space and the Lagrange equations are verified to be recovered for them, which assures us of the selfconsistency [7]. We stress that in our discussion we do *not* employ any gauge-fixing. The same result for the Hamiltonian could be alternatively obtained[13], however, if we did introduce the gauge-fixing constraint $A_+ \approx 0$ and made further modification on $\{,\}_D$ in order to implement $A_+ \approx 0, \pi^+ \approx 0$ as well. That it is accessible on the phase space to take care of the remaining first class constraint, but not in the bosonized Lagrangian, follows from the Hamiltons eqns. of motion. We recall [8] that in the SM φ, ω, and $\pi_\omega = (e/\sqrt{\pi})\int dx A_-$ were shown to be the independent operators and that the matter field φ appeared instead in the LF Hamiltonian.

The *canonical quantization* is peformed via the correspondence $i\{f,g\}_D \rightarrow [f,g]$ and we find the following equal-τ commutators

$$\begin{aligned}
[E(x), E(y)] &= iK(x,y)a^2e^2/(a-1), \\
[G(x), E(y)] &= 0, \\
[G(x), G(y)] &= ia^2e^2K(x,y).
\end{aligned} \tag{17}$$

For $a > 1$, when the tachyons are absent as seen from (6), these commutators are also physical and the independent field operators E and G generate the Hilbert space with a tensor product structure of the Fock spaces F_E and F_G of these fields with the positive definite metric.

We can make, in view of (12), the following LF momentum space expansions

$$\begin{aligned}
E(x,\tau) &= \frac{ae}{\sqrt{(a-1)}\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\ \frac{\theta(k)}{\sqrt{2k}}\left[d(k,\tau)e^{-ikx}\ +\ d^\dagger(k,\tau)e^{ikx}\right], \\
G(x,\tau) &= \frac{ae}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\ \frac{\theta(k)}{\sqrt{2k}}\left[g(k,\tau)e^{-ikx}\ +\ g^\dagger(k,\tau)e^{ikx}\right],
\end{aligned} \tag{18}$$

where the operators $(d, g, d^\dagger, g^\dagger)$ satisfy the canonical commutation relations of two independent harmonic oscillators; the well known set of Schwinger's bosonic oscillators, often

[13] A similar discussion is encountered also in the LF quantized Chern-Simons-Higgs system [23].

employed in the angular momentum theory. The expression for the Hamiltonian becomes

$$H^{lf}_{CSM} = \delta(0)\frac{a^2e^2}{2(a-1)} \int_{k>0}^{\infty} \frac{dk}{2k}\, N_d(k,\tau) \tag{19}$$

where we have dropped the infinite zero-point energy term and note that [3] $\left[d^\dagger(k,\tau), d(l,\tau)\right] = -\delta(k-l)$, $d^\dagger(k,\tau)d(k,\tau) = \delta(0)N_d(k,\tau)$ etc. with similar expressions for the independent g-oscillators. We verify that $[N_d(k,\tau), N_d(l,\tau)] = 0$, $[N_d(k,\tau), N_g(l,\tau)] = 0$, $\left[N_d(k,\tau), d^\dagger(k,\tau)\right] = d^\dagger(k,\tau)$ etc..

The Fock space can hence be built on a basis of eigenstates of the hermitian number operators N_d and N_g. The ground state of CSM is degenerate and described by $|0>= |E=0)\otimes|G\}$ and it carries vanishing LF energy in agreement with the conventional theory discusion [21, 22]. For a fixed k these states, $|E=0)\otimes({g^\dagger(k,\tau)}^n/\sqrt{n!})|0\}$, are labelled by the integers $n = 0, 1, 2, \cdots$. The θ-vacua are absent in the CSM. However, we recall [8] that in the SM the degenerate *chiral vacua* are also labelled by such integers. We remark also that on the LF we work in the Minkowski space and that in our discussion we do *not* make use of the Euclidean space theory action, where the (classical) vacuum configurations of the Euclidean theory gauge field, belonging to the distinct topological sectors, are useful, for example, in the functional integral quantization of the gauge theory.

Conclusions

The LF hyperplane is seen to be *equally appropriate* as the conventional one for quantizing field theory. The *front form* formulation is found to be quite transparent and the physical contents following from the *quantized* theory agree with those known in the conventional *instant form* treatment. Evidently, they should not depend on whether we employ the conventional or the LF coordinates to span the Minkowski space and study the temporal evolution of the quantum dynamical system in t or τ respectively.

We note that in our context the (LF) hyperplanes $x^{\pm} = 0$ define the characteristic surfaces of hyperbolic partial differential equation. It is known from their mathematical theory [12] that a solution exists if we specify the initial data on both of the hyperplanes. From the present discussion and the earlier works [2, 8] we conclude that it is sufficient in the *front form* treatment to choose one of the hyperplanes, as proposed by Dirac [1], for canonically quantizing the theory. The equal-τ *commutators of the field operators*, at a fixed initial LF-time, form now a part of the initial data instead and we deal with operator differential equations. The information on the commutators on the other characteristic hyperplane seems to be already contained [8] in the quantized theory and need not be specified separately. As a side comment, the well accepted notion that a classical model field theory must be upgraded first through quantization, before we confront it with the experimental data, finds here a theoretical re-affirmation.

The *physical* Hilbert space is obtained in a direct fashion in the LF quantized CSM and SM gauge theories, once the constraints are eliminated and the appreciably reduced set of independent operators on the LF phase space identified. CSM has in it both the massive and the massless scalar excitations while only the massive one appears in the SM. There are no condensate or θ-vacua in CSM but they both have degenerate vacuum structure. In the conventional treatment [18] an extended phase space is employed and suitable constraints are required to be imposed in order to define the *physical* Hilbert space which would then lead to the description of the physical vacuum state. The functional integral method together with the LF quantization may be an efficient tool for handling the nonperturbative calculations.

A discussion parallel to the one given here can also be made in the *front form* theory of the gauge invariant formulation [22] of the CSM. In an earlier work [24], where the BRST-BFV functional integral quantization was employed, it was demonstrated that this formulation and the gauge noninvariant one in fact lead to the same effective action. Also the BRS-BFT quantization method proposed [13] recently can be extended to the *front form* theory as illustrated in the Appendix D for the CSM on the LF and where different equivalent actions are obtained following the method.

Appendix A: Poincaré Generators on the LF

The Poincaré generators in coordinate system (x^0, x^1, x^2, x^3), satisfy $[M_{\mu\nu}, P_\sigma] = -i(P_\mu g_{\nu\sigma} - P_\nu g_{\mu\sigma})$ and $[M_{\mu\nu}, M_{\rho\sigma}] = i(M_{\mu\rho} g_{\nu\sigma} + M_{\nu\sigma} g_{\mu\rho} - M_{\nu\rho} g_{\mu\sigma} - M_{\mu\sigma} g_{\nu\rho})$ where the metric is $g_{\mu\nu} = diag\,(1, -1, -1, -1)$, $\mu = (0, 1, 2, 3)$ and we take $\epsilon_{0123} = \epsilon_{-+12} = 1$. If we define $J_i = -(1/2)\epsilon_{ikl} M^{kl}$ and $K_i = M_{0i}$, where $i, j, k, l = 1, 2, 3$, we find $[J_i, F_j] = i\epsilon_{ijk} F_k$ for $F_l = J_l, P_l$ or K_l while $[K_i, K_j] = -i\epsilon_{ijk} J_k$, $[K_i, P_l] = -iP_0 g_{il}$, $[K_i, P_0] = iP_i$, and $[J_i, P_0] = 0$.

