Symplectic Matrix, Gauge Invariance and Dirac Brackets for Super-QED

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Abstract

The calculation of Dirac brackets (DB) using a symplectic matrix approach but in a Hamiltonian framework is discussed, and the calculation of the DB for the supersymmetric extension of QED (super-QED) is shown. The relation between the zero-mode of the pre-symplectic matrix and the gauge transformations admitted by the model is verified. A general prescription to construct Lagrangeans linear in the velocities is also presented.

Introduction

The quantization of singular field theories can be formulated, following Dirac [1], in the framework of a Hamiltonian formalism, by mapping the so-called Dirac brackets (DB) into quantum commutators. In a more recent work [2], Faddeev-Jackiw (FJ) showed that using a Lagrangean formalism, in which the Lagrangean is of first degree in the velocities, these DB can also be obtained as the elements of the inverse of the symplectic matrix of the model. In [3], Barcelos-Wotzasek (BW) showed how the FJ approach can also be consistently used to obtain these DB even when the model under consideration is constrained from the geometric point of view[1]. The ideas presented in [2, 3] provided a useful framework for further work too, as for instance the quantization of singular systems in superspace [4], and establishing the connection between the gauge invariance of a given model and the zero-modes of its pre-symplectic matrix [5].

In this paper, the calculation of Dirac brackets using a symplectic matrix approach but in a Hamiltonian framework is discussed, and the calculation of the DB for the supersymmetric extension of QED (super-QED) [6] is presented.

The exposition is organized as follows. In sec. 1, it is shown that the symplectic matrix of a model can be obtained directly from its Hamiltonian, hence without having to describe the model in terms of a Lagrangean of first degree in the velocities (as is implicit, for instance, in [10]). This possibility is then exploited in sec. 2, where we work on super-QED in a Hamiltonian framework by first explicitly determining the form of the gauge transformations admitted by the theory from the zero modes of the associated pre-symplectic matrix. The gauge is then fixed and the DB of the model are obtained. In addition, for when a Lagrangean description is preferred, in sec. 3 a general prescription for obtaining a first degree Lagrangean equivalent to another given one is shown. This prescription is independent of the original Lagrangean's degree in the velocities, and appears as an appropriate alternative when working with singular models defined in superspace.

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[1]This case occurs when the model has second-class constraints in the Dirac approach, and was not covered in [2]

1 Symplectic matrix and Hamiltonian formalism for singular systems

Consider a system with N degrees of freedom described by a singular Lagrangean $L(q_i, \dot{q}_i)$, i.e., a system with the Hessian matrix rank $R < N$. After defining the momenta, we can express[2] $N - R$ momenta p_ρ $(\rho : R + 1 \to N)$ and R velocities $\dot{q}_a$ $(a : 1 \to R)$ as in [7]

$$\dot{q}_a \approx f_a(q_i, p_b, \dot{q}_\rho), \tag{1}$$

$$p_\rho \approx g_\rho(q_i, p_b) \tag{2}$$

where $i : 1 \to N$ and $b : 1 \to R$. We will call the $\dot{q}_a$ *invertible* and the $\dot{q}_\rho$ *non-invertible*. Following [7], the Hamiltonian can then be written as[3]

$$H = p_a f_a(q_i, p_b, \dot{q}_\rho) + g_\rho(q_i, p_b)\dot{q}_\rho - L(q_i, f_a(q_i, p_b, \dot{q}_\rho), \dot{q}_\rho). \tag{3}$$

and the corresponding Hamilton equations as

$$\dot{q}_a \approx \frac{\partial H}{\partial p_a} - \dot{q}_\rho \frac{\partial g_\rho}{\partial p_a} \tag{4}$$

$$\dot{p}_a \approx \dot{q}_\rho \frac{\partial g_\rho}{\partial q_a} - \frac{\partial H}{\partial q_a} \tag{5}$$

$$\frac{dg_\gamma(q_i, p_b)}{dt} \approx \dot{q}_\rho \frac{\partial g_\rho}{\partial q_\gamma} - \frac{\partial H}{\partial q_\gamma} \qquad (\gamma : R + 1 \to N) \tag{6}$$

