NOTAS DE FÍSICA

VOLUME XX

Nº 3

ON THE INTRODUCTION OF MASSIVE SYSTEMS IN TWISTOR THEORY

by

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RIO DE JANEIRO, BRASIL

1973

INTRODUCTION

The twistor formalism has recently received a large progress in its foundations, mainly due to the works of Penrose. This formalism in flat spacetime, that is for special relativity, is a procedure for treating with conformal invariant theories, that is, with null rest mass particles and with fields presenting such symmetry. However, for the specification of certain problems it is necessary to know how the interaction with a massive source may be introduced in the formalism, as for instance in the scattering of zero rest mass particles by a massive spinning source (by means of the linearized Schwarzschild's or Kerr's solution giving the gravitational field). Penrose has proposed a process for introducing massive systems in twistor theory by means of the superposition of intermediate null rest mass systems (i.e. superposition of momenta and angular momenta of null rest mass systems)(1). As result, a massive system is described by a second rank twistor. Presently we consider a perturbative approach which introduces non null rest masses in this formalism in such form that the mass appears as an independent quantity, not bounded to the values assumed by the intermediate auxiliary systems as is the case for Penrose's method. This process allows for a direct limit to the case of a null rest mass system, very much similar, for instance to the limit from the Dirac equation for a massive particle to Weyl's equation.

In the first section a brief definition of a twistor is given, and the definition of the real quadratic form in twistor space which gives the scalar product of twistors is also given. Presently we show how to write this quadratic form as the matrix elements of the second rank Hermitian singular spinor associated to null rest mass systems. In the second section we give

Penrose's way of introducing non null rest mass systems, and finally in the third section we treat with the problem of introducing non null rest mass systems by means of perturbations acting on some given null rest mass system.

The notation used is the following: spinor indices are denoted by capital latin letters, they range from 0 to 1. Two types of such indices exist: unprimed and primed indices, both transforming as representations of the group SL(2,C). The twistor indices are indicated by greek letters, formally they range from 0 to 3 in the complex domain, or they may be thought as representing the effect of considering simultaneously two sets of two--component spinors of the above type (the particular form in which this combination appears will be seen later). Tensor indices, referring to the Lorentz group, are denoted by latin letters, and the signature of the flat spacetime metric is taken equal to -2.

1 - TWISTORS IN FLAT SPACETIME

In this section the concept of a twistor will be slightly treated, for a more extensive presentation see the reference given previously. Given a null rest mass particle with the momentum four-vector P^a(which is taken as real quantities), we can introduce the Hermitian singular spinor matrix $\overset{o}{p}{}^{AA'} = \bar{\pi}^A \pi^{A'}$ such that

$$P^a = \sigma^a{}_{AA'} \overset{o}{P}{}^{AA'} , \quad \overset{o}{P}{}^{AA'} = 1/2 \ \sigma_a{}^{AA'} P^a$$

In this event, P^a is a future pointing real null four-vector. In our notation the spin matrices σ^a are defined by

$$\sigma_{a\,AA'}\,\sigma^{b}_{\;BB'} + \sigma^{b}_{\;AA'}\,\sigma_{a\,BB'} = \delta^{b}_{a}\,\varepsilon_{AB}\,\varepsilon_{A'B'}$$

which imply in

$$\sigma_{a\,AA'}\,\sigma^{a}_{\;BB'} = 2\,\varepsilon_{AB}\,\varepsilon_{A'B'}$$

the major part of the spinor notation used here is similar to that used by Penrose[2], for instance,

$$\bar{\eta}^{A'} = (\overline{\eta^{A'}})$$

for any two-component spinor η. Associated to this particle we also have a spin vector S_a such that

$$S_a = s\,P_a \tag{1-1}$$

the quantity s being the helicity, and $|s|$ is the spin of the particle. Since,

$$S_a = -1/2\;\eta_{abcd}\,P^b\,M^{cd} \tag{1-2}$$

where M^{cd} is the angular momentum tensor. Using the spinors associated to η_{abcd} and to M^{cd}, which have the form

$$\eta_{abcd} \leftrightarrow i(\varepsilon_{AD}\,\varepsilon_{BC}\,\varepsilon_{A'C'}\,\varepsilon_{B'D'} - \varepsilon_{AC}\,\varepsilon_{BD}\,\varepsilon_{A'D'}\,\varepsilon_{B'C'})$$

$$M^{cd} \leftrightarrow \mu^{(CD)}\,\varepsilon^{C'D'} + \varepsilon^{CD}\,\bar{\mu}^{(C'D')}$$