The LF generators are P_+, P_-, P_1, P_2, $M_{12} = -J_3$, $M_{+-} = -K_3$, $M_{1-} = -(K_1 +$

$J_2)/\sqrt{2} \equiv -B_1$, $M_{2-} = -(K_2 - J_1)/\sqrt{2} \equiv -B_2$, $M_{1+} = -(K_1 - J_2)/\sqrt{2} \equiv -S_1$ and $M_{2+} = -(K_2 + J_1)/\sqrt{2} \equiv -S_2$. We find $[B_1, B_2] = 0$, $[B_a, J_3] = -i\epsilon_{ab}B_b$, $[B_a, K_3] = iB_a$, $[J_3, K_3] = 0$, $[S_1, S_2] = 0$, $[S_a, J_3] = -i\epsilon_{ab}S_b$, $[S_a, K_3] = -iS_a$ where $a, b = 1, 2$ and $\epsilon_{12} = -\epsilon_{21} = 1$. Also $[B_1, P_1] = [B_2, P_2] = iP^+$, $[B_1, P_2] = [B_2, P_1] = 0$, $[B_a, P^-] = iP_a$, $[B_a, P^+] = 0$, $[S_1, P_1] = [S_2, P_2] = iP^-$, $[S_1, P_2] = [S_2, P_1] = 0$, $[S_a, P^+] = iP_a$, $[S_a, P^-] = 0$, $[B_1, S_2] = -[B_2, S_2] = -iJ_3$, $[B_1, S_1] = [B_2, S_2] = -iK_3$. For $P_\mu = i\partial_\mu$, and $M_{\mu\nu} \to L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$ we find $B_a = (x^+P^a - x^aP^+)$, $S_a = (x^-P^a - x^aP^-)$, $K_3 = (x^-P^+ - x^+P^-)$ and $J_3 = (x^1P^2 - x^2P^1)$. Under the conventional *parity* operation $\mathcal{P}$: $(x^\pm \leftrightarrow x^\mp, x^{1,2} \to -x^{1,2})$ and $(p^\pm \leftrightarrow p^\mp, p^{1,2} \to -p^{1,2})$, we find $\vec{J} \to \vec{J}$, $\vec{K} \to -\vec{K}$, $B_a \to -S_a$ etc.. The six generators P_l, M_{kl} leave $x^0 = 0$ hyperplane invariant and are called *kinematical* while the remaining P_0, M_{0k} the *dynamical* ones. On the LF there are *seven* kinematical generators : $P^+, P^1, P^2, B_1, B_2, J_3$ and K_3 which leave the LF hyperplane, $x^0 + x^3 = 0$, invariant and the three *dynamical* ones S_1, S_2 and P^- form a mutually commuting set. The K_3 which was dynamical becomes now a kinematical; it generates scale transformations of the LF components of x^μ, P^μ and $M^{\mu\nu}$. We note that each of the set $\{B_1, B_2, J_3\}$ and $\{S_1, S_2, J_3\}$ generates an $E_2 \simeq SO(2) \otimes T_2$ algebra; this will be shown below to be relevant for defining the *spin* for massless particle. Including K_3 in each set we find two subalgebras each with four elements. Some useful identities are $e^{i\omega K_3}\, P^\pm\, e^{-i\omega K_3} = e^{\pm\omega}\, P^\pm$, $e^{i\omega K_3}\, P^\perp\, e^{-i\omega K_3} = P^\perp$, $e^{i\bar{v}.\bar{B}}\, P^-\, e^{-i\bar{v}.\bar{B}} = P^- + \bar{v}.\bar{P} + \frac{1}{2}\bar{v}^2 P^+$, $e^{i\bar{v}.\bar{B}}\, P^+\, e^{-i\bar{v}.\bar{B}} = P^+$, $e^{i\bar{v}.\bar{B}}\, P^\perp\, e^{-i\bar{v}.\bar{B}} = P^\perp + v^\perp P^+$, $e^{i\bar{u}.\bar{S}}\, P^+\, e^{-i\bar{u}.\bar{S}} = P^+ + \bar{u}.\bar{P} + \frac{1}{2}\bar{u}^2 P^-$, $e^{i\bar{u}.\bar{S}}\, P^-\, e^{-i\bar{u}.\bar{S}} = P^-$, $e^{i\bar{u}.\bar{S}}\, P^\perp\, e^{-i\bar{u}.\bar{S}} = P^\perp + u^\perp P^-$ where $P^\perp \equiv \bar{P} = (P^1, P^2)$, $v^\perp \equiv \bar{v} = (v_1, v_2)$ and $(v^\perp.P^\perp) \equiv (\bar{v}.\bar{P}) = v_1P^1 + v_2P^2$ etc. Analogous expressions with P^μ replaced by X^μ can be obtained if we use $[P^\mu, X_\nu] \equiv [i\partial^\mu, x_\nu] = i\delta^\mu_\nu$.

Appendix B: LF Spin Operator. Hadrons in LF Fock Basis

The Casimir generators of the Poincaré group are : $P^2 \equiv P^\mu P_\mu$ and W^2, where $W_\mu = (-1/2)\epsilon_{\lambda\rho\nu\mu}M^{\lambda\rho}P^\nu$ defines the Pauli-Lubanski pseudovector. It follows from $[W_\mu, W_\nu] = i\epsilon_{\mu\nu\lambda\rho}W^\lambda P^\rho$, $[W_\mu, P_\rho] = 0$ and $W.P = 0$ that in a representation charactarized by particular eigenvalues of the two Casimir operators we may simultaneously diagonalize P^μ along with just one component of W^μ. We have $W^+ = -[J_3P^+ + B_1P^2 - B_2P^1]$, $W^- = J_3P^- + S_1P^2 - S_2P^1$, $W^1 = K_3P^2 + B_2P^- - S_2P^+$, and $W^2 = -[K_3P^1 + B_1P^- - S_1P^+]$ and it shows that W^+ *has a special place* since it contains only the kinematical generators [8]. On the LF we define $\mathcal{J}_3 = -W^+/P^+$ as the *spin operator*. It may be shown to commute with P_μ, B_1, B_2, J_3, and K_3. For $m \neq 0$ we may use the parametrizations $p^\mu : (p^- = (m^2 + {p^\perp}^2)/(2p^+), p^+ = (m/\sqrt{2})e^\omega, p^1 = -v_1p^+, p^2 = -v_2p^+)$ and $\tilde{p}^\mu : (1, 1, 0, 0)(m/\sqrt{2})$ in the rest frame. We have $P^2(p) = m^2 I$ and $W(p)^2 = W(\tilde{p})^2 = -m^2[J_1^2 + J_2^2 + J_3^2] = -m^2 s(s+1)I$ where s assumes half-integer values. Starting from the rest state $|\tilde{p}; m, s, \lambda, ..\rangle$ with $J_3\, |\tilde{p}; m, s, \lambda, ..\rangle = \lambda\, |\tilde{p}; m, s, \lambda, ..\rangle$ we may build an arbitrary eigenstate of $P^+, P^\perp, \mathcal{J}_3$ (and P^-) on the LF by

$$|p^+, p^\perp; m, s, \lambda, ..\rangle = e^{i(\bar{v}.\bar{B})}e^{-i\omega K_3}|\tilde{p}; m, s, \lambda, ..\rangle$$

