CBPF-NF-003/01

# ON THE INTERPRETATION OF LEVI-CIVITA SPACETIME FOR $0 \leq \sigma < \infty$

*L. Herrera*[1*], *N. O. Santos*[2,3†], *A. F. F. Teixeira*[3‡] *and A. Z. Wang*[4§]

[1]Area de Física Teórica, Facultad de Ciencias,
Universidad de Salamanca, 37008 Salamanca, Spain.

[2]Laboratório de Astrofísica e Radioastronomia,
Centro Regional Sul de Pesquisas Espaciais - INPE/MCT,
Cidade Universitária, 97105-900 Santa Maria-RS, Brazil.

[3] LAFEX, Centro Brasileiro de Pesquisas Físicas, 22290-180 Rio de Janeiro-RJ, Brazil.

[4]Departamento de Física Teórica, Universidade do Estado do Rio de Janeiro,
20550-013 Rio de Janeiro-RJ, Brazil.

**Abstract**

We study Levi-Civita metric for values of its $\sigma$ parameter in the range $0 \leq \sigma < \infty$. We show that the value $\sigma = 1/2$ makes the axial and angular coordinates switch meaning. We present its geodesics and a physical source satisfying the energy conditions for all the range of $\sigma$. This source allows us to obtain an energy per unit length which agrees with the behaviour of the geodesics and the fact that the solution has no event horizon.

# 1 Introduction

The cylindrically symmetric static vacuum solution of Einstein's field equations was obtained by Levi-Civita (LC) [1] in 1919. Ever since much was written by researchers trying to grasp its physical and geometrical interpretations. But this endeavour proved to be difficult and uncertain. Only in 1958 Marder [2] established that the solution contains two arbitrary independent parameters usually called $\sigma$ and $a$. Understanding the origin, geometry and physics that lies behind these two parameters is the big challenge to understand the solution.

For small values of $\sigma$, as noticed by LC himself, the corresponding Newtonian field is the external gravitational field produced by an infinitely long homogeneous line mass, with $\sigma$ representing the mass per unit length.

In this approximation the parameter $a$ is also associated to the constant arbitrary potential that exists in the Newtonian solution. In 1979 Bonnor [3] pointed out that $a$ is also dressed with a relevant global topological meaning, and cannot be removed by scale transformations.

There is a series of obstacles and apparently contradictory properties of $\sigma$ to allow possible interpretations (see a discussion in [4]). In this article we present some results concerning $\sigma$ that suggest certain interpretations but, we are aware, collide with other results. In section 2 we study the nature of the coordinates LC solution related to the range of different values of $\sigma$ between 0 and $\infty$. Circular geodesics are studied in this same range for $\sigma$ in section 3. A cylindrical shell source of anisotropic fluid is matched to LC solution with $0 \leq \sigma < \infty$ satisfying the energy conditions in section 4. The definition of energy per unit length is studied in section 5 for the shell source and compared to the properties of the geodesics studied in section 3. We end the article with a short conclusion.

# 2 The Levi-Civita metric

The general static cylindrically symmetric vacuum spacetime satisfying Einstein's field equations is given by Levi-Civita (LC) metric [1], which we write in the form

$$ds^2 = \varrho^{4\sigma}dt^2 - \varrho^{4\sigma(2\sigma-1)}\left(d\varrho^2 + \frac{1}{a_m^2}dm^2\right) - \frac{1}{a_n^2}\varrho^{2(1-2\sigma)}dn^2, \qquad (1)$$

where $-\infty < t < \infty$ is the time and $0 \leq \varrho < \infty$ the radial coordinates, and $\sigma$, $a_m$ and

$a_n$ are constants. This spacetime clearly has the Killing vectors $\xi^\mu_{(t)} = \delta^\mu_t$, $\xi^\mu_{(m)} = \delta^\mu_m$ and $\xi^\mu_{(n)} = \delta^\mu_n$.

The nature of the coordinates $m$ and $n$, so far unspecified, depends upon the behaviour of the metric coefficients. Either $a_m$ or $a_n$ can be transformed away by a scale transformation depending upon the behaviour of the coordinates $m$ and $n$, thus leaving the metric with only two independent parameters.