We obtain from (1-1) and (1-2),

$$\mu_{(CA)} = i\,\omega_{(C}\,\bar{\pi}_{A)} \qquad (1\text{-}3)$$

for some spinor ω.

The twistor associated to this system is defined by the pair of two-component spinors ω^A and $\pi_{A'}$ as[1]

$$Z^{\alpha} \leftrightarrow (\omega^A, \pi_{A'}) \qquad (1\text{-}4)$$

This pair of spinors (up to a phase factor) describe the pair of tensors P^a, M^{ab} which describe the system. In spite of looking similar to a Dirac spinor, the Z^{α} is really not similar to such four-component spinor[3] since it may be proved that Z^{α} possess well defined properties of transformation under translations.

Defining the conjugate twistor by

$$\bar{Z}_{\alpha} \leftrightarrow (\bar{\pi}_{A'}, \bar{\omega}^{A'})$$

The Hermitian form

$$Z^{\alpha}\,\bar{Z}_{\alpha} = \omega^A\,\bar{\pi}_A + \pi_{A'}\,\bar{\omega}^{A'} = 2s \qquad (1\text{-}5)$$

defines the inner product in twistor space. This form has a signature (++--).

We have, excluding the possibility that ω is parallel to π

$$\pi_{A'} = \frac{\bar{\omega}^{A} \overset{o}{P}_{AA'}}{\bar{\omega}^{B} \bar{\pi}_{B}}$$

taking complex conjugate,

$$\bar{\pi}_{A} = \frac{\omega^{A'} \overset{o}{P}_{A'A}}{\omega^{B'} \pi_{B'}}$$

so that

$$Z^{\alpha} \bar{Z}_{\alpha} = \frac{\omega^{A'} \overset{o}{P}_{A'A} \omega^{A}}{\omega^{B'} \pi_{B'}} + \frac{\bar{\omega}^{A} P_{AA'} \bar{\omega}^{A'}}{\bar{\omega}^{B} \pi_{B}} \tag{1-6}$$

Defining,

$$\tilde{\omega}^{A} = \frac{\omega^{A}}{\sqrt{\omega^{B'} \pi_{B'}}} , \quad \tilde{\omega}^{A'} = \frac{\omega^{A'}}{\sqrt{\omega^{B'} \pi_{B'}}}$$

which gives,

$$\bar{\tilde{\omega}}^{A'} = \frac{\bar{\omega}^{A}}{\sqrt{\bar{\omega}^{B} \bar{\pi}_{B}}} , \quad \bar{\tilde{\omega}}^{A} = \frac{\bar{\omega}^{A}}{\sqrt{\bar{\omega}^{B} \bar{\pi}_{B}}}$$

We get for (1-6)

$$Z^{\alpha}\ \overline{Z}_{\alpha} = \hat{\omega}^{A'}\ \overset{o}{P}_{A'A}\ \hat{\omega}^{A} + \overline{\hat{\omega}}^{A'}\ P_{AA'}\ \overline{\hat{\omega}}^{A} \tag{1-7}$$

Introducing the notation

$$<\xi\ |\overset{o}{P}|\eta> = \overline{\xi}^{A}\ \overset{o}{P}_{AA'}\ \eta^{A'} \tag{1-8}$$

holding for any pair of spinors ξ and η. This expression will give a real number if we consider only the "diagonal" matrix elements:

$$<\xi\ |\overset{o}{P}|\ \xi> = \overline{<\xi\ |\overset{o}{P}|\ \xi>}$$

(for proving this use the property that $\overset{o}{P}$ is Hermitian). For our case,

$$\overline{<\xi\ |\overset{o}{P}|\ \xi>} = |\ \pi^{A'}\ \xi_{A'}|^{2} \tag{1-9}$$

