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Dimensional Hypotheses for the n-dimensional Electromagnetic and Gravitational Fine Structure Constants

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Abstract

We use Vaschy-Buckhingham Theorem as a systematic tool to build univocal n-dimensional extensions of the electric and gravitational fine structure constants and show that their ratio is dimensionally invariant. The results allow us to obtain the dimensionality of the non-compact space as three with a relative standard uncertainty of 1.08×10^{-13}.

Keywords: Vaschy-Buckingham Theorem; Dimensionality; Fundamental Constants

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Dimensional Analysis can be considered a cornerstone of the syntaxis of Physics. Its axiomatic foundations, now well established, are based on the structure of a finite multiplicative group. A fundamental result is the Π or Vaschy-Buckingham Theorem [1, 2] from which we can infer the existence of a definite number of Πs, i. e., independent adimensional quantities, associated to the physical processes under consideration, expressed as products of dimensional ones. Fundamental physical theories - Electrodynamics, Quantum Mechanics and General Relativity - involve fundamental physical constants: e, the electron charge, $\hbar$, the Planck constant, c, the velocity of light and G, the newtonian gravitational constant. In many applications of those theories, these constants appear in adimensional multiplicative combinations, such as $\alpha = e^2/\hbar c$, the fine structure constant for electromagnetic interactions introduced by Sommerfeld when he extended Bohr theory to include elliptical orbits and relativistic effects, in order to explain the observed splitting of the energy levels of hydrogen atoms, in four-dimensional spacetimes. Its intriguing numerical value, $\sim 1/137$, has motivated the until now vain search for its deduction from fundamental principles. The importance of α has inspired also the construction of analogous fine structure constants involving other interactions [3].

We shall now discuss the dimensionality of the (non-compact) space itself. The connection between the dimensionality of physical space and the mathematical structure of physical laws was first established by Ehrenfest [4, 5] who solved the n-dimensional Kepler problem showing that stable solutions are possible only for n = 2 or n = 3. He treated also the stability of the Bohr model of the atom. This approach was later explored by other authors [6, 7]. An alternative procedure to the stability arguments has been presented [8]. The idea of using nointeger dimensionalities and measuring ϵ, the deviation from the usual, or non-compact, three-dimensionality was first proposed, according to Jammer [9], by Zeilinger and Svozil [10] who analyzed the experimentally measured values of the anomalous magnetic moment of the electron. It has also been suggested that Mössbauer effect could be used to measure ϵ, in a Pound-Rebka device, by using the gravitational redshift [11].

Now, to define our problem we introduce and interpret Π numbers associated to interactions in n-dimensional non-compact spaces, using the M, L, T basis. By Vaschy-Buckingham Theorem, given i physical quantities a_i, with k independent dimensions, we can define $i - k$ adimensional Π numbers [12]

$$\Pi_j \;=\; \frac{a_{k+j}}{a_1^{p_{k+j}} ... a_k^{r_{k+j}}}. \tag{1}$$

This allows us to elect each a_{k+j} as a directive quantity according to the physical process under study. Directive quantities physically guide the construction of Π numbers and correspond to Buckingham's derived quantities [1]. In n-dimensional physical spaces, c, a ratio between space and time, and $\hbar$, related to unidimensional oscillators (the only possible unidimensional structure), maintain their dimensional formulas LT^{-1} and ML^2T^{-1}, respectively. This implies that their numerical values are independent of n. This important result will be used in the construction of Figure 1. From a similarity principle[1], we obtain the n-dimensional Laplace Equation whose solution yields interactions proportional to $1/L^{(n-1)}$. Then, if we preserve, also by a similarity principle, Newton's second law as second derivative of displacement, by the Dimensional Homogeneity Principle, a straightforward calculation shows that the n-dimensional newtonian gravitational and electromagnetic constants, G_n and e_n, have as dimensional formulas $L^nM^{-1}T^{-2}$ and ML^nT^{-2}.

[1]The similarity principle itself, which plays a fundamental role on these results, will be discussed in detail elsewhere.