If we make use of the following *identity* [2]

$$\mathcal{J}_3(p) = J_3 + v_1B_2 - v_2B_1 = e^{i(\bar{v}.\bar{B})}\, J_3\, e^{-i(\bar{v}.\bar{B})}$$

we find $\mathcal{J}_3\,|p^+,p^\perp;m,s,\lambda,..\rangle = \lambda\,|p^+,p^\perp;m,s,\lambda,..\rangle$. Introducing also the operators $\mathcal{J}_a = -(\mathcal{J}_3 P^a + W^a)/\sqrt{P^\mu P_\mu}$, $a=1,2$, which do, however, contain dynamical generators, we verify that $[\mathcal{J}_i,\mathcal{J}_j] = i\epsilon_{ijk}\mathcal{J}_k$.

For $m=0$ case when $p^+\neq 0$ a convenient parametrization is $p^\mu : (p^- = p^+ {v^\perp}^2/2,\ p^+, p^1 = -v_1 p^+, p^2 = -v_2 p^+)$ and $\tilde{p} : (0, p^+, 0^\perp)$. We have $W^2(\tilde{p}) = -(S_1^2 + S_2^2){p^+}^2$ and $[W_1,W_2](\tilde{p}) = 0$, $[W^+,W_1](\tilde{p}) = -ip^+W_2(\tilde{p})$, $[W^+,W_2](\tilde{p}) = ip^+W_1(\tilde{p})$ showing that W_1, W_2 and W^+ generate the algebra $SO(2)\otimes T_2$. The eigenvalues of W^2 are hence not quantized and they vary continuously. This is contrary to the experience so we impose that the physical states satisfy in addition $W_{1,2}|\,\tilde{p};\, m=0,..\rangle = 0$. Hence $W_\mu = -\lambda P_\mu$ and the invariant parameter λ is taken to define as the *spin* of the massless particle. ¿From $-W^+(\tilde{p})/\tilde{p}^+ = J_3$ we conclude that λ assumes half-integer values as well. We note that $W^\mu W_\mu = \lambda^2 P^\mu P_\mu = 0$ and that on the LF the definition of the spin operator appears unified for massless and massive particles. A parallel discussion based on $p^-\neq 0$ may also be given.

As an illustration consider the three particle state on the LF with the total eigenvalues p^+, λ and $p^\perp$. In the *standard frame* with $p^\perp = 0$ it may be written as $(|x_1p^+, k_1^\perp;\lambda_1\rangle|x_2p^+, k_2^\perp;\lambda_2\rangle|x_3p^+, k_3^\perp;\lambda_3\rangle\ )$ with $\sum_{i=1}^3 x_i = 1$, $\sum_{i=1}^3 k_i^\perp = 0$, and $\lambda = \sum_{i=1}^3 \lambda_i$. Aplying $e^{-i(\bar{p}.\bar{B})/p^+}$ on it we obtain $(|x_1p^+, k_1^\perp + x_1p^\perp;\lambda_1\rangle|x_2p^+, k_2^\perp + x_2p^\perp;\lambda_2\rangle|x_3p^+, k_3^\perp + x_3p^\perp;\lambda_3\rangle\ )$ now with $p^\perp\neq 0$. The x_i and $k_i^\perp$ indicate relative (invariant) parameters and do not depend upon the reference frame. The x_i is the fraction of the total longitudinal momentum carried by the i^{th} particle while $k_i^\perp$ its transverse momentum. The state of a pion with momentum $(p^+,p^\perp)$, for example, may be expressed as an expansion over the LF Fock states constituted by the different number of partons [5]

$$|\pi : p^+, p^\perp\rangle = \sum_{n,\lambda}\int \bar{\Pi}_i \frac{dx_i d^2{k^\perp}_i}{\sqrt{x_i}\,16\pi^3}|n :\ x_ip^+, x_ip^\perp + {k^\perp}_i, \lambda_i\rangle\ \psi_{n/\pi}(x_1, {k^\perp}_1, \lambda_1; x_2, ...)$$

where the summation is over all the Fock states n and spin projections λ_i, with $\bar{\Pi}_i dx_i = \Pi_i dx_i\ \delta(\sum x_i - 1)$, and $\bar{\Pi}_i d^2k_i^\perp = \Pi_i dk_i^\perp\ \delta^2(\sum k_i^\perp)$. The wave function of the parton $\psi_{n/\pi}(x,k^\perp)$ indicates the probability amplitude for finding inside the pion the partons in the Fock state n carrying the 3-momenta $(x_ip^+, x_ip^\perp + k_i^\perp)$. The Fock state of the pion is also off the energy shell : $\sum k_i^- > p^-$.

The *discrete symmetry* transformations may also be defined on the LF Fock states [5, 8] For example, under the conventional parity $\mathcal{P}$ the spin operator $\mathcal{J}_3$ is not left invariant. We may rectify this by defining *LF Parity operation* by $\mathcal{P}^{lf} = e^{-i\pi J_1}\mathcal{P}$. We find then $B_1 \to -B_1, B_2 \to B_2, P^\pm \to P^\pm, P^1 \to -P^1, P^2 \to P^2$ etc. such that $\mathcal{P}^{lf}|p^+,p^\perp;m,s,\lambda,..\rangle \simeq |p^+,-p^1,p^2;m,s,-\lambda,..\rangle$. Similar considerations apply for charge conjugation and time inversion. For example, it is straightforward to construct the free *LF Dirac spinor* $\chi(p) = [\sqrt{2}p^+\Lambda^+ + (m-\gamma^a p^a)\,\Lambda^-]\tilde{\chi}/\sqrt{\sqrt{2}p^+ m}$ which is also an eigenstate of $\mathcal{J}_3$ with eigenvalues $\pm 1/2$. Here $\Lambda^\pm = \gamma^0\gamma^\pm/\sqrt{2} = \gamma^\mp\gamma^\pm/2 = (\Lambda^\pm)^\dagger$, $(\Lambda^\pm)^2 = \Lambda^\pm$, and $\chi(\tilde{p}) \equiv \tilde{\chi}$ with $\gamma^0\tilde{\chi} = \tilde{\chi}$. The conventional (equal-time) spinor can also be constructed by the procedure analogous to that followed for the LF spinor and it has the well known form $\chi_{con}(p) = (m+\gamma.p)\tilde{\chi}/\sqrt{2m(p^0+m)}$. Under the conventional parity operation $\mathcal{P} : \chi'(p') = c\gamma^0\chi(p)$ (since we must require $\gamma^\mu = {L^\mu}_\nu\, S(L)\gamma^\nu S^{-1}(L)$, etc.). We find $\chi'(p) = c[\sqrt{2}p^-\Lambda^- +$

$(m - \gamma^a p^a)\,\Lambda^+]\,\tilde{\chi}/\sqrt{\sqrt{2}p^- m}$. For $p \neq \tilde{p}$ it is not proportional to $\chi(p)$ in contrast to the result in the case of the usual spinor where $\gamma^0\chi_{con}(p^0, -\vec{p}) = \chi_{con}(p)$ for $E > 0$ (and $\gamma^0\eta_{con}(p^0, -\vec{p}) = -\eta_{con}(p)$ for $E < 0$). However, applying parity operator twice we do show $\chi''(p) = c^2\chi(p)$ hence leading to the usual result $c^2 = \pm 1$. The LF parity operator over spin 1/2 Dirac spinor is $\mathcal{P}^{lf} = c\,(2J_1)\,\gamma^0$ and the corresponding transform of χ is shown to be an eigenstate of $\mathcal{J}_3$.