We also have, for a general "matrix element" of $\overset{o}{P}$,

$$<\xi\ |\overset{o}{P}|\eta> = \overline{<\eta\ |\overset{o}{P}|\ \xi>}$$

Note also the result: (which follows directly from (1-9)).

$$<\pi\ |\overset{o}{P}|\pi> = 0 \tag{1-10}$$

We have for (1-7)

$$Z^{\alpha} \bar{Z}_{\alpha} = < \bar{\hat{\omega}} \,|\overset{o}{P}|\hat{\omega} > + < \hat{\omega} \,|\overset{o}{P}|\bar{\hat{\omega}} >$$

which in turn is equivalent to(*)

$$Z^{\alpha} \bar{Z}_{\alpha} = 2 \text{ Re} < \bar{\hat{\omega}} \,|\overset{o}{P}|\hat{\omega} > \qquad (1\text{-}11)$$

Thus, the value for the helicity takes the form

$$s = \text{Re} < \bar{\hat{\omega}} \,|\overset{o}{P}|\, \hat{\omega} >$$

From (1-10) we see that presently the inner product $Z^{\alpha} \bar{Z}_{\alpha}$ gives all the information on the physical quantities associated to the null rest mass particle. It is simple to verify that the other possible matrix element of $\overset{o}{P}$ vanishes:

$$< \hat{\omega} \,|\overset{o}{P}|\pi > = 0$$

2. MASSIVE SYSTEMS AS A SECOND RANK TWISTOR

Given a timelike momentum fourvector P_a, associate to it we have an Hermitian second order non-singular spinor $P_{AA'}$. Therefore, in this situation it is not possible to select one spinor quantity for the description of this

(*) The case for null twistors is excluded from the present analysis, since as was noted ω cannot be here proportional to π.

motion. As consequence the results of the twistor formalism do not apply here in a direct way. Nevertheless, Penrose has shown that this situation may be regarded as the superposition of several (at least two null momenta, the twistor method being applied to each one of such intermediate momenta. If we select all such auxiliary null momenta as future pointing, we obtain as result of the superposition of such momenta a future pointing timelike momentum. Besides, as before we take all null momenta as real quantities. The same superposition is taken for the angular momenta of the null systems. Thus, we have (from here on we take just two auxiliary null systems)

$$P_a = \overset{1}{P}_a + \overset{2}{P}_a$$

$$M^{ab} = \overset{1}{M}{}^{ab} + \overset{2}{M}{}^{ab}$$

to each pair $(\overset{i}{P}_a, \overset{i}{M}{}^{ab})$ of tensors it is associated a pair of spinors $(\overset{i}{\omega}{}^A\ \overset{i}{\pi}_{A'})$ according to the prescription of the last section. Then, associated to the massive system with momentum P_a and angular momentum M^{ab} we get the spinors $P_{AA'}$ and $M^{AA'\,BB'}$ such that

$$P_{AA'} = \overline{\pi}{}^1{}_A\, \pi^1{}_{A'} + \overline{\pi}{}^2{}_A\, \pi^2{}_{A'}$$

$$M^{AA'\,BB'} = i\left\{ \left[\omega^1{}^{(A}\, \overline{\pi}{}^{1B)} + \omega^2{}^{(A}\, \overline{\pi}{}^{2B)} \right] \varepsilon^{A'B'} - \varepsilon^{AB} \left[\overline{\omega}{}^1{}^{(A'}\, \pi^{1B')} + \overline{\omega}{}^2{}^{(A'}\, \pi^{2B')} \right] \right\}$$

We have

$$P^2 = 2\overset{1}{P}_a\, \overset{2}{P}{}^a = m^2$$

giving,

$$m^2 = 4\ |\pi^{1\ B'}\ \pi^2{}_{B'}|^2$$

The case for m = 0 is recovered for $\pi^2{}_{B'} = \lambda\pi^1{}_{B'}$ where λ is a complex number. In this case we get

$$\overset{2}{P}_a = |\lambda|^2\ \overset{1}{P}_a$$

giving the well known result that the superposition of two parallel null vectors is again a null vector.