In order to find that behaviour in a comprehensive way, we first transform the radius $\varrho$ into a proper radius $r$ by defining

$$\varrho^{2\sigma(2\sigma-1)} d\varrho = dr, \tag{2}$$

so obtaining

$$\varrho = R^{1/\Sigma}, \quad R = \Sigma r, \quad \Sigma = 4\sigma^2 - 2\sigma + 1; \tag{3}$$

the metric (1) then becomes

$$ds^2 = f(r)dt^2 - dr^2 - h(r)dm^2 - l(r)dn^2, \tag{4}$$

with

$$f(r) = R^{4\sigma/\Sigma}, \quad h(r) = \frac{1}{a_m^2} R^{4\sigma(2\sigma-1)/\Sigma}, \quad l(r) = \frac{1}{a_n^2} R^{2(1-2\sigma)/\Sigma}. \tag{5}$$

When $\sigma = 0$ we have $\Sigma = 1$ and considering $a_m = a_n = 1$, then (4) becomes

$$ds^2 = dt^2 - dr^2 - dm^2 - r^2 dn^2, \tag{6}$$

corresponding to a flat four-geometry; by inspection we interprete $m$ as an axial $z$ coordinate, and $n$ as an azimuthal $\phi$ coordinate, and write

$$ds^2 = dt^2 - dr^2 - dz^2 - r^2 d\phi^2. \tag{7}$$

It is customary to assume that $\phi$ ranges from 0 to $2\pi$, and topologically identify these two extremes; in so doing, the picture of a cartesian two-dimensional plane naturally follows, in surfaces where $t$ and $z$ are constants. Equivalently, one may assume that $\phi$ ranges from $-\infty$ to $\infty$, together with the topological identification of every $\phi$ with $\phi + 2\pi$.

However, sometimes a two-dimensional flat conical structure with $t$ and $z$ constants appears more suitable to represent the physical situation, and some modification of the previous assumptions is demanded.

One could simply maintain (7) as the line element, with the fact that $\phi$ now ranges from 0 to $2\pi \sin\alpha$, where $\alpha$ is the half-angle of the cone, together with the topological identification of 0 and $2\pi \sin\alpha$.

However, the most common strategy in these cases is to set $g_{\phi\phi} = -r^2 \sin^2\alpha \, d\phi^2$, while maintaining $\phi$ going from 0 to $2\pi$ with the topological identification of 0 and $2\pi$ as before.

In this paper we often deal with two-dimensional surfaces $t$ and $z$ constants, endowed with rotational symmetry $\xi^{\mu}_{(\phi)}$, but with radially varying gaussian curvature.

In all circumstances we shall follow the convention that $-\infty < \phi < \infty$ with the equivalence $\phi \sim \phi + 2\pi$. Concerning the variable $z$ in (7), the most common assumption is that it goes from $-\infty$ to $\infty$ and that points with different $z$ are always different; however, this last statement is an unnecessary topological limitation, since the topological identification of any $z$ with $z + \zeta$ is admissible without destroying the flatness of the four-space.

Assuming $z \sim z + \zeta$ promotes compactification of the space in the $z$ direction (for $\zeta$ finite and nonzero), an occasionally desirable operation.

Before proceeding, a few words seem worthwhile concerning the cylindrical coordinates $(r, z, \phi)$ in a curved three-space with cylindrical symmetry.

We consider the line element

$$dl^2 = dr^2 + h(r)dm^2 + l(r)dn^2, \tag{8}$$

which clearly has the commuting vector fields $\xi^{\mu}_{(m)}$ and $\xi^{\mu}_{(n)}$; certainly $r$ is the radial coordinate, but which of the coordinates $m$ and $n$ is the angular $\phi$ and which is the axial $z$? By analogy with the flat case (where $g_{\phi\phi} = r^2$ and $g_{zz} = 1$), it seems appropriate to call angular that coordinate whose metric coefficient vanishes at $r = 0$, and call axial the other coordinate when the corresponding metric coefficient does not vanish at $r = 0$.