Appendix C: SSB Mechanism. Continuum Limit of Discretized LF Quantized Theory. Nonlocality of LF Hamiltonian.

In order to keep the discussion[14] simple we would assume ω to be a consant background field. so that $\mathcal{L} = \dot{\varphi}\varphi' - V(\phi)$. Dirac procedure is applied now to construct Hamiltonian field theory which may be quantized. We may avoid using distribuitions if we restrict x to a finite interval from $-R/2$ to $R/2$. The *limit to the continuum ($R \to \infty$)*, however, must [25] be taken later to remove the spurious finite volume effects. Expanding φ by Fourier series we obtain $\phi(\tau, x) \equiv \omega + \varphi(\tau, x) = \omega + \frac{1}{\sqrt{R}}q_0(\tau) + \frac{1}{\sqrt{R}}\sum'_{n\neq 0}\; q_n(\tau)\; e^{-ik_n x}$ where $k_n = n(2\pi/R)$, $n = 0, \pm 1, \pm 2, \ldots$ and the *discretized theory* Lagrangian becomes $i\sum_n k_n\, q_{-n}\,\dot{q}_n - \int dx\; V(\phi)$. The momenta conjugate to q_n are $p_n = ik_n q_{-n}$ and the canonical LF Hamiltonian is found to be $\int\, dx\, V(\omega + \varphi(\tau, x))$. The primary constraints are thus $p_0 \approx 0$ and $\Phi_n \equiv p_n - ik_n q_{-n} \approx 0$ for $n \neq 0$. We follow the *standard* Dirac procedure [5] and find three weak constraints $p_0 \approx 0$, $\beta \equiv \int dx\, V'(\phi) \approx 0$, and $\Phi_n \approx 0$ for $n \neq 0$ on the phase space and they are shown to be second class. We find for $n \neq 0$ and $m \neq 0$: $\{\Phi_n, p_0\} = 0$, $\{\Phi_n, \Phi_m\} = -2ik_n\delta_{m+n,0}$, $\{\Phi_n, \beta\} = \{p_n, \beta\} = -(1/\sqrt{R})\int dx\; [\,V''(\phi) - V''([\omega + q_0]/\sqrt{R})\,]\, e^{-ik_n x} \equiv -\alpha_n/\sqrt{R}$, $\{p_0, p_0\} = \{\beta, \beta\} = 0$, $\{p_0, \beta\} = -(1/\sqrt{R}) \int dx\;\, V''(\phi) \equiv -\alpha/\sqrt{R}$. Implement first the pair of constraints $p_0 \approx 0$, $\beta \approx 0$ by modifying the Poisson brackets to the star bracket $\{\}^*$ defined by $\{f, g\}^* = \{f, g\} - [\{f, p_0\}\; \{\beta, g\} - (p_0 \leftrightarrow \beta)](\alpha/\sqrt{R})^{-1}$. We may then set $p_0 = 0$ and $\beta = 0$ as strong equalities. We find by inspection that the brackets $\{\}^*$ of the remaining variables coincide with the standard Poisson brackets except for the ones involving q_0 and p_n $(n \neq 0)$: $\{q_0, p_n\}^* = \{q_0, \Phi_n\}^* = -(\alpha^{-1}\alpha_n)$. For example, if $V(\phi) = (\lambda/4)\,(\phi^2 - m^2/\lambda)^2$, $\lambda \geq 0, m \neq 0$ we find $\{q_0, p_n\}^*\; [\{\, 3\lambda\,(\omega + q_0/\sqrt{R})^2 - m^2\,\}R\, + 6\lambda(\omega + q_0/\sqrt{R}) \int\, dx\varphi\, + 3\lambda\, \int\, dx\,\varphi^2\,] = -\,3\lambda\,[\,2(\omega + q_0/\sqrt{R})\,\sqrt{R}q_{-n} + \int\, dx\,\varphi^2\, e^{-ik_n x}\,]$.

Implement next the constraints $\Phi_n \approx 0$ with $n \neq 0$. We have $C_{nm} = \{\Phi_n, \Phi_m\}^* = -2ik_n\delta_{n+m,0}$ and its inverse is given by $C^{-1}{}_{nm} = (1/2ik_n)\delta_{n+m,0}$. The Dirac bracket which takes care of all the constraints is then given by

$$\{f, g\}_D = \{f, g\}^* - \sum\nolimits'_n \frac{1}{2ik_n}\{f,\, \Phi_n\}^*\, \{\Phi_{-n},\, g\}^*$$

where we may now in addition write $p_n = ik_n q_{-n}$. It is easily shown that $\{q_0, q_0\}_D = 0, \{q_0, p_n\}_D = \{q_0,\, ik_n q_{-n}\}_D = \frac{1}{2}\,\{q_0, p_n\}^*, \{q_n, p_m\}_D = \frac{1}{2}\delta_{nm}$.

The limit to the continuum, $R \to \infty$ is taken as usual: $\Delta = 2(\pi/R) \to dk$, $k_n = n\Delta \to k$, $\sqrt{R}q_{-n} \to lim_{R\to\infty} \int_{-R/2}^{R/2} dx\, \varphi(x)\, e^{ik_n x} \equiv \int_{-\infty}^{\infty} dx\, \varphi(x)\, e^{ikx} = \sqrt{2\pi}\tilde{\varphi}(k)$ for all n, $\sqrt{2\pi}\varphi(x) = \int_{-\infty}^{\infty} dk\, \tilde{\varphi}(k) e^{-ikx}$, and $(q_0/\sqrt{R}) \to 0$. ¿From $\{\sqrt{R}\, q_m, \sqrt{R}\, q_{-n}\}_D = R\,\delta_{nm}/(2ik_n)$ following from $\{q_n, p_m\}_D$ for $n, m \neq 0$ we derive, on using $R\delta_{nm} \to$

[14] see [15] and Nuovo Cimento A108 (1995) 35.