A second order twistor is of the form

$$Q^{\alpha\beta} = \begin{pmatrix} Q^{AB} & Q^A{}_{B'} \\ Q_{A'}{}^B & Q_{A'B'} \end{pmatrix}$$

and may be considered as a direct product of a twistor

$$Z^\alpha = \begin{pmatrix} Z^A \\ Z_{A'} \end{pmatrix}$$

by another one of the form

$$\overset{T\beta}{U} = (\ U^B \quad U_{B'}\ )$$

That is,

$$Q^{\alpha\beta} = Z^\alpha\ \overset{T\beta}{U}$$

An important example of a second order twistor is the unit twistor $I^{\alpha\beta}$ given by

$$I^{\alpha\beta} = \begin{pmatrix} \varepsilon^{AB} & 0 \\ 0 & 0 \end{pmatrix} = - I^{\beta\alpha}$$

The $I^{\alpha\beta}$ may be considered as the direct product of the two null twistors (skew symmetrized product) associated to the base spinors o^A and c^B:

$$\overset{\alpha}{I} = \begin{pmatrix} o^A \\ 0 \end{pmatrix} , \quad J^{\beta} = \begin{pmatrix} c^B \\ 0 \end{pmatrix}$$

$$\overset{\alpha\beta}{I} = \overset{\alpha}{I}\,\overset{T\beta}{J} - \overset{\beta}{I}\,\overset{T\alpha}{J}$$

Then, according to Penrose[1], a massive system is described by the second order twistor $A^{\alpha\beta}$ given by

$$A^{\alpha\beta} = 2(E^{\alpha}{}_{\gamma}\, I^{\beta\gamma} + E^{\beta}{}_{\gamma}\, I^{\alpha\gamma}) \qquad (2\text{-}1)$$

where

$$E^{\alpha}{}_{\gamma} = \sum_{i=1}^{2} \overset{i}{Z}{}^{\alpha}\, \overset{i}{\bar{Z}}_{\gamma} \quad , \quad \overset{i}{Z}{}^{\alpha} \leftrightarrow (\overset{i}{\omega}{}^{A} , \overset{i}{\pi}_{A'})$$

a direct calculation gives,

$$A^{\alpha\beta} = \begin{pmatrix} -2i\,\mu^{(AB)} & 2P^{A}{}_{B'} \\ 2P_{A'}{}^{B} & 0 \end{pmatrix}$$

for

$$\mu^{(AB)} = i \sum_{k=1}^{2} \overset{k}{\omega}{}^{(A} \overset{k}{\pi}{}^{B)}$$

$$P^{A}{}_{B'} = \sum_{k=1}^{2} \overset{k}{P}{}^{A}{}_{B'} = \overset{1}{\pi}{}^{A} \overset{1}{\pi}{}_{B'} + \overset{2}{\pi}{}^{A} \overset{2}{\pi}{}_{B'}$$

Note that from the definition of the rest mass, and from the relation giving the conformal changing into the components $\overset{i}{\pi}{}^{A}$:

$$\tilde{\overset{i}{\pi}}{}_{A'} = \overset{i}{\pi}{}_{A'} - \frac{i}{2} \Upsilon_{AA'} \overset{i}{\omega}{}^{A}$$

$$\Upsilon_{AA'} = \nabla_{AA'} \log \Omega \, , \qquad d\tilde{s}^2 = \Omega^2 \, ds^2$$

we get the following conformal changing in m^2

$$\tilde{m}^2 = m^2 + \delta m^2 \quad ,$$

where,

$$\delta m^2 = |w|^2 + 2 \operatorname{Re} (w \overset{1}{\pi}{}^{B'} \overset{2}{\pi}{}_{B'})$$

for

$$w = - \frac{i}{2} \Upsilon_{AB'} (\overset{1}{\pi}{}^{B'} \overset{2}{\omega}{}^{A} + \overset{2}{\pi}{}^{B'} \overset{1}{\omega}{}^{A}) - \frac{1}{4} \Upsilon_{A}{}^{B'} \Upsilon_{CB'} \overset{1}{\omega}{}^{A} \overset{2}{\omega}{}^{C}$$