We now return to the line element given by (4) and (5), and consider $0 < \sigma < 1/2$. For this range of $\sigma$ we always have $h(r)$ diverging when $r \to 0$, and always $l(0) = 0$; we then visualize $m$ as the axial coordinate $z$, and $n$ as the angular coordinate $\phi$:

$$ds^2 = R^{4\sigma/\Sigma} dt^2 - dr^2 - R^{-4\sigma(1-2\sigma)/\Sigma} dz^2 - \frac{1}{a^2} R^{2(1-2\sigma)/\Sigma} d\phi^2, \tag{9}$$

with possible topological identifications in $z$ and $\phi$. When $\sigma = 1/2$ the two metric coefficients $h$ and $l$ in (5) are constant, unitary. Then neither $m$ nor $n$ is entitled to be an angular coordinate, and the three coordinates $(r,m,n)$ are better visualized as cartesian coordinates $(x,y,z)$. We have, *e.g.*, the Rindler flat spacetime [5], whose $t$=const sections have planar symmetry:

$$ds^2 = z^2dt^2 - dx^2 - dy^2 - dz^2, \tag{10}$$

with possible topological identifications in the coordinates $x$ and $y$. We next consider $1/2 < \sigma < \infty$; in this range of $\sigma$ we always have $h(0) = 0$ and $l(r)$ diverging when $r \to 0$; so we now interprete $m$ as an angular coordinate $\phi$, and $n$ as an axial coordinate $z$:

$$ds^2 = R^{4\sigma/\Sigma}dt^2 - dr^2 - R^{-2(2\sigma-1)/\Sigma}dz^2 - \frac{1}{a^2}R^{4\sigma(2\sigma-1)/\Sigma}d\phi^2, \tag{11}$$

with possible topological identifications in $z$ and $\phi$, and where we replaced $a_m$ for $a$.

The Kretschmann scalar $\mathcal{R} = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ for the metric (4) is [6]

$$\mathcal{R} = \frac{64\sigma^2(2\sigma-1)^2}{\Sigma^3 r^4}; \tag{12}$$

from (12) we see that the spacetime (4) is locally flat only for $\sigma = 0, 1/2$ and $\infty$. In section **3** we match (9) and (11) to a cylindrical anisotropic shell of matter.

The interior of the shell cylinder is assumed to be Minkowski spacetime, hence there the coordinates have a well defined meaning. The matching condition, of the continuity of the metrics, relates the respective metric coefficients for each coordinate, in particular $g_{zz}$ and $g_{\phi\phi}$, thus giving a further support to the coordinates chosen in (9) and (11).

# 3 Circular geodesics

For the circular geodesics [7] we have $\dot{r} = \dot{z} = 0$ and $g_{\phi\phi,r}\dot{\phi}^2 + g_{tt,r}\dot{t}^2 = 0$ where the dot stands for differentiation with respect to $s$. The angular velocity $\omega$ of a particle moving along a geodesics is $\omega = \dot{\phi}/\dot{t}$ and its tangential velocity $W^\mu$ is given by $W^\phi = \omega/\sqrt{g_{tt}}$ which is the only non null component.

In the case $0 \le \sigma < 1/2$, from (9) we obtain

$$\omega^2 = \frac{2\sigma}{1-2\sigma}a^2R^{2(4\sigma-1)/\Sigma}, \tag{13}$$

$$W^2 = \frac{2\sigma}{1-2\sigma}; \tag{14}$$

and, in the case $1/2 < \sigma < \infty$, from (11) we have

$$\omega^2 = \frac{1}{2\sigma - 1} a^2 R^{8\sigma(1-\sigma)/\Sigma}, \tag{15}$$

$$W^2 = \frac{1}{2\sigma - 1}. \tag{16}$$

It is worthwhile noting that for a given system (i.e. a fixed $\sigma$) the squared velocity $W^2$ is the same for all circular geodesics, in agreement with the corresponding Newtonian gravitation. We see from (14) that $W$ monotonically increases with $\sigma$, that is from $\sigma = 0$ producing $W = 0$, to $\sigma = 1/4$ attaining $W = 1$ (the speed of light), and finally $\sigma = 1/2$ producing geodesics with $W = \infty$. With $\sigma$ increasing beyond $1/2$, we note from (16) that $W$ diminishes, attaining $W = 1$ for $\sigma = 1$ and $W = 0$ for $\sigma = \infty$.

In other words, the circular geodesics are timelike when either $0 < \sigma < 1/4$ or $\sigma > 1$, are lightlike when $\sigma = 1/4$ or $\sigma = 1$, and are spacelike when $1/4 < \sigma < 1$.