$\int_{-\infty}^{\infty} dx e^{i(k-k')x} = 2\pi\delta(k-k')$, that $\{\tilde{\varphi}(k), \tilde{\varphi}(-k')\}_D = \delta(k-k')/(2ik)$ where $k, k' \neq 0$. If we use the integral representation of the sgn function the well known LF Dirac bracket $\{\varphi(x,\tau), \varphi(y,\tau)\}_D = -\frac{1}{4}\epsilon(x-y)$ is obtained. The expressions of $\{q_0, p_n\}_D$ (or $\{q_0, \varphi'\}_D$) show that the DLCQ is harder to work with here[15]. The continuum limit of the *constraint eq.* $\beta = 0$ is

$$\omega(\lambda\omega^2 - m^2) + lim_{R\to\infty}\frac{1}{R}\int_{-R/2}^{R/2} dx\Big[(3\lambda\omega^2 - m^2)\varphi + \lambda(3\omega\varphi^2 + \varphi^3)\Big] = 0$$

while that for the LF Hamiltonian

$$P^- = \int dx\,\Big[\omega(\lambda\omega^2 - m^2)\varphi + \frac{1}{2}(3\lambda\omega^2 - m^2)\varphi^2 + \lambda\omega\varphi^3 + \frac{\lambda}{4}\varphi^4\Big]$$

These results follow immediately if we worked directly in the continuum formulation; we do have to handle generalized functions now. In the LF Hamiltonian theory we have an additional new ingredient in the form of the *constraint equation*. Elimination of ω using it would lead to a *nonlocal LF Hamiltonian* corresponding to the local one in the equal-time formulation. At the tree or classical level the integrals appearing in in the constraint eq. are convergent and when $R \to \infty$ it leads to $V'(\omega) = 0$. In equal-time theory this is essentially *added* to it as an external constraint based on physical considerations. In the renormalized theory [15] the constraint equation describes the high order quantum corrections to the tree level value of the condensate.

The quantization is performed via the correspondence $i\{f,g\}_D \to [f,g]$. Hence $\varphi(x,\tau) = (1/\sqrt{2\pi})\int dk\;\theta(k)\;[a(k,\tau)\,e^{-ikx} + a^\dagger(k,\tau)e^{ikx}]/(\sqrt{2k})$, were $a(k,\tau)$ and $a^\dagger(k,\tau)$ satisfy the canonical equal-τ commutation relations, $[a(k,\tau), a(k',\tau)^\dagger] = \delta(k-k')$ etc.. The vacuum state is defined by $a(k,\tau)|vac\rangle = 0$, $k > 0$ and the tree level description of the *SSB* is given as follows. The values of $\omega = \langle|\phi|\rangle_{vac}$ obtained from $V'(\omega) = 0$ the different vacua in the theory. Distinct Fock spaces corresponding to different values of ω are built as usual by applying the creation operators on the corresponding vacuum state. The $\omega = 0$ corresponds to a *symmetric phase* since the Hamiltonian is then symmetric under $\varphi \to -\varphi$. For $\omega \neq 0$ this symmetry is violated and the system is in a *broken or asymmetric phase.*

The extension to $3+1$ dimensions and to global continuous symmetry is straightforward[16]. Consider real scalar fields $\phi_a (a = 1, 2, ..N)$ which form an isovector of global internal symmetry group $O(N)$. We now write $\phi_a(x, x^\perp, \tau) = \omega_a + \varphi_a(x, x^\perp, \tau)$ and the Lagrangian density is $\mathcal{L} = [\dot{\varphi}_a\varphi'_a - (1/2)(\partial_i\varphi_a)(\partial_i\varphi_a) - V(\phi)]$, where $i = 1, 2$ indicate the transverse space directions. The Taylor series expansion of the constraint equations $\beta_a = 0$ gives a set of coupled equations $R\,V'_a(\omega) + V''_{ab}(\omega)\int dx\varphi_b + V'''_{abc}(\omega)\int dx\varphi_b\varphi_c/2 + ... = 0$. Its discussion at the tree level leads to the conventional theory results. The LF symmetry generators are found to be $G_\alpha(\tau) = -i\int d^2x^\perp dx\varphi'_c(t_\alpha)_{cd}\varphi_d$ $= \int d^2k^\perp\, dk\,\theta(k)\, a_c(k, k^\perp)^\dagger(t_\alpha)_{cd}a_d(k, k^\perp)$ where $\alpha, \beta = 1, 2, .., N(N-1)/2$, are the group indices, t_α are hermitian and antisymmetric generators of $O(N)$, and $a_c(k, k^\perp)^\dagger$

[15] However, we do require it if we use numerical computations on the computer.

[16] Nuovo Cimento A107 (1994) 549 and ref. [2, 15]

($a_c(k, k^{\perp})$) is creation (destruction) operator contained in the momentum space expansion of φ_c. These are to be contrasted with the generators in the equal-time theory, $Q_\alpha(x^0) = \int d^3x\, J^0 = -i\int d^3x(\partial_0\varphi_a)(t_\alpha)_{ab}\varphi_b - i(t_\alpha\omega)_a\int d^3x(d\varphi_a/dx_0)$. All the symmetry generators thus annihilate the LF vacuum and the SSB is now seen in the broken symmetry of the quantized theory Hamiltonian. The criteria for the counting of the number of Goldstone bosons on the LF is found to be the same as in the conventional theory. In contrast, the first term on the right hand side of $Q_\alpha(x^0)$ does annihilate the conventional theory vacuum but the second term gives now non-vanishing contributions for some of the (broken) generators. The symmetry of the conventional theory vacuum is thereby broken while the quantum Hamiltonian remains invariant. The physical content of SSB in the *instant form* and the *front form*, however, is the same though achieved by different descriptions. Alternative proofs on the LF, in two dimensions, can be given of the Coleman's theorem related to the absence of Goldstone bosons; we are unable to implement the second class constraints over the phase space.

Appendix D: BRS-BFT Quantization on the LF of the CSM[17]

Recently, it was shown [8] that the well known *condensate* or θ-vacua in the SM could be obtained by a straightforward quantization of the theory on the light-front (LF). The procedure adopted was the one proposed earlier in connection with the *front form* description of the SSB as described earlier in these lectures. The scalar field of the equivalent bosonized SM is separated, based on physical considerations, into the *dynamical bosonic condensate* and the quantum fluctuation fields. The Dirac procedure is then followed in order to construct the Hamiltonian formulation and the quantized theory. The θ-vacua were shown [8] to come out naturally along with their continuum normalization. It is then rather important to understand as to how and why the vacuum structure in the LF quantized CSM should come out to be quite different; as is known from the rather elaborate studies on CSM in the conventional framework. We could work with the standard Dirac method but the recently proposed BFT procedure which is elegant and avoids the computation of Dirac brackets. It would thus get tested on the LF as well and it also allows for constructing (new) effective Lagrangian theories.

We convert the two second class constraints of the bosonized CSM with $a > 1$ into first class constraints according to the BFT formalism. We obtain then the first class Hamiltonian from the canonical Hamiltonian and recover the DB using Poisson brackets in the extended phase space. The corresponding first class Lagrangian is then found by performing the momentum integrations in the generating functional.