3 - AN ALTERNATIVE APPROACH FOR INCLUDING MASSIVE PARTICLES IN THE TWISTOR THEORY

From the mathematical viewpoint the method used in the section 2 for incorporating the rest mass in the formalism of twistors is very elegant and self consistent. However, this method gives a well determined expression for the rest mass as function of the square of the modulus of the complex number $\overset{1}{\pi}{}^{B'} \overset{2}{\pi}{}_{B'}$. Since both quantities $\overset{1}{\pi}{}^{B'}$ and $\overset{2}{\pi}{}^{B'}$ are associated to null momenta $\overset{1}{P}$ and $\overset{2}{P}$, it follows, in principle, that the spinors $\overset{1}{\pi}$ and $\overset{2}{\pi}$ may assume any value (of course excluding the trivial value zero), thus giving to m a domain of variation much larger than is necessary. In special relativity the twistor formalism is a process for treating with conformal invariant particles and fields, thus, the introduction of the rest mass in this formalism is connected to a break of the conformal symmetry. According to section 2 such process is obtained by selecting a particular class of spinors such that for every given value of m the previous relation holds. But for other values of the spinors we may form values for m which do not exist in the spectrum of the elementary particles.

In what follows we will propose another process for introducing the rest mass in the twistor formalism in such way that m appears as an independent quantity, not related to other quantities directly connected to the twistor concept. The process which will be followed allows for a direct limit to the case of null rest mass. We start by writing

$$P^{AA'} = \overset{\circ}{P}{}^{AA'} + f(m^2)\, H^{AA'} \tag{3-1}$$

$$\omega^{(AB)} = \omega^{(A}{}_{\overline{\pi}}{}^{B)} + f(m^2)\, \tau^{(AB)} \tag{3-2}$$

where $\overset{o}{P}{}^{AA'}$ is the singular Hermitian spinor introduced before. Since we are interested in a real quantity P_a the spinor $P^{AA'}$ is taken as a second rank Hermitian spinor. In the present problem we are directly interested in the spinor representation of a massive particle, therefore we need two spinors for the definition of $P^{AA'}$. Then, the general form for $H^{AA'}$ is:

$$H^{AA'} = \varepsilon_1(\bar{\xi}^A \eta^{A'} + \xi^{A'} \bar{\eta}^A) + \varepsilon_2(\bar{\xi}^A \xi^{A'} + \bar{\eta}^A \eta^{A'}) \qquad (3\text{-}3)$$

where ε_1 and ε_2 are constants with the values (-1, 0, 1). This formula may, eventually, be simplified by considering that one of the spinors is proportional to $\bar{\pi}^A$, as for instance, $\bar{\eta}^A = \alpha\bar{\pi}^A$, for α a complex. However, in all subsequent discussions we will keep the general form (3-3). From (3-3) we have

$$H^2 = -4\,\varepsilon_1^2\,|\xi^{A'}\eta_{A'}|^2 + 4\,\varepsilon_2^2\,|\xi^{A'}\eta_{A'}|^2$$

$$H_o = \varepsilon_2(\,|\xi_{o'}|^2 + |\xi_{1'}|^2 + |\eta_{o'}|^2 + |\eta_{1'}|^2) + 2\,\varepsilon_1\,\mathrm{Re}(\bar{\xi}_o\,\eta_{o'} + \bar{\xi}_1\,\eta_{1'})$$

Therefore, for the choices $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$, H_a will be a null fourvector. For $\varepsilon_1 = 0$, $\varepsilon_2 = \pm 1$, H_a is timelike, and for $\varepsilon_2 = 0$, $\varepsilon_1 = \pm 1$, H_a is spacelike.