Defining the $N - R$ *primary constraints* ϕ_ρ of such a model as

$$\phi_\rho = p_\rho - g_\rho(q_i, p_b) \approx 0 \tag{7}$$

the Poisson Brackets (PB) of ϕ_ρ with the Hamiltonian and between themselves are given by

$$
\begin{aligned}
h_\rho &\equiv \{\phi_\rho, H\} = -\frac{\partial g_\rho}{\partial q_b}\frac{\partial H}{\partial p_b} + \frac{\partial g_\rho}{\partial p_b}\frac{\partial H}{\partial q_b} - \frac{\partial H}{\partial q_\rho}, \\
P_{\rho\gamma} &\equiv \{\phi_\rho, \phi_\gamma\} = \frac{\partial g_\rho}{\partial q_b}\frac{\partial g_\gamma}{\partial p_b} - \frac{\partial g_\rho}{\partial p_b}\frac{\partial g_\gamma}{\partial q_b} - \frac{\partial g_\rho}{\partial q_\gamma} + \frac{\partial g_\gamma}{\partial q_\rho}
\end{aligned}
\tag{8}
$$

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[2]The definition of weak equalities used along this paper follows that given in [1].

[3]Throughout this paper we use the convention of sum over repeated indices.

Roughly speaking, the strategy in Dirac's method consists of substituting the expressions (4) and (5) into (6) to obtain a description involving only the non-invertible velocities:

$$P_{\rho\gamma}\dot{q}_{\gamma} = -h_{\rho} \tag{9}$$

Now, when trying to invert this system to express the velocities $\dot{q}_{\gamma}$ as functions of the coordinates and momenta, we encounter two different cases [7].

1.1 Case $\det P \neq 0$

The simplest case occurs when the matrix P is invertible (not all the h_{ρ} are weakly null), and we have

$$\dot{q}_{\gamma} \approx -P^{-1}{}_{\gamma\rho}h_{\rho} \tag{10}$$

Using this result, the time derivative of an arbitrary function $A(q_i, p_i, t)$ can be expressed as

$$\dot{A} \approx \underbrace{\{A, H\} - \{A, \phi_{\gamma}\}P^{-1}{}_{\gamma\rho}\{\phi_{\rho}, H\}}_{\text{Dirac bracket between } A \text{ and } H} + \frac{\partial A}{\partial t} \tag{11}$$

and the definition of Dirac brackets can be seen as an extension of the formula above for the case of two arbitrary functions A_1 and A_2:

$$\{A_1, A_2\}_D = \{A_1, A_2\} - \{A_1, \phi_{\gamma}\}P^{-1}{}_{\gamma\rho}\{\phi_{\rho}, A_2\} \tag{12}$$

These brackets, in turn, are the cornerstone both in the Dirac and the FJ methods. Now, still in a Hamiltonian framework, if instead of removing the invertible velocities as done in (9), we work with all the velocities, as in the FJ approach, and rewrite equations (4), (5) and (6) as

$$\underbrace{\begin{pmatrix} 0 & \frac{\partial g_{\rho}}{\partial q_b} & -\delta_{ba} \\ -\frac{\partial g_{\gamma}}{\partial q_a} & -\frac{\partial g_{\gamma}}{\partial q_{\rho}} + \frac{\partial g_{\rho}}{\partial q_{\gamma}} & -\frac{\partial g_{\gamma}}{\partial p_a} \\ \delta_{ba} & \frac{\partial g_{\rho}}{\partial p_b} & 0 \end{pmatrix}}_{F^{(0)}{}_{(N+R)\times(N+R)}} \begin{pmatrix} \dot{q}_a \\ \dot{q}_{\rho} \\ \dot{p}_a \end{pmatrix} \approx \begin{pmatrix} \frac{\partial H}{\partial q_b} \\ \frac{\partial H}{\partial q_{\gamma}} \\ \frac{\partial H}{\partial p_b} \end{pmatrix} \tag{13}$$

we directly arrive at the symplectic matrix ($F^{(0)}$ above) for such a model. Actually, from (8), (12) and (13), it follows that

$$(F^{(0)})^{-1}_{\sigma,\omega} = \{y_{\sigma}, y_{\omega}\}_D \tag{14}$$

where $\{y_\sigma, y_\omega\}_D$ are the Dirac Brackets for y_σ, y_ω, $(\sigma, \omega) : 1 \to N + R$, $y_i = q_i$ and $y_{N+a} = p_a$.