(a) Conversion to First Class Constrained Dynamical System

The bosonized CSM model (for $a > 1$) is described by the action

$$S_{CSM} \;=\; \int d^2x\,\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + eA_\nu(\eta^{\mu\nu} - \epsilon^{\mu\nu})\partial_\mu\phi + \frac{1}{2}ae^2A_\mu A^\mu\right], \qquad (20)$$

[17]Contribuited paper LP-002, Session P 17, *International Symposium on Lepton-Photon Interactions-LP'97*, Desy, Hamburg, July 1997 (available as .ps file on the Desy database). Presented in a talk given at the *8th Workshop on Light-Cone QCD and Nonperturbative Hadronic Physics*, Lutsen, Minnesota, August 1997.

where a is a regularization ambiguity which enters when we calculate the fermionic determinant in the fermionic CSM. The action in the LF coordinates takes the form

$$S_{CSM} \;=\; \int d^2x^- \; \left[\frac{1}{2}(\partial_+ A_- - \partial_- A_+)^2 + \partial_-\phi\,\partial_+\phi + 2eA_-\partial_+\phi + ae^2 A_+ A_-\right], \qquad (21)$$

We now make the *separation*, in the scalar field (a generalized function) : $\phi(\tau, x^-) = \omega(\tau) + \varphi(\tau, x^-)$. The Lagrangian density then becomes

$$\mathcal{L} = \frac{1}{2}(\partial_+ A_- - \partial_- A_+)^2 + \partial_-\varphi\,\partial_+\varphi + ae^2[A_+ + \frac{2}{ae}(\partial_+\varphi + \partial_+\omega)]A_-, \qquad (22)$$

We note that the dynamical fields are A_- and φ while A_+ has no kinetic term. On making a redefinition of the (auxiliary) field A_+ we can recast the action on the LF in the following form

$$S_{CSM} \;=\; \int dx^- \; \left[\dot{\varphi}\varphi' + \frac{1}{2}(\dot{A}_- - A_+{}')^2 - 2e\dot{A}_-\varphi + ae^2 A_+ A_-\right], \qquad (23)$$

The canonical momenta are given by

$$\begin{aligned} \pi^+ &= 0, \\ \pi^- &= \dot{A}_- \;-\; A'_+ - 2e\varphi, \\ \pi_\varphi &= \varphi'. \end{aligned} \qquad (24)$$

We follow now the Dirac's standard procedure in order to build an Hamiltonian framework on the LF. The definition of the canonical momenta leads to two primary constraints

$$\pi^+ \approx 0, \qquad (25)$$

$$\Omega_1 \equiv (\pi_\varphi - \varphi') \approx 0 \qquad (26)$$

and we derive one secondary constraint

$$\Omega_2 \equiv \partial_-\pi^- + +2e\varphi' + ae^2 A_- \approx 0. \qquad (27)$$

This one follows when we require the τ independence (e.g., the persistency) of the primary constraint π^+ with respect to the preliminary Hamiltonian

$$H' = H_c{}^{l.f.} + \int dx\; u_+\pi^+ + \int dx\; u_1\Omega_1, \qquad (28)$$

where H_c is the canonical Hamiltonian

$$H_c{}^{l.f.} \;=\; \int dx \;\left[\; \frac{1}{2}(\pi^- + 2e\varphi)^2 + (\pi^- + 2e\varphi)A'_+ - ae^2 A_+ A_-\right], \qquad (29)$$

and we employ the standard equal-τ Poisson brackets. The u_+ and u_1 denote the Lagrange multiplier fields. The persistency requirement for Ω_1 give conditions to determine u_1. The Hamiltonian is next extended to include also the secondary constraint

$$H_e{}^{l.f.} = H_c{}^{l.f.} + \int dx\; u_+\pi^+ + \int dx\; u_1\Omega_1 + \int dx\; u_2\Omega_2 \qquad (30)$$

and the procedure is now repeated with respect to the extended Hamiltonian. For the case $a > 1$, no more secondary constraints are seen to arise and we are left only with the persistency conditions which determine the multipliers u_1 and u_2 while u_+ is left undetermined. We also find[18] $\{\Omega_i, \Omega_j\} = D_{ij} \ (-2\partial_x \delta(x-y))$ where $i, j = 1, 2$ and $D_{11} = 1$, $D_{22} = ae^2$, $D_{12} = D_{21} = -e$ and π^+ is shown to have vanishing brackets with $\Omega_{1,2}$. The $\pi^+ \approx 0$ constitutes a first class constraint on the phase space; it generates local transformations of A_+ which leave the H_e invariant, $\{\pi^+, H_e\} = \Omega_2 \approx 0$. The Ω_1, Ω_2 constitute a set of second class constraints and do not involve A_+ or π^+. It is very convenient, though not necessary, to add to the set of constraints on the phase space the (accessible) gauge fixing constraint $A_+ \approx 0$. It is evident from that such a gauge freedom is *not* available at the Lagrangian level. We will also implement (e.g., turn into strong equalities) the (trivial) pair of weak constraints $A_+ \approx 0$, $\pi^+ \approx 0$ by defining the Dirac brackets with respect to them. It is easy to see that for the other remaining dynamical variables the corresponding Dirac brackets coincide with the standard Poisson brackets. The variables A_+, π^+ are thus removed from the discussion, leaving behind a constrained dynamical system with the two second class constraints Ω_1, Ω_2 and the light-front Hamiltonian

$$H^{l.f.} = \frac{1}{2} \int dx \ (\pi^- + 2e\varphi)^2 + \int dx \ u_1 \Omega_1 + \int dx \ u_2 \Omega_2 \tag{31}$$

which will be now handled by the BFT procedure.

We introduce the following linear combinations $\top_i$, $i = 1, 2$, of the above constraints

$$\begin{aligned} \top_1 &= c_1(\Omega_1 + \frac{1}{M}\Omega_2) \\ \top_2 &= c_2(\Omega_1 - \frac{1}{M}\Omega_2) \end{aligned} \tag{32}$$

where $c_1 = 1/\sqrt{2(1 - e/M)}$, $c_2 = 1/\sqrt{2(1 + e/M)}$, $M^2 = ae^2$, and $a > 1$. They satisfy

$$\{\top_i, \top_j\} = \delta_{ij}(-2\partial_x \delta(x-y)) \tag{33}$$

and thus diagonalize the constraint algebra.