The equation (3-1) is equivalent to

$$P_a = \overset{o}{P}_a + R_a, \quad R_a = fH_a$$

then,

$$P^2 = 2\overset{o}{P}_a R^a + R^2$$

For R_a timelike (H_a timelike) we can map to a frame where $\vec{R} = 0$, giving

$$P^2 = R_o^2 + 2\overset{o}{P}_o R_o$$

since $\overset{o}{P}_o$ is greater than zero, P_a is timelike if $R_o > 0$ or if $R_o < 0$ but in the last situation we have:

$$R_o = -b, \quad b > 0; \quad b > 2\overset{o}{P}_o > \overset{o}{P}_o$$

and this implies that P_a is past pointing. This case will not be considered, that means we take $R_o > 0$ if R_a is timelike (this implies that for this case $\varepsilon_2 = +1)^{(*)}$. For R_a a null fourvector we have,

$$P^2 = 2\,\overset{o}{P}_a R^a$$

taking R_a as non parellel to $\overset{o}{P}_a$, such that $\overset{o}{P}_o > 0$, $R_o > 0$, we get $P^2 > 0$ with positive P_o (a situation similar to that of section 2). If R_a is spacelike we can map to a frame where $R_o = 0$, giving

$$P_o = \overset{o}{P}_o > 0$$

$$P^2 = -2\vec{N}\,.\,\vec{R} - \vec{R}^2, \quad \vec{N} = \overset{o}{\vec{P}}$$

(*) later it will be shown that $R_o > 0$ implies in $H_o > 0$.

for some choices of the components $\vec{N}$ and $\vec{R}$ we can make P^2 positive, for instance, taking $\vec{N} = (0, n, 0) = N^i (i = 1, 2, 3)$, $\vec{R} = (0, q, 0)$ with q negative, $q = -s$, $s > 0$, we get

$$2ns - s^2 = m^2$$

(for real m). Given m and n the solution for s is,

$$s = n \pm ((n+m)(n-m))^{1/2}$$

and selecting the situation for $n > m$, the quantity

$$s = n + ((n+m)(n-m))^{1/2}$$

is positive, and $P^2 > 0$. Thus, we have: (for all situations giving $P^2 > 0$)

(i) H_a is a null fourvector if $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$, for $R_o > 0$, $H_a \overset{o}{P}{}^a \neq 0$.

(ii) H_a is timelike future pointing, $\varepsilon_1 = 0$, $\varepsilon_2 = +1$

(iii) H_a is spacelike, $\varepsilon_2 = 0$, $\varepsilon_1 = \pm 1$.

Introducing the notation

$$\bar{\psi}^A = \begin{pmatrix} \bar{\xi}^A \\ \\ \bar{\eta}^A \end{pmatrix}$$

Then

$$\bar{\psi}^{A} \psi^{T\,A'} = \begin{pmatrix} \bar{\xi}^{A} \xi^{A'} & \bar{\xi}^{A} \eta^{A'} \\ \bar{\eta}^{A} \xi^{A'} & \bar{\eta}^{A} \eta^{A'} \end{pmatrix}$$

In this notation (3-3) takes the form

$$H^{AA'} = \mathrm{Tr}\left[(\varepsilon_2 \, .I + \varepsilon_1 \, S) \, \bar{\psi}^{A} \psi^{T\,A'} \right] \qquad (3\text{-}4)$$

where I is the 2 x 2 identity matrix and S is the matrix

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Given the quantities $H^{AA'}$, that means, given the fourvector H_a, the quantities $\bar{\psi}^{A}$ are not uniquely determined by (3-4). Indeed, under $\bar{\psi}^{A} \to \bar{\psi}'^{A'}$ (*) for

$$\bar{\psi}^{A} = U \bar{\psi}'^{A}, \quad U^{\dagger} U = 1$$

(U is a 2 x 2 unitary matrix) the same values for $H^{AA'}$ are obtained:

$$H'^{AA'} = H^{AA'}$$

Due to this indetermination in the form of the $\bar{\psi}^{A}$ (and consequently in the form of the $\bar{\xi}^{A}$ and $\bar{\eta}^{A}$), these quantities will be used only through the invariant combination written in the right hand side of (3-4).