Let us consider now e_n, $\hbar$, c, G_n and, as the gravitational "charge", the proton mass m_p. As we have a three-dimensional basis and five quantities, by the Vaschy-Buckingham Theorem, we can construct only two independent Π numbers. Interactions are always given by the product of their characteristic charges. This leads us necessarily to choose ${e_n}^2$ and m_p^2 as the directive quantities. Then, Eq. (1) gives us

$$\Pi_q = [\hbar]^{\epsilon_1} \; [G_n]^{\epsilon_2} \; [c]^{\epsilon_3} \; q^2, \tag{2}$$

where q represents the charges. The solutions of the respective systems are

$$\Pi_{e_n} = \left[\hbar^{2(2-n)} \; {G_n}^{3-n} \; c^{2(n-4)}\right]^{\frac{1}{n-1}} \; {e_n}^2, \tag{3}$$

$$\Pi_{m_p} = \left[\hbar^{(2-n)} \; G_n \; c^{(n-4)}\right]^{\frac{2}{n-1}} \; {m_p}^2. \tag{4}$$

Without the Vaschy-Buckingham Theorem, we could not have obtained these univocal solutions. The remarkable singularity at $n = 1$ appears also in the n-dimensional extension of the Planck scales [13]. Note that complex structures only can be built from interactions if open or closed orbits can exist. This is possible only for $n > 1$.

Projecting Π_{e_n} and Π_{m_p} in a space with three dimensions, i. e., for $n = 3$, we have

$$\Pi_{e_n\,(n=3)} = \frac{e^2}{\hbar\, c}, \qquad \Pi_{m_p\,(n=3)} = \frac{G\, {m_p}^2}{\hbar\, c}. \tag{5}$$

The above result shows that the Π numbers given by (3) and (4) are adimensional quantities obtained from a systematic use for interacting charges of Vaschy-Buckingham Theorem, satisfying Laplace Equation in n-dimensions, that reduces, respectively, to α and its gravitational analogue α_G for $n = 3$. It can be concluded that these Πs are in fact the generalized fine structure constants α_n and α_{Gn}. This interpretation and the fact that for $n \neq 3$ it is impossible to construct an adimensional quantity with only e_n, $\hbar$ and c suggest indeed that gravitation may play a relevant role in atomic structure for $n \neq 3$.

The first discussion of this problem was presented by Barrow and Tipler [14]. In their words "... *in N dimensions, the dimensionless constant of Nature is proportional to*

$$\hbar^{(2-n)} G^{\frac{3-n}{2}} c^{(n-4)} e^{n-1} \quad ''.$$

Note that these authors, by this way, avoided to identify their result with the fine-structure constant. In fact, it corresponds to $\alpha_n^{\frac{(n-1)}{2}}$, one among the infinite possible adimensional combinations of the four fundamental n-dimensional constants. In their aproach, the singularity at $n = 1$ disappears and for n = 1, 2, 3 and 4 each one of the fundamental constants is absent. As their result does not contemplate the Vaschy-Buckingham Theorem, we cannot atribute a physical meaning to their constant and to the absence of the electromagnetic constant at $n = 1$.

Now, there is no general agreement about the dependence of the values of e_n^2 and G_n on dimensionality. However, we can, at least, assume that the variation of both e_n^2 and $G_n m_p^2$ follow the same trend so that the curves of α_n and α_{Gn} versus n have the same shape. The plots of both α_n and α_{Gn} are given by Figure 1 in which e_n^2 and $G_n m_p^2$ are taken as constants. In the interval $[0, 1[$ the value of α_n begins at 10^{194} and diverges positively. Figure (1) shows the extreme sensibility of α_n and α_{Gn} on n in the interval $]1, 10]$. Only in a very short neighborhood of $n = 3$, $\alpha_n \sim 1$ (for example $\alpha_2 \sim 10^{-68}$, $\alpha_4 \sim 10^{19}$ and α_n

= 1 at n ∼ 3.067). So, the error on the experimental value of α yields the narrow range in which we can find the dimensionality of the space in which we live.