1.2   Case $\det P = 0$

The interesting case is that in which the determinant of P is weakly null; that is, the system is constrained from a geometric point of view, and P has $M < (N - R)$ null eigenvectors. In such a case, multiplying both sides of (9) by these eigenvectors, we obtain M relations (constraints) of the form

$$\chi_A(q_i, p_b) \approx 0 \qquad (A : 1 \to M) \tag{15}$$

Differentiating these expressions with respect to time and using (4) and (5), we obtain

$$\dot{\chi}_A \approx \{\chi_A, H\} + \dot{q}_\gamma \{\chi_A, \phi_\gamma\} \approx 0 \tag{16}$$

These M expressions can be used to extend the system (9) as follows:

$$\underbrace{\begin{pmatrix} \{\phi_\rho, \phi_\gamma\} \\ \{\chi_A, \phi_\gamma\} \end{pmatrix}}_{C^{(0)}(N-R+M)\times(N-R)} \underbrace{(\dot{q}_\rho)}_{(N-R)\times 1} \approx \begin{pmatrix} \{H, \phi_\rho\} \\ \{H, \chi_A\} \end{pmatrix} \tag{17}$$

Although $C^{(0)}$ above is not a square matrix, it is possible to add M columns to it with the purpose of making it square without altering the content of the system, via

$$\begin{pmatrix} \{\phi_\rho, \phi_\gamma\} & J_{\rho B} \\ \{\chi_A, \phi_\gamma\} & K_{AB} \end{pmatrix} \begin{pmatrix} \dot{q}_\rho \\ 0 \end{pmatrix} \approx \begin{pmatrix} \{H, \phi_\rho\} \\ \{H, \chi_A\} \end{pmatrix} \tag{18}$$

where $B : 1 \to M$, and $J_{\rho B}$ and K_{AB} are arbitrary. Now, to obtain the generalized antisymmetric Dirac brackets, we proceed as in [1] and take $J_{\rho B} = \{\phi_\rho, \chi_B\}$ and $K_{AB} = \{\chi_A, \chi_B\}$, so that C becomes

$$C = \begin{pmatrix} \{\phi_\rho, \phi_\gamma\} & \{\phi_\rho, \chi_B\} \\ \{\chi_A, \phi_\gamma\} & \{\chi_A, \chi_B\} \end{pmatrix} \tag{19}$$

Using ζ_μ (or ζ_ν) to represent any of the constraints ϕ_ρ or χ_A, the system (17) can be written as

$$\{\zeta_\mu, H\} + \{\zeta_\mu, \zeta_\rho\}\dot{q}_\rho \approx 0 \tag{20}$$

and provided the model has no gauge invariance, by repeating the steps represented by equations (15) to (19) it is always possible to extend C so as to have $\det(C) \neq 0$, in turn leading to

$$\dot{q}_\rho \approx -C^{-1}{}_{\rho\mu}\{\zeta_\mu, H\}, \tag{21}$$

$$0 \approx C^{-1}{}_{A\mu}\{\zeta_\mu, H\} \tag{22}$$

Using (4), (5), (22) and (21), the time derivative of an arbitrary function A can then be expressed as

$$\dot{A} \approx \underbrace{\{A, H\} - \{A, \zeta_\mu\}C^{-1}{}_{\mu\nu}\{\zeta_\nu, H\}}_{\text{Dirac bracket between } A \text{ and } H} + \frac{\partial A}{\partial t} \tag{23}$$

from where the DB between two arbitrary functions A_1 and A_2 in the case $\det P = 0$ becomes

$$\{A_1, A_2\}_D = \{A_1, A_2\} - \{A_1, \zeta_\mu\}C^{-1}{}_{\mu\nu}\{\zeta_\nu, A_2\} \tag{24}$$