We now introduce new auxiliary fields Φ^i in order to convert the second class constraint $\top_i$ into first class ones in the extended phase space. Following BFT [13] we require these fields to satisfy

$$\begin{aligned} \{A^\mu(\text{or } \pi_\mu), \Phi^i\} &= 0, \quad \{\varphi(\text{or } \pi_\varphi), \Phi^i\} = 0, \\ \{\Phi^i(x), \Phi^j(y)\} &= \omega^{ij}(x,y) = -\omega^{ji}(y,x), \end{aligned} \tag{34}$$

where ω^{ij} is a constant and antisymmetric matrix. The strongly involutive modified constraints $\widetilde{\top}_i$ satisfying the abelian algebra

$$\{\widetilde{\top}_i, \widetilde{\top}_j\} = 0 \tag{35}$$

[18]We make the convention that the first variable in an equal- τ bracket refers to the longitudinal coordinate $x^- \equiv x$ while the second one to $y^- \equiv y$

as well as the boundary conditions, $\widetilde{\top}_i \mid_{\Phi^i=0} = \top_i$ are then postulated to take the form of the following expansion

$$\widetilde{\top}_i(A^\mu, \pi_\mu, \varphi, \pi_\varphi; \Phi^j) = \top_i + \sum_{n=1}^{\infty} \widetilde{\top}_i^{(n)}, \qquad \top_i^{(n)} \sim (\Phi^j)^n. \tag{36}$$

The first order correction terms in this infinite series are written as

$$\widetilde{\top}_i^{(1)}(x) = \int dy X_{ij}(x,y)\Phi^j(y). \tag{37}$$

The first class constraint algebra of $\widetilde{\top}_i$ then leads to the following condition:

$$\{\top_i, \top_j\} + \{\widetilde{\top}_i^{(1)}, \widetilde{\top}_i^{(1)}\} = 0 \tag{38}$$

or

$$(-2\partial_x\delta(x-y))\delta_{ij} + \int dw\ dz\ X_{ik}(x,w)\omega^{kl}(w,z)X_{jl}(y,z) = 0. \tag{39}$$

There is clearly some arbitrariness in the appropriate choice of ω^{ij} and X_{ij} which corresponds to the canonical transformation in the extended phase space. We can take without any loss of generality the simple solutions,

$$\begin{aligned} \omega^{ij}(x,y) &= -\delta^{ij}\epsilon(x-y) \\ X_{ij}(x,y) &= \delta_{ij}\partial_x\delta(x-y), \end{aligned} \tag{40}$$

Their inverses are easily shown to be

$$\begin{aligned} {\omega^{-1}}_{ij}(x,y) &= -\frac{1}{2}\delta_{ij}\partial_x\delta(x-y) \\ (X^{-1})^{ij}(x,y) &= \frac{1}{2}\delta^{ij}\epsilon(x-y), \end{aligned} \tag{41}$$

With the above choice, we find up to the first order

$$\begin{aligned} \widetilde{\top}_i &= \top_i + \widetilde{\top}_i^{(1)} \\ &= \top_i + \partial\Phi^i, \end{aligned} \tag{42}$$

and a strongly first class constraint algebra

$$\{\top_i + \widetilde{\top}_i^{(1)}, \top_j + \widetilde{\top}_j^{(1)}\} = 0. \tag{43}$$

The higher order correction terms (suppressing the integration operation)

$$\widetilde{\top}_i^{(n+1)} = -\frac{1}{n+2}\Phi^l {\omega^{-1}}_{lk}(X^{-1})^{kj}B_{ji}^{(n)} \qquad (n \geq 1) \tag{44}$$

with

$$B_{ji}^{(n)} \equiv \sum_{m=0}^{n}\{\widetilde{\top}_j^{(n-m)}, \widetilde{\top}_i^{(m)}\}_{(A,\pi,\varphi,\pi_\varphi)} + \sum_{m=0}^{n-2}\{\widetilde{\top}_j^{(n-m)}, \widetilde{\top}_i^{(m+2)}\}_{(\Phi)} \tag{45}$$

automatically vanish as a consequence of the proper choice of ω^{ij} made above. The Poisson brackets are to be computed here using the standard canonical definition for A_μ and φ as postulated above. We have now only the first class constraints in the extended phase space and in view of the proper choice only $\widetilde{\top}_i^{(1)}$ contributes in the infinite series above.

(b)- First Class Hamiltonian and Dirac Brackets

We next introduce modified ("gauge invariant") dynamical variables $\widetilde{F} \equiv (\widetilde{A}_\mu, \widetilde{\pi}^\mu, \widetilde{\varphi}, \widetilde{\pi}_\varphi)$ corresponding to $F \equiv (A_\mu, \pi^\mu, \varphi, \pi_\varphi)$ over the phase space by requiring the the following strong involution condition for $\widetilde{F}$ with the first class constraints in our extended phase space, viz,

$$\{\widetilde{\top}_i, \widetilde{F}\} = 0 \tag{46}$$

with

$$\widetilde{F}(A_\mu, \pi^\mu, \varphi, \pi_\varphi; \Phi^j) \;=\; F + \sum_{n=1}^{\infty} \widetilde{F}^{(n)}, \qquad \widetilde{F}^{(n)} \sim (\Phi^j)^n \tag{47}$$

and which satisfy the boundary conditions, $\widetilde{F}\mid_{\Phi^i=0} = F$.

The first order correction terms are easily shown to be given by

$$\widetilde{F}^{(1)}(x) = -\int du\, dv\, dz\;\; \Phi^j(u){\omega^{-1}}_{jk}(u,v){X^{-1}}^{kl}(v,z)\;\{\top_l(z), F(x)\}_{(A,\pi,\varphi,\pi_\varphi)}. \tag{48}$$

We find

$$\begin{aligned}
\widetilde{A}_-^{(1)} &= \frac{1}{2M}\partial(c_1\Phi^1 - c_2\Phi^2)\\
\widetilde{\pi}^{-(1)} &= \frac{M}{2}(c_1\Phi^1 - c_2\Phi^2)\\
\widetilde{\varphi}^{(1)} &= -\frac{1}{2}(c_1\Phi^1 + c_2\Phi^2),\\
\widetilde{\pi}_\varphi^{(1)} &= \frac{1}{2}\partial\left[c_1(1-\frac{2e}{M})\Phi^1 + c_2(1+\frac{2e}{M})\Phi^2\right]
\end{aligned} \tag{49}$$

where only the combinations $(c_1\Phi^1 \pm c_2\Phi^2)$ of the auxiliary fields are seen to occur. Furthermore, since the modified variables $\widetilde{F} = F + \widetilde{F}^{(1)} + ...$, up to the first order corrections, are found to be strongly involutive as a consequence of the proper choice made above, the higher order correction terms

$$\widetilde{F}^{(n+1)} \;=\; -\frac{1}{n+1}\Phi^j\omega_{jk}X^{kl}G_l^{(n)}, \tag{50}$$

with

$$G_l^{(n)} \;=\; \sum_{m=0}^{n}\{\top_i^{(n-m)}, \widetilde{F}^{(m)}\}_{(A,\pi,\phi,\pi_\phi)} + \sum_{m=0}^{n-2}\{\top_i^{(n-m)}, \widetilde{F}^{(m+2)}\}_{(\Phi)} + \{\top_i^{(n+1)}, \widetilde{F}^{(1)}\}_{(\Phi)} \tag{51}$$

again vanish. In principle we may follow similar procedure for any functional of the phase space variables; it may get, however, involved.