The $\tau^{(AB)}$ of Eq. (3-2) is of the form

$$\tau^{(AB)} = \mu^{(A} \underset{\nu}{-} {}^{B)} \tag{3-5}$$

The $f(m^2)$ in equations (3-1) and (3-2) is taken as a real quantity restricted by the condition

$$\lim_{m \to 0} f(m^2) = 0 \tag{3-6}$$

From (3-1) we get, by imposing that $P^2 = m^2$ with real m(the mass term):

$$f^2 H^2 + 4af - m^2 = 0 \tag{3-7}$$

where,

$$a = 1/2 \overset{o}{P}_a H^a$$

Eq. (3-7) is an algebraic equation for determining the function $f(m^2)$. If H_a is timelike or spacelike the solution of this equation which satisfies the condition (3-6) is

$$f(m^2) = -\frac{2a}{H^2} + \frac{1}{H^2} (4a^2 + H^2 m^2)^{1/2} \tag{3-8}$$

For a null H_a, ($H^2 = 0$), the solution for (3-6), (3-7) is simply

$$f(m^2) = \frac{m^2}{4a} \tag{3-9}$$

This last case is similar to Penrose's method. By taking $f = 1$ in (3-9) we obtain Penrose's result.

Considering (3-8) for H_a timelike, and transforming to a inertial frame where $\vec{H} = 0$, we obtain (taking $H_o > 0$),

$$P_o = K + f H_o = + (K^2 + m^2)^{1/2} > 0$$

(K is here a short $\overset{o}{P}_o$). Since a similar result holds for $R_o = fH_o > 0$, we conclude that $R_o > 0$, $H_o > 0$. This in turn implies that here $\varepsilon_2 = + 1$.

Another condition obtained from (3-8) is:

$$4a^2 + H^2 m^2 \geq 0$$

The case when this quantity vanishes may be verified by some spacelike H_a, but it has no interest since then $\underline{m}$ is not present in (3-8). Thus, this possibility is not considered. A timelike or spacelike H_a may satisfy the condition $4a^2 + H^2 m^2 > 0$. For H_a spacelike, we have

$$H^2 = - n^2 > \frac{-4a^2}{m^2}$$

Given the base spinors $\overline{o}^A$, $\overline{\tau}^A$ such that $\overline{o}_A \overline{\tau}^A = 1$, we may write for the two spinors $\overline{\pi}^A$, ω^A associated to the null rest mass initial system:

$$\overline{\pi}^A = \alpha\, \overline{o}^A + \beta\, \overline{\tau}^A$$

$$\omega^A = \gamma\, \overline{o}^A + \delta\, \overline{\tau}^A$$

defining another spinor $\bar{\rho}^A$,

$$\bar{\rho}^A = \mu\,\bar{o}^A + \nu\,\bar{\iota}^A$$

such that

$$\bar{\rho}_A \; \bar{\pi}^A = 1 \qquad (3\text{-}10\text{-}1)$$

$$\bar{\rho}_A \; \omega^A = 0 \qquad (3\text{-}10\text{-}2)$$

which is equivalent to introduce two 2 x 2 matrices, M and N, the first unimodular and the second singular:(*)

$$M = \begin{pmatrix} \mu & \nu \\ \alpha & \beta \end{pmatrix}, \qquad N = \begin{pmatrix} \mu & \nu \\ \gamma & \delta \end{pmatrix}$$

$$||M|| = 1, \qquad ||N|| = 0$$

($||M||$ = det M).