To construct the symplectic matrix directly from the Hamiltonian when $\det P = 0$, we proceed as follows. First, as done in the case $\det P \neq 0$, we rewrite the system composed by (4), (5) and (6) as in (13). The resulting matrix $F^{(0)}$ is now singular[4]. We then multiply both sides of (13) by the M null eigenvectors of $F^{(0)}$, obtaining the M relations (15). Taking the time derivative of these constraints, we extend our system in the same way as in (17), obtaining

$$\underbrace{\begin{pmatrix} 0 & \frac{\partial g_\rho}{\partial q_b} & -\delta_{ba} \\ -\frac{\partial g_\gamma}{\partial q_a} & -\frac{\partial g_\gamma}{\partial q_\rho} + \frac{\partial g_\rho}{\partial q_\gamma} & -\frac{\partial g_\gamma}{\partial p_a} \\ \delta_{ba} & \frac{\partial g_\rho}{\partial p_a} & 0 \\ -\frac{\partial \chi_A}{\partial q_a} & -\frac{\partial \chi_A}{\partial q_\rho} & -\frac{\partial \chi_A}{\partial p_a} \end{pmatrix}}_{F^{(1)}{}_{(N+R+M)\times(N+R)}} \begin{pmatrix} \dot{q}_a \\ \dot{q}_\rho \\ \dot{p}_a \end{pmatrix} \approx \begin{pmatrix} \frac{\partial H}{\partial q_b} \\ \frac{\partial H}{\partial q_\gamma} \\ \frac{\partial H}{\partial p_b} \\ 0 \end{pmatrix} \tag{25}$$

The procedure for turning square the matrix $F^{(1)}$ above is the same as that used for the matrix $C^{(0)}$ in the Dirac method[5]: we add M columns without altering the contents of

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[4] $F^{(0)}$ is called pre-symplectic in order to point out that it is not invertible.

[5] The procedure used here to turn F a square matrix is also equivalent to the one adopted by Barcelos-Wotzasek [3] in order to enlarge the pre-symplectic matrix in the context of the FJ method.

the system, via

$$
\begin{pmatrix}
0 & \frac{\partial g_\rho}{\partial q_b} & -\delta_{ba} & \frac{\partial \chi_A}{\partial q_b} \\
-\frac{\partial g_\gamma}{\partial q_a} & -\frac{\partial g_\gamma}{\partial q_\rho} + \frac{\partial g_\rho}{\partial q_\gamma} & -\frac{\partial g_\gamma}{\partial p_a} & \frac{\partial \chi_A}{\partial q_\gamma} \\
\delta_{ba} & \frac{\partial g_\rho}{\partial p_a} & 0 & \frac{\partial \chi_A}{\partial p_b} \\
\frac{-\partial \chi_A}{\partial q_a} & \frac{-\partial \chi_A}{\partial q_\rho} & \frac{-\partial \chi_A}{\partial p_a} & 0
\end{pmatrix}
\underbrace{}_{F^{(1)}{}_{(N+R+M)\times(N+R+M)}}
\begin{pmatrix}
\dot{q}_a \\
\dot{q}_\rho \\
\dot{p}_a \\
0
\end{pmatrix}
\approx
\begin{pmatrix}
\frac{\partial H}{\partial q_b} \\
\frac{\partial H}{\partial q_\gamma} \\
\frac{\partial H}{\partial p_b} \\
0
\end{pmatrix}
\tag{26}
$$

Now $F^{(1)}$ is antisymmetric. If the model has no gauge invariance, either $F^{(1)}$ is invertible, or enlarging $F^{(1)}$ as explained, again, a finite number of times, will lead to the required invertible matrix. The resulting matrix is the symplectic matrix of the model, and the elements of its inverse are then the fundamental DB (see [3]). If the model has gauge invariance of some type, we then need to remove it by fixing the gauge (i.e., introducing related terms in the formulation of the model), after which the procedure just explained will render the symplectic invertible matrix we are looking for (see sec. 2).