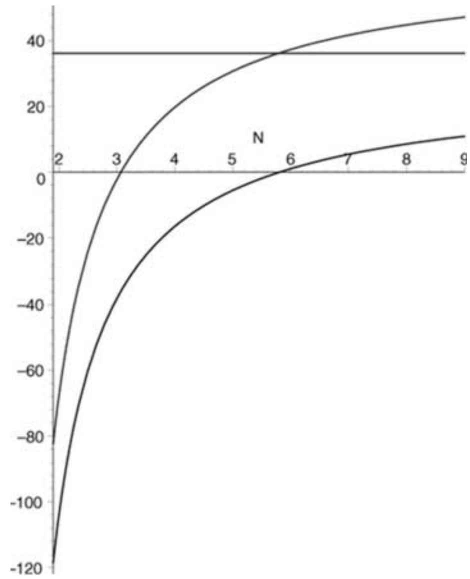

Figure 1: Dependence of $\log_{10}[\alpha_n]$ (upper curve), $\log_{10}[\alpha_{Gn}]$ (lower curve) and $\log_{10}[\alpha_n/\alpha_{Gn}]$ (straight line) on n, the dimensionality of space. The values of α_n and α_{Gn} increase very radiply and tend to $\sim 10^{63}$ and $\sim 10^{27}$, respectively.

In addition, the ratio between α_n and α_{Gn} is a constant given by $e_n^2/(G_n m_p^2) = e^2/(G m_p^2) \sim 10^{36}$. This means that in systems in which the balance between electric and gravitational forces plays a major role - such as planets [16],[3] - there are parameters - such as planetary radii - insensitive to dimensional change. This fact has deep astronomical, physical and biological implications. Note that, if the dimensionality of space had evolved as a function of time, the fine tuning of electromagnetic interactions at atomic and molecular levels, necessary to the existence of life, could not be maintained. As a final remark, we note that the relative standard uncertainty of 3.7×10^{-9} on the experimental value of α [15] implies a relative standard uncertainty of $1.08\ 10^{-13}$ on the three-dimensionality of the non-compact physical space. Note that this result is maintained even if we cannot accept the value of G_n as a dimensional invariant as discussed above, because its hypothetical variation should be neglected in the vicinity of 3.

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References

[1] A. Vaschy, Ann. Tél. 12 (1892) 25.

[2] E. Buckingham, Phys. Rev. IV (1914) 345.

[3] B. J. Carr and M. J. Rees, Nature 278 (1979) 605 .

[4] P. Ehrenfest, Proc. Amsterdam Acad. 20 (1917) 200.

[5] P. Ehrenfest, Ann. Phys. 61 (1920) 440.

[6] F. R. Tangherlini, Nuovo Cimento 27 (1963) 636.

[7] L. Gurevich and V. Mostepanenko, Phys. Lett. 35A (1971) 201.

[8] F. Caruso and R. Moreira Xavier, Fund. Sci. 8 (1987) 73.

[9] M. Jammer, Concepts of Space, Dover, 1993.

[10] A. Zeilinger and K. Svozil, Phys. Rev. Lett. 54 (1985) 2553.

[11] F. Caruso and R. Moreira Xavier, in *Essays on Interdisciplinary Topics in Natural Sciences - Memorabilia: Jacques A. Danon.1*, Ed. Frontières, 1997.

[12] G. I. Barenblatt, Dimensional Analysis, Gordon and Breach, 1987.

[13] J. B. Bastos Filho and R. Moreira Xavier, in Advances in Fundamental Physics, eds. M. Barone and F. Selleri, Hadronic Press, 1995.

[14] J. D. Barrow and F. J. Tipler, The Anthropic Cosmological Principle Oxford University Press, 1986, p. 269.

[15] P. J. Mohr and B. N. Taylor, Rev. Mod. Phys 72 (2000) 351.

[16] V. F. Weisskopf, Science 187 (1975) 609.