We have from (3-1), (3-2), (3-5) and (3-10):

(*) Given the complex numbers α, β, γ and δ (components of $\bar{\pi}^A, \omega^A$) the two conditions (3-10) (or $||M|| = 1$, $||N|| = 0$) fix uniquely the four real parameters contained in the components μ, ν of $\bar{\rho}^A$. Another way for showing this is to note that from (3-10-2) $\bar{\rho}_A = \lambda\omega_A$ for λ a complex, then (3-10-1) gives $\lambda\omega_A \bar{\pi}^A = 1$, since the product $\omega_A \bar{\pi}^A$ is known, we determine λ from this relation. Note also that $s = \mathrm{Re}(\lambda^{-1})$ where $\underline{s}$ is the helicity of null rest mass initial system.

$$\Lambda^{A'} \equiv \bar{\rho}_A \; P^{AA'} = \pi^{A'} + f\, \bar{\rho}_A \, H^{AA'} \tag{3-11}$$

$$\Omega^{B} \equiv \bar{\rho}_A \, \omega^{(AB)} = \omega^{B} + f\, \bar{\rho}_A \, \mu^{(A \, \bar{\nu} \, B)} \tag{3-12}$$

It is also possible to form combinations such as

$$T^{B} \equiv \frac{\omega_A \, \omega^{(AB)}}{\omega_C \, \bar{\pi}^{C}} = \omega^{B} + f\, \frac{\omega_A \, \mu^{(A \, \bar{\nu} \, B)}}{\omega_C \, \bar{\pi}^{C}}$$

$$\Delta^{A'} \equiv \frac{\omega_A \, P^{AA'}}{\omega_C \, \bar{\pi}^{C}} = \pi^{A'} + f\, \frac{\omega_A \, H^{AA'}}{\omega_B \, \bar{\pi}^{B}}$$

$$K^{B} \equiv \frac{\bar{\pi}_A \, \omega^{(AB)}}{\bar{\pi}_C \, \omega^{C}} = \bar{\pi}^{B} + f\, \frac{\bar{\pi}_A \, \mu^{(A \, \bar{\nu} \, B)}}{\bar{\pi}_C \, \omega^{C}}$$

but in this case we have to suppose that $\omega_A \, \bar{\pi}^{A}$ is different from zero, consequently this process will not apply for null twistors (the initial system is given by a null twistor). Due to this conclusion we will not use these type of relations, but only the expressions (3-11) and (3-12). Calling,

$$S^{A'} = \bar{\rho}_A \, H^{AA'}$$

$$R^{B} = \bar{\rho}_A \, \mu^{(A \, \bar{\nu} \, B)}$$

we have

$$\Lambda^{A'} = \pi^{A'} + f\, S^{A'} \qquad (3\text{-}13)$$

$$\Omega^{B} = \omega^{B} + f\, R^{B} \qquad (3\text{-}14)$$

with these spinors it is possible to construct the twistor

$$\overset{\alpha}{Y} = (\omega^{A} + f\, R^{A},\ \pi_{A'} + f\, S_{A'})$$

or

$$\overset{\alpha}{Y} = \overset{\alpha}{Z} + f\, J^{\alpha} \ , \quad J^{\alpha} = (R^{A},\ S_{A'})$$

The twistor J is associated to some null rest mass particle with momentum $W_{AA'} = \bar{S}_{A}\, S_{A'} = \bar{\rho}^{\,B}\, H_{BA'}\, H_{B'A}\, \rho^{B'}$, and with angular momentum $N_{AA'BB'} = \equiv i(R_{(A}\, \bar{S}_{B)}\ \varepsilon_{A'B'} + \bar{R}_{(A'}\, S_{B')}\ \varepsilon_{AB})$. Introducing the quantity

$$C^{\alpha}{}_{\gamma} = 1/2\ (\overset{\alpha}{Z}\, \bar{Z}_{\gamma} + \overset{\alpha}{Y}\, \bar{Y}_{\gamma})$$

which corresponds to the factor $E^{\alpha}{}_{\gamma}$ of the section 2, and such that

$$\lim_{m \to 0} C^{\alpha}{}_{\gamma} = \overset{\alpha}{Z}\, \bar{Z}_{\gamma}$$

We may form a combination similar to (2-1),

$$B^{\alpha\beta} = 2(C^{\alpha}{}_{\gamma}\, I^{\beta\gamma} + C^{\beta}{}_{\gamma}\, I^{\alpha\gamma}) \;.$$

This second rank twistor describes the massive particle according to the process presently considered. The fundamental characteristic of this method is that the mass m appears as an independent quantity, not bounded to the values assumed by the spinors under consideration.

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