The procedure outlined shows how the symplectic matrix (13) can be built directly from the Hamiltonian, as opposed to setting up a first degree Lagrangean (FJ method). This amounts to working out a constrained system having secondary constraints, using a Hamiltonian description, but involving all the velocities, instead of just the non-invertible ones as in Dirac's method. Furthermore, using equivalent arguments, one can work out the system using only the non-invertible velocities, but describing the model with a Lagrangean of first degree in the velocities. In summary, from these considerations, what appears relevant is the choice of whether or not to work with all the velocities, as opposed to the choice between a Hamiltonian or Lagrangean framework.

Example

As an example of the use of the technique just described, consider the case of a relativistic point particle. This example is interesting because the description is gauge (scale) invariant; the Lagrangean is given by [6]

$$
L = -m \sqrt{u_0{}^2 - u_1{}^2 - u_2{}^2 - u_3{}^2},
\tag{27}
$$

_____

[6]The space-time signature is $(-1, +1, +1, +1)$.

where $u_\mu = \frac{dx_\mu}{d\tau}$, and τ is the arbitrary monotonic parameter [8]. In this model, there are three relations of the form (1) and one relation of the form (2):

$$u_0 \quad \approx \quad \frac{p^0 u_3}{\sqrt{Z}}, \qquad u_1 \quad \approx \quad -\frac{p^1 u_3}{\sqrt{Z}}, \qquad u_2 \quad \approx \quad -\frac{p^2 u_3}{\sqrt{Z}}, \tag{28}$$

$$p^3 \quad \approx \quad \overset{+}{-} \sqrt{p^{0\,2} - p^{1\,2} - p^{2\,2} - m^2} \tag{29}$$

where $Z = (p^0)^2 - (p^1)^2 - (p^2)^2 - m^2$. These equations lead to the canonical Hamiltonian (3): $H = 0$. The related Hamilton equations are: those shown in (28), of the form (4); three other ones of the form (5)

$$\frac{dp^0}{d\tau} \approx 0, \qquad \frac{dp^1}{d\tau} \approx 0, \qquad \frac{dp^2}{d\tau} \approx 0; \tag{30}$$

and one of the form (6)

$$p^2 \frac{dp^2}{d\tau} - p^0 \frac{dp^0}{d\tau} + p^1 \frac{dp^1}{d\tau} \quad \approx \quad 0 \tag{31}$$

From these equations, the pre-symplectic matrix $F^{(0)}$ (see (13)) is given by

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\dfrac{p^0}{\sqrt{Z}} & \dfrac{p^1}{\sqrt{Z}} & \dfrac{p^2}{\sqrt{Z}} \\ -1 & 0 & 0 & \dfrac{p^0}{\sqrt{Z}} & 0 & 0 & 0 \\ 0 & -1 & 0 & -\dfrac{p^1}{\sqrt{Z}} & 0 & 0 & 0 \\ 0 & 0 & -1 & -\dfrac{p^2}{\sqrt{Z}} & 0 & 0 & 0 \end{bmatrix}}_{F^{(0)}} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ \frac{dp^0}{d\tau} \\ \frac{dp^1}{d\tau} \\ \frac{dp^2}{d\tau} \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{32}$$

$F^{(0)}$ is singular and has one null eigenvector

$$\left[\; -\frac{p^0}{\sqrt{Z}}, \quad \frac{p^1}{\sqrt{Z}}, \quad \frac{p^2}{\sqrt{Z}}, \quad 1, \quad 0, \quad 0, \quad 0, \; \right] \tag{33}$$