These facts suggest that, while $\sigma$ increases from zero to $1/2$, the effective energy density per unit length, $\mu$, of the line source that produces the spacetime (9) increases too. So that for a test particle to remain in a circular motion its velocity has to increase in order to make a balance between the attracting gravitational force, that increases with $\mu$, and the centrifugal force, that increases with $W$.

On the other hand, when $\sigma$ further increases from $1/2$ to $\infty$ in the spacetime (11), it appears that the effective energy per unit length monotonically decreases to zero value.

# 4 Matching LC spacetime to a cylindrical shell source

Here we follow the same procedure as in [6]. We consider an infinitely thin cylindrical shell of anisotropic fluid matter with a finite radius and we match it to the exterior LC spacetime given by either (9) or (11). For simplicity we shall assume that the source is static.

For the interior of the shell cylinder we assume Minkowski spacetime, since it is the only static spacetime deprived of energy density [8].

In order to do the matching we require the continuity of the metric coefficients across the shell [9], allowing us to obtain the most general anisotropic shell fluid source.

Using the same coordinate system as in (9) or (11), we have for the interior $0 \leq r < r_0$ of the shell cylinder with radius $r = r_0$ the Minkowski spacetime,

$$ds_-^2 = dt^2 - dr^2 - dz^2 - r^2 d\phi^2. \tag{17}$$

Indices $-$ and $+$ refer to interior and exterior spacetimes, respectively. In order to have the general matching at $r = r_0$ satisfied for the exterior LC metric, we make a reparametrization of $t$ and $z$ like

$$t \rightarrow \frac{t}{A}, \ \ z \rightarrow \frac{z}{B}, \tag{18}$$

where $A$ and $B$ are constants. Then (9) with (18) becomes, for $0 \leq \sigma \leq 1/2$,

$$ds_+^2 = \frac{1}{A^2} R^{4\sigma/\Sigma} dt^2 - dr^2 - \frac{1}{B^2} R^{4\sigma(2\sigma-1)/\Sigma} dz^2 - \frac{1}{a^2} R^{2(1-2\sigma)/\Sigma} d\phi^2, \tag{19}$$

and (11) with (18), for $1/2 \leq \sigma < \infty$,

$$ds_+^2 = \frac{1}{A^2} R^{4\sigma/\Sigma} dt^2 - dr^2 - \frac{1}{B^2} R^{2(1-2\sigma)/\Sigma} dz^2 - \frac{1}{a^2} R^{4\sigma(2\sigma-1)/\Sigma} d\phi^2. \tag{20}$$

Then, considering the junction condition [9]

$$g_{\mu\nu}^+ \mid_{r_0} = g_{\mu\nu}^- \mid_{r_0}, \tag{21}$$

we obtain from (17) and (19), for $0 \leq \sigma \leq 1/2$,

$$A = R_0^{2\sigma/\Sigma}, \ \ B = R_0^{2\sigma(2\sigma-1)/\Sigma}, \ \ ar_0 = R_0^{(1-2\sigma)/\Sigma}, \tag{22}$$

where $R_0 = \Sigma r_0$; and from (17) and (20), for $1/2 \leq \sigma < \infty$,

$$A = R_0^{2\sigma/\Sigma}, \ \ B = R_0^{(1-2\sigma)/\Sigma}, \ \ ar_0 = R_0^{2\sigma(2\sigma-1)/\Sigma}. \tag{23}$$

Taub has shown [9] that if (21) is satisfied then the first derivatives of the metric are in general discontinuous across $r = r_0$, giving rise to a shell of matter. Following him,

$$g_{\mu\nu,\lambda}^+ \mid_{r_0} - g_{\mu\nu,\lambda}^- \mid_{r_0} = n_\lambda b_{\mu\nu}, \tag{24}$$

where $n_\lambda$ is the normal to the hypersurface $r = r_0$, directed outwards, giving $n_\lambda = \delta_\lambda^r$. From (24) we calculate $b_{\mu\nu}$ and obtain the energy momentum tensor $T_{\mu\nu}$ of the shell, which is given by

$$T_{\mu\nu} = \tau_{\mu\nu} \delta(r - r_0), \tag{25}$$

where

$$\tau_{\mu\nu} = \frac{1}{16\pi}[b(ng_{\mu\nu} - n_\mu n_\nu) + n_\lambda(n_\mu b^\lambda_\nu + n_\nu b^\lambda_\mu) - nb_{\mu\nu} - n_\lambda n_\delta b^{\lambda\delta} g_{\mu\nu}], \tag{26}$$