We make a side remark on the Dirac formulation for dealing with the systems with second class constraints by using the Dirac bracket (DB), rather than extending the phase space. In fact, the Poisson brackets of the modified (gauge invariant) variables $\widetilde{F}$ in the BFT formalism are related [13] to the DB, which implement the constraints $\top_i \approx 0$ in the problem under discussion, by the relation $\{f,g\}_D = \{\widetilde{f},\widetilde{g}\}\mid_{\Phi^i=0}$. In view of only the linear first order correction in CSM the computation of the right hand side is quite simple. We list some of the Dirac brackets

$$
\begin{aligned}
\{\pi^-,\pi^-\}_D &= \{\widetilde{\pi^-},\widetilde{\pi^-}\}|_{\Phi=0} \\
&= \{\widetilde{\pi^-}^{(1)},\widetilde{\pi^-}^{(1)}\} = \frac{a^2e^2}{(a-1)}(-\frac{1}{4}\epsilon(x-y)), \\
\{\varphi,\varphi\}_D &= \{\widetilde{\varphi},\widetilde{\varphi}\}|_{\Phi=0} \\
&= \{\widetilde{\varphi}^{(1)},\widetilde{\varphi}^{(1)}\} = \frac{a}{(a-1)}(-\frac{1}{4}\epsilon(x-y)) \\
\{\varphi,\pi^-\}_D &= \{\widetilde{\varphi}^{(1)},\widetilde{\pi^-}^{(1)}\} = \frac{ae}{(a-1)}(-\frac{1}{4}\epsilon(x-y)).
\end{aligned} \tag{52}
$$

The other ones follow on using the now strong relations $\Omega_1=\Omega_2=0$ with respect to $\{,\}_D$ and from $H^{l.f}$ it follows that the LF Hamiltonian reduces effectively to

$$
H_D^{l.f.} = \frac{1}{2}\int dx\,(\pi^- + 2e\varphi)^2. \tag{53}
$$

The first class LF Hamiltonian $\widetilde{H}$ which satisfies the boundary condition $\widetilde{H}\mid_{\Phi^i=0} = H_D^{l.f.}$ and is in strong involution with the constraints $\widetilde{\top}_i$, e.g., $\{\widetilde{\top}_i,\widetilde{H}\}=0$, may be constructed following the BT procedure or simply guessed for the CSM. It is given by

$$
\widetilde{H} = \frac{1}{2}\int dx(\widetilde{\pi}^- + 2e\widetilde{\varphi})^2 \tag{54}
$$

which is just the expression in of $H_D^{l.f.}$ with field variables F replaced by the $\widetilde{F}$ variables, which already commute with the constraints $\widetilde{T}_i$. We do also check that $\{\widetilde{H},\widetilde{H}\}=0$ and we may identify $\widetilde{H}$ with the BRS Hamiltonian. This completes the operatorial conversion of the original second class system with the Hamiltonian H_c and constraints Ω_i into the first class one with the Hamiltonian $\widetilde{H}$ and (abelian) constraints $\widetilde{T}_i$.

(c)- First Class Lagrangian

We consider now the partition function of the model in order to construct the Lagrangian corresonding to $\widetilde{H}$ in the canonical Hamiltonian formulation discussed above.

We start by representing each of the auxiliary field Φ^i by a pair of fields $\pi^i,\theta^i,\ i=1,2$ defined by

$$
\Phi^i = \frac{1}{2}\pi^i - \int du\ \ \epsilon(x-u)\ \theta^i(u) \tag{55}
$$

such that π^i, θ^i satisfy

$$\{\pi^i, \theta^j\} = -\delta^{ij}\delta(x-y) \quad etc., \tag{56}$$

e.g., the (standard Heisenberg type) canonical Poisson brackets.

The phase space partition function is given By the Faddeev formulae

$$Z = \int \mathcal{D}A_- \mathcal{D}\pi^- \mathcal{D}\varphi \mathcal{D}\pi_\varphi \mathcal{D}\theta^1 \mathcal{D}\pi^1 \mathcal{D}\theta^2 \mathcal{D}\pi^2 \prod_{i,j=1}^{2} \delta(\widetilde{\Upsilon}_i)\delta(\Gamma_j) \det \mid \{\widetilde{\Upsilon}_i, \Gamma_j\} \mid e^{iS}, \tag{57}$$

where

$$S = \int d^2x \left(\pi^- \dot{A}_- + \pi_\varphi \dot{\varphi} + \pi^1 \dot{\theta}^1 + \pi^2 \dot{\theta}^2 - \widetilde{\mathcal{H}}\right) \equiv \int d^2x\, \mathcal{L}, \tag{58}$$

with the Hamiltonian density $\widetilde{\mathcal{H}}$ corresponding to the Hamiltonian $\widetilde{H}$ which is now expressed in terms of (θ^i, π_i) rather than in terms of Φ^i. The gauge-fixing conditions Γ_i are chosen such that the determinants occurring in the functional measure are nonvanishing. Moreover, Γ_i may be taken to be independent of the momenta so that they correspond to the Faddeev-Popov type gauge conditions.

We will now verify in the *unitary gauge*, defined by the original second class constraints: $\Gamma_i \equiv \Omega_i = 0$, i=1,2 being employed in the partition function, do in fact lead to the original Lagrangian. We check that the determinants in the functional measure are non-vanishing and field independent while the product of delta functionals reduces to

$$\delta(\pi_\varphi - \varphi')\delta(\pi^{-\prime} + 2e\varphi' + M^2 A_-)\delta(\pi^{1\prime} - 4\theta^1)\delta(\pi^{2\prime} - 4\theta^2) \tag{59}$$

Since π_φ is absent from $\widetilde{H}$ we can perform functional integration over it using the first delta functional. The second delta functional is exponentiated as usual and we name the integration variable as A_+ for convenience. The functional integral over θ^1 and θ^2 are easily performed due to the presence of the delta functionals and it also reduces $\widetilde{\mathcal{H}}$ to $(\pi^- + 2e\varphi)^2/2$. The functional integrations over the then decoupled variables π^1 and π^2 give rise to constant factors which are absorbed in the normalization. The partition function in the unitary gauge thus becomes

$$Z = \int \mathcal{D}A_- \mathcal{D}\pi^- \mathcal{D}\varphi \mathcal{D}A_+ e^{iS}, \tag{60}$$

with

$$S = \int d^2x \left[\pi^- \dot{A}_- + \varphi' \dot{\varphi} + (\pi^{-\prime} + 2e\varphi' + M^2 A_-)A_+ - \frac{1}{2}(\pi^- + 2e\varphi)^2\right], \tag{61}$$

Performing the shift $\pi^- \to \pi^- - 2e\varphi$ and doing subsequently a Gaussian integral over π^- we obtain the original bosonized Lagrangian with ω eliminated by the field redefinition of A_+. It is interesting to recall that while constructing the LF Hamiltonian framework we eliminated the variable A_+ making use of the gauge freedom on the LF phase space and it gave rise to appreciable simplification. However, on going over to the first class Lagrangian formalism using the partition functional this variable reappears as it should, since the initial bosonized action is not gauge invariant due to the presence of the mass term for the gauge field. Making other acceptable choices for gauge-functions we can arrive at different effective Lagrangians for the system under consideration. It is interesting to recall that in the fermionic Lagrangian the right-handed component of the fermionic field describes a free field and only the left-handed one is gauged. field while only the left component is gauged. It is also clear from our discussion that $\widetilde{H}$ proposed above is not unique and we could modify it so that it still lead to the original Lagrangian in the unitary gauge. The corresponding first class Lagrangian would produce still other gauge-fixed effective Lagrangians. It will be interesting to study the models on the LF with more flavours and accompanying non-abelian gauge symmetry using the BFT-BFV formalism.

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