Multiplying both sides of (32) by this eigenvector does not lead to new constraints. This is so because the model is gauge (scale) invariant, so that to proceed further we need to first

fix the gauge. This is done here as in [8], by introducing $x_0 - \tau \approx 0$; the corresponding new equation is $u_0 = 1$ and leads to the addition of one line to $F^{(0)}$. The system of equations then becomes

$$
\underbrace{\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -\dfrac{p^0}{\sqrt{Z}} & \dfrac{p^1}{\sqrt{Z}} & \dfrac{p^2}{\sqrt{Z}} \\
-1 & 0 & 0 & \dfrac{p^0}{\sqrt{Z}} & 0 & 0 & 0 \\
0 & -1 & 0 & -\dfrac{p^1}{\sqrt{Z}} & 0 & 0 & 0 \\
0 & 0 & -1 & -\dfrac{p^2}{\sqrt{Z}} & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}}_{F^{(1)}}
\begin{bmatrix}
u_0 \\ u_1 \\ u_2 \\ u_3 \\ \frac{dp^0}{d\tau} \\ \frac{dp^1}{d\tau} \\ \frac{dp^2}{d\tau}
\end{bmatrix}
\approx
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1
\end{bmatrix} ,
\tag{34}
$$

The matrix $F^{(1)}$ is not a square matrix, but it is possible to make it so by adding to it one column without altering the content of the system. We choose the elements of this new column so that the resulting matrix, $F^{(2)}$, is antisymmetric. It now turns out that $F^{(2)}$ is invertible; hence, no more extensions are required and $F^{(2)}$ is already the symplectic matrix of the model, and the elements of its inverse

$$
\underbrace{\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & -\dfrac{p^1}{p^0} & -1 & 0 & \dfrac{p^1}{p^0} \\
0 & 0 & 0 & 0 & -\dfrac{p^2}{p^0} & 0 & -1 & \dfrac{p^2}{p^0} \\
0 & 0 & 0 & 0 & \dfrac{\sqrt{Z}}{p^0} & 0 & 0 & -\dfrac{\sqrt{Z}}{p^0} \\
0 & \dfrac{p^1}{p^0} & \dfrac{p^2}{p^0} & -\dfrac{\sqrt{Z}}{p^0} & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & -\dfrac{p^1}{p^0} & -\dfrac{p^2}{p^0} & \dfrac{\sqrt{Z}}{p^0} & 0 & 0 & 0 & 0
\end{bmatrix}}_{(F^{(2)})^{-1}}
\tag{35}
$$

are the fundamental DB for the problem

$$
\{p_\mu, x_\nu\}_D = -g_{\mu\nu} + g_{\mu 0}\frac{p_\nu}{p_0},
\tag{36}
$$

$$
\{p_\mu, p_\nu\}_D = \{x_\mu, x_\nu\}_D = 0.
\tag{37}
$$

These results are in agreement with those shown in [8] obtained using the Dirac method.

2 Dirac brackets for super-QED

In this section we derive the Dirac brackets for super-QED as the elements of the inverse of the symplectic matrix, in turn built directly from the Hamiltonian, as explained in the previous section. Besides, super-QED is a supersymmetric gauge invariant model, and hence it is not possible to set up the symplectic matrix until this gauge invariance is determined and fixed. This obstacle is reflected by the fact that the pre-symplectic matrix $F^{(0)}$ has null eigenvectors which do not lead to new constraints. For such a model, it has been shown [5] that there is a relation between these zero modes of $F^{(0)}$ and the gauge invariance of the theory. We then proceed as follows. First, for completeness, this connection between zero modes and gauge invariance is briefly reviewed. The gauge

transformations admitted by super-QED are then derived and the gauge fixed. Finally, we use the technique explained in the previous section to obtain the symplectic matrix of the model and compute the DB as the elements of its inverse.

2.1   Zero-modes of the pre-symplectic matrix and gauge invariance

Consider a system with N degrees of freedom described by a singular Lagrangean $L(q_i, \dot{q}_i)$ (rank $R < N$). As shown in sec. 3, such a system can always be described by a Lagrangean $L^{(1)}$ linear in the velocities[7]

$$L^{(1)}(q_i, \dot{q}_i, p_b) = p_a \dot{q}_a + g_\rho(q_i, p_b)\dot{q}_\rho - H(p_b, q_i) \qquad (38)$$