and where $\delta(r - r_0)$ denotes the Dirac delta function, $n = n_\lambda n^\lambda$, and $b = b^\lambda_\lambda$. Considering (17) and (19) we obtain from (24) the nonvanishing components of $b_{\mu\nu}$ for $0 \leq \sigma \leq 1/2$,

$$b_{\mu\nu} = \frac{4\sigma}{R_0} diag(1, 0, 1 - 2\sigma, 2\sigma r_0^2), \tag{27}$$

where the order of the coordinates is $(t, r, z, \phi)$; and from (17), (20), and (24) we find for $1/2 \leq \sigma < \infty$

$$b_{\mu\nu} = \frac{2}{R_0} diag(2\sigma, 0, 2\sigma - 1, r_0^2). \tag{28}$$

With (27) and (28) substituting into (26) we can write the shell energy momentum ancillary tensor as

$$\tau_{\mu\nu} = diag(\rho, 0, p_z, r_0^2 p_\phi), \tag{29}$$

where $\rho$ is the energy density and $p_z$ and $p_\phi$ are the pressures in the $z$ and $\phi$ directions respectively.

For $0 \leq \sigma \leq 1/2$ these quantities measure

$$\rho = \frac{\sigma}{4\pi R_0}, \quad p_z = \frac{\sigma(1 - 2\sigma)}{4\pi R_0}, \quad p_\phi = \frac{\sigma^2}{2\pi R_0}, \tag{30}$$

while for $1/2 \leq \sigma < \infty$ they are

$$\rho = \frac{\sigma}{4\pi R_0}, \quad p_z = \frac{2\sigma - 1}{8\pi R_0}, \quad p_\phi = \frac{1}{8\pi R_0}. \tag{31}$$

It can easily be shown that the anisotropic fluids described by (30) and (31) satisfy the weak, strong and dominant energy conditions [10].

# 5 The energy per unit length of the shell

In the Newtonian limit of the LC spacetime $\sigma$ can be interpreted as the energy per unit length of the source. Considering Israel's definition of energy density per unit length [11] we obtain, for both (30) and (31),

$$\mu = \int_0^\infty \int_0^{2\pi} (\rho + p_z + p_\phi)\delta(r - r_0)\sqrt{-g}drd\phi = \frac{\sigma}{\Sigma}, \tag{32}$$

where $g$ is the determinant of the metric. From (32) we see that for small $\sigma$ we have $\mu \approx \sigma$, which is consistent with the Newtonian limit.

As $\sigma$ increases $\mu$ increases too, reaching a maximum at $\mu_{max} = \sigma = 1/2$; then $\mu$ starts diminishing with further increase of $\sigma$, becoming $\mu = 0$ for $\sigma \to \infty$.

These results are consistent with (14) and (16), showing that the tangential speed steadily increases with $\sigma$ and $\mu$ up to a maximum $W \to \infty$ at $\sigma = \mu_{max} = 1/2$, and then decreases with increasing $\sigma$ and decreasing $\mu$ up to $W = 0$, when also $\mu = 0$.

Furthermore, when $W = 1$ the circular geodesics are null. From (14) and (16) we can see that this corresponds to $\sigma = 1/4$ and $\sigma = 1$, for which we have $\mu(1/4) = \mu(1) = 1/3$. These properties between $\mu$ and $W$ can be seen too from the expression of $\mu$ in terms of $W$ which is, for $\sigma < 1/2$, from (14), as well as for $\sigma > 1/2$, from (16),

$$\mu = \frac{W^2(1+W^2)}{2(1+W^2+W^4)}. \tag{33}$$

It is worth noticing that the relations above between $\omega$, $\sigma$ and $\mu$ are consistent with the behaviour of a gyrosocope moving along a circular path (not necessarily a geodesic) with angular velocity $\omega$.

Indeed, using the Rindler-Perlick method [12], it is not difficult to find the rate of precession $\Omega$ of such gyroscope for the line element (4),

$$\Omega = \frac{\omega(l'f - lf')}{2\sqrt{fl}(f - \omega^2 l)}, \tag{34}$$

where $l = g_{\phi\phi}$ and $\omega$ is any angular velocity of the gyroscope around the line source.

For $0 \leq \sigma \leq 1/2$ we have from (9),

$$\Omega = \frac{a\omega(1-4\sigma)R^{-2\sigma(1+2\sigma)/\Sigma}}{a^2 - \omega^2 R^{2(1-4\sigma)/\Sigma}}; \tag{35}$$

whereas for $1/2 \leq \sigma < \infty$ we have from (11),

$$\Omega = \frac{4a\omega\sigma(\sigma-1)R^{-(1+2\sigma)/\Sigma}}{a^2 - \omega^2 R^{8\sigma(\sigma-1)/\Sigma}}. \tag{36}$$

Thus we see that the gyroscope is locked at the lattice, $\Omega = 0$, for $\sigma = 1/4$ and 1 respectively, as expected for null paths [12], and both values of $\sigma$ produce from (32) $\mu(1/4) = \mu(1) = 1/3$.

Also, for $\sigma = 0$ we obtain the Thomas precession (modified by the effect of $a$)

$$\Omega = \frac{a\omega}{a^2 - \omega^2 r^2}; \tag{37}$$

and similarly for $\sigma \gg 1$,

$$\Omega \approx \frac{a\tilde{\omega}}{a^2 - \tilde{\omega}^2 r^2}, \tag{38}$$

where $\tilde{\omega} = 4\omega\sigma^2$.

However, for $\sigma = 1/2$ the situation is not so clear (see [4] and the discussion below).

Also observe that $\mu$ has only one maximum at $\sigma = 1/2$ , and the maximum is finite, with the value $= 1/2$. Then, since in cylindrical sources no black holes are formed, one might conclude that the minimum mass per unit length to form a black hole satisfies the constraint $\mu > 1/2$.

Indeed, according to our present understanding of the black hole physics, there should exist a lower mass limit to form them. For example, it is accepted that the mass of a neutron star is between $1.2M_\odot$ and $1.7M_\odot$, where $M_\odot$ denotes the solar mass [14].

When the mass of a star is $M \gg 1.7M_\odot$, it might exist only in the state of a black hole.

There are two other expressions for mass per unit length besides Israel's (32). One is given by Marder [2], which is

$$\mu_M = \int_0^\infty \int_0^{2\pi} \rho\delta(r - r_0)\sqrt{g_{(2)}}drd\phi, \tag{39}$$

where $g_{(2)}$ is the determinant of the induced metric on the 2-surface defined by $t = z =$const. The other definition is given by Vishveshwara and Winicour [13], based on the Killing vectors of time translation $\xi^\mu_{(0)} = \delta^\mu_t$ and rotation $\xi^\mu_{(3)} = \delta^\mu_\phi$, which is

$$\mu_{VW} = -\frac{1}{2\tau}(\lambda_{33}\lambda_{00,\tau} - \lambda_{03}\lambda_{03,\tau}), \tag{40}$$

where

$$\lambda_{00} = \xi^\mu_{(0)}\xi_{\mu(0)}, \quad \lambda_{03} = \xi^\mu_{(0)}\xi_{\mu(3)}, \quad \lambda_{33} = \xi^\mu_{(3)}\xi_{\mu(3)}, \tag{41}$$

$$\tau^2 = -2(\lambda_{00}\lambda_{33} - \lambda^2_{03}). \tag{42}$$

Using (19,30) or (20),31) we obtain

$$\mu_M = \frac{\sigma}{2\Sigma}, \quad \mu_{VW} = \sigma; \tag{43}$$

we see that $\mu_M$ does dot produce the Newtonian limit, and $\mu_{VW}$ does not explain the circular geodesics behaviour. Hence we discard both definitions. (In [6] the expressions (15) and (16) should be interchanged with their respective analises.)

# 6 Conclusion

We have presented the cylindrically symmetric static vacuum solution of Einstein's field equations, obtained by LC in its general form, by only specifying the time and radial coordinates and not specifying the other two.

We showed that the nature of the two coordinates is closely linked with the range of $\sigma$.

There are two ranges, $0 \leq \sigma < 1/2$ and $1/2 < \sigma < \infty$, where the coordinates, being $z$ and $\phi$, switch their nature [15]. We calculated the circular geodesics for these two ranges and it appears, from their behaviour, that the energy per unit length increases by increasing $\sigma$ up to $1/2$ and then diminishes while $\sigma$ increases to $\infty$.