When the system has secondary constraints, its pre-symplectic matrix $F^{(0)}$ is singular, so that we proceed as explained in [3], enlarging $F^{(0)}$ by using its null eigenvectors until the enlarged $F^{(0)}$ (denoted as $F^{(1)}$ in equation (26)) has no more null eigenvectors or such eigenvectors exist but they generate no more constraints. According to [5], we then write the functional variation of the action in terms of $F^{(1)}$ as

$$\delta S = \int \delta y_\omega \left(F^{(1)}_{\omega\sigma} \dot{y}_\sigma - \frac{\partial H}{\partial y_\omega} \right) \, dt = 0, \qquad (39)$$

where $(\sigma, \omega) : 1 \to N + R$, and y_ω represents both q_i and p_b. If the matrix $F^{(1)}$ is singular with n zero modes $V^{(n)}$, taking $\delta y_\omega = \varsigma V^{(n)}_\omega$ (ς is an infinitesimal parameter) we have

$$\delta y_\omega \frac{\partial H}{\partial y_\omega} \;=\; \varsigma V^{(n)}_\omega \frac{\partial H}{\partial y_\omega} \;\approx\; 0 \qquad (40)$$

Recalling that these $V^{(n)}$ don't lead to new constraints, a transformation δy satisfying the above is by definition a gauge transformation with $V^{(n)}$ playing the role of the infinitesimal generators.

2.2   Zero-modes and gauge invariance in super-QED

Following [6], we write the Lagrangean for the four-dimensional supersymmetric generalization of QED (WZ gauge) in terms of the field components of the supersymmetric

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[7]For the symbols and indices entering (38) see the conventions introduced with equations (1), (2) and (3).

multiplet as[8]

$$\mathcal{L} = \tfrac{1}{2}D^2 - \tfrac{1}{4}v_{mn}v^{mn} - i\lambda\sigma^m\partial_m\bar{\lambda} + F_+F^*_+ + F_-F^*_- + A^*_+\Box A_+ + +A^*_-\Box A_-$$

$$+i(\partial_n\bar{\psi}_+\bar{\sigma}^n\psi_+ + \partial_n\bar{\psi}_-\bar{\sigma}^n\psi_-) + \tfrac{1}{2}ev_n(\bar{\psi}_+\bar{\sigma}^n\psi_+ - \bar{\psi}_-\bar{\sigma}^n\psi_-)$$

$$+\tfrac{1}{2}iev^n(A^*_+\partial_n A_+ - \partial_n A^*_+ A_+ - A^*_-\partial_n A_- + \partial_n A^*_- A_-)$$

$$-\tfrac{1}{2}ie\sqrt{2}(A_+\bar{\psi}_+\bar{\lambda} - A^*_+\psi_+\lambda - A_-\bar{\psi}_-\bar{\lambda} + A^*_-\psi_-\lambda)$$

$$+\tfrac{1}{2}eD(A^*_+A_+ - A^*_-A_-) - \tfrac{1}{4}e^2v_nv^n(A^*_+A_+ + A^*_-A_-)$$

$$+m(A_+F_- + A_-F_+ - \psi_+\psi_- - \bar{\psi}_+\bar{\psi}_- + A^*_+F^*_- + A^*_-F^*_+)$$

$$(41)$$

where m is the electron mass, D is a real scalar field, v^n is a real vector field, $v_{mn} = \partial_m v_n - \partial_n v_m$, F_+, F_-, A_+ and A_- are complex scalar fields, ψ_+, $\bar{\psi}_-$, ψ_-, $\bar{\psi}_+$, $\bar{\lambda}$ and λ are Weyl spinors (2 components), and σ^m are the Pauli matrices[9]. Now, instead of proceeding by setting up a Lagrangean of first degree in the velocities, we will work on the model using a Hamiltonian description, for which we define the momenta

$$\tau \approx \frac{\partial\mathcal{L}}{\partial(\partial_0 A_+)} = \tfrac{1}{2}iev^0 A^*_+ + \partial_0 A^*_+ \qquad \tau_1 \approx \frac{\partial\mathcal{L}}{\partial(\partial_0 A^*_+)} = -\tfrac{1}{2}iev^0 A_+ + \partial_0 A_+,$$