We matched a shell source to LC solution satisfying the energy conditions for the whole range of $\sigma$. The energy per unit length $\mu$ calculated from the definition given by Israel reproduces the behaviour of the circular geodesics and, furthermore, while producing a maximum for $\mu$ suggests a possible explanation for the non existence of event horizons in LC spacetime. Of course, we are aware that $\mu$ is model dependent and cannot be given a general meaning but none the less it satisfies part of the expected properties.

Some questions, however, remain unanswered, which leaves the puzzle incomplete.

Indeed, observe that for $\sigma = 1/2$ the energy density $\rho$ as well as $p_\phi$ and $\mu$ are nonvanishing, but the spacetime is flat. Now, if the source would be a plane (as seems to be the case in [16]) then one could invoke the equivalence principle to explain the vanishing of the Riemann tensor.

However in our case the proper radius of the cylinder (unlike the case analyzed in [16]) remains constant and finite for any value of $\sigma$. So the question is: why do a cylinder with positive energy density and pressure distribution produces vanishing curvature? This last question, together with the fact that a gyroscope moving along the $\phi$-coordinate in the LC spacetime with $\sigma = 1/2$ behaves in an unexpected way (see [4]), brings out the difficulties in interpreting the $\sigma = 1/2$ case as due to a cylinder, in spite of the fact that the source presented in section 4 is physically satisfactory.

The situation described above might have its origins in the following two facts:

1. The source is a shell and therefore the second fundamental form is discontinuous across the boundary surface. It is unclear whether demanding both fundamental

forms to be continuous is compatible with a reasonable cylindrical source or if, instead, it leads always to the same results as in [16] when $\sigma = 1/2$.

2. Axial and angular coordinates switch meaning at $\sigma = 1/2$, so it is reasonable to ask if for this value of $\sigma$, $r = r_0$ describes a cylinder.

At any rate it is clear that this point requires further discussion.

# 7 Acknowledgment

NOS is grateful to Professor Alberto Santoro for his kind hospitality at the Centro Brasileiro de Pesquisas Físicas, where this work was developed. This work has been partially supported by the Spanish Ministry of Education under grant PB96-1306. The financial assistance from CNPq (NOS, AZW) and FAPERJ (AZW) is also gratefully acknowledged.

# References

[1] Levi-Civita T 1919 *Rend. Acc. Lincei* **28**, 101

[2] Marder L 1958 *Proc. R. Soc.* A **244**, 524

[3] Bonnor W B 1979 *J. Phys. A: Math. Gen.* **12**, 847

[4] Herrera L, Ruifernández J and Santos N O *The Static Cylinder, Gyroscopes, the C-metric, and All That*, to appear in *Gen. Rel. Grav.*

[5] da Silva M F A, Wang A Z and Santos N O 1998 *Phys. Lett.* A **244**, 462

[6] Wang A Z, da Silva M F A and Santos N O 1997 *Class. Quantum Grav.* **14**, 2417

[7] da Silva M F A, Herrera L, Paiva F M and Santos N O 1995 *J. Math. Phys.* **36**, 3625

[8] Apostolatos T A and Thorne K S 1992 *Phys. Rev.* D **46**, 2435

[9] Taub A H 1980 *J. Math. Phys.* **21**, 1423

[10] Hawking S W and Ellis G F R 1973 *The Large Scale Structure of Spacetime* (Cambridge University Press, Cambridge)

[11] Israel W 1977 *Phys. Rev.* D **15**, 935

[12] Rindler W and Perlick V 1990 *Gen. Rel. Grav.* **22**, 1067; Herrera L, Paiva F M and Santos N O 2000 *Class. Quantum Grav.* **17**, 1549

[13] Vishveshwara C V and Winicour J 1977 *J. Math. Phys.* **18**, 1280

[14] Glendenning N K 1997 *Compact Stars: Nuclear Physics, Particle Physics, and General Relativity* (Springer-Verlag, New York)

[15] Miguelote A Y, da Silva M F A, Wang A Z and Santos N O *Levi-Civita Solutions Coupled with Electromagnetic Fields*, to appear in *Class. Quantum Grav.*

[16] Bonnor W B 1999 *The Static Cylinder in General Relativity* in *On Einstein's Path*, editor Alex Harvey (Springer-Verlag, New York) , page 113