$$\pi_i \approx \frac{\partial\mathcal{L}}{\partial(\partial_0 v^i)} = \partial_i v^0 + \partial_0 v^i \qquad \pi_{\bar{\lambda}^b} \approx \frac{\partial\mathcal{L}}{\partial(\partial_0 \bar{\lambda}^b)} = i\sigma^0{}_{ab}\lambda^a$$

$$\pi_{\bar{\psi}^b_+} \approx \frac{\partial\mathcal{L}}{\partial(\partial_0 \bar{\psi}^b_+)} = i\bar{\sigma}^0_{b,a}\psi^a_+ \qquad \pi_{\bar{\psi}^b_-} \approx \frac{\partial\mathcal{L}}{\partial(\partial_0 \bar{\psi}^b_-)} = i\bar{\sigma}^0_{b,a}\psi^a_-$$

$$\kappa \approx \frac{\partial\mathcal{L}}{\partial(\partial_0 A_-)} = -\tfrac{1}{2}iev^0 A^*_- + \partial_0 A^*_- \qquad \kappa_1 \approx \frac{\partial\mathcal{L}}{\partial(\partial_0 A^*_-)} = \tfrac{1}{2}iev^0 A_- + \partial_0 A_-$$

$$(42)$$

and note that the following momenta are all primary constraints:

$$0 = \frac{\partial\mathcal{L}}{\partial(\partial_0\lambda^b)} = \frac{\partial\mathcal{L}}{\partial(\partial_0 v^0)} = \frac{\partial\mathcal{L}}{\partial(\partial_0 D)} = \frac{\partial\mathcal{L}}{\partial(\partial_0 F_+)} = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi^b_+)} = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi^b_-)} = \frac{\partial\mathcal{L}}{\partial(\partial_0 F_-)}$$

$$= \frac{\partial\mathcal{L}}{\partial(\partial_0 F^*_+)} = \frac{\partial\mathcal{L}}{\partial(\partial_0 F^*_-)}$$

$$(43)$$

---

[8]In (41) $(m,n) : 0 \to 3$ and the space-time signature is $(-1, +1, +1, +1)$.

[9]The rule for omitted contracted spinor indices is $\psi\eta = \eta\psi = \eta^\alpha\psi_\alpha$, $\bar{\psi}\bar{\eta} = \bar{\eta}\bar{\psi} = \bar{\eta}_{\dot{\alpha}}\psi^{\dot{\alpha}}$.

The canonical Hamiltonian is then given by

$$\mathcal{H} = \tfrac{1}{4}v^{i2}A_-A_-^* + \tfrac{1}{2}ie\sqrt{2}A_-^*\epsilon_{ab}\psi_-^a\lambda^b + \tfrac{1}{4}e^2v^{i2}A_+A_+^*$$

$$-\tfrac{1}{2}iev^i\partial_iA_-^*A_- + \tfrac{1}{2}eDA_-A_-^* + \tfrac{1}{2}iev^i\partial_iA_-A_-^* - \tfrac{1}{2}eDA_+A_+^*$$

$$+\tfrac{1}{2}ie\sqrt{2}A_-\epsilon_{\dot{a}\dot{b}}\bar{\psi}_-^{\dot{a}}\bar{\lambda}^{\dot{b}} - \tfrac{1}{2}\partial_iv^j\partial_jv^i + \tfrac{1}{2}\partial_iv^{j2} - i\partial_i\bar{\psi}_-^{\dot{a}}\bar{\sigma}_{\dot{a}a}^i\psi_-^a$$

$$+\left(i\lambda^a\sigma_{a\dot{a}}^i\partial_i\bar{\lambda}^{\dot{a}}\right) + \tfrac{1}{2}\pi_i^2 - F_-F_-^* - \tfrac{1}{2}ie\sqrt{2}A_+^*\epsilon_{ab}\psi_+^a\lambda^b$$

$$-\tfrac{1}{2}ie\sqrt{2}A_+\epsilon_{\dot{a}\dot{b}}\bar{\psi}_+^{\dot{a}}\bar{\lambda}^{\dot{b}} + \partial_iA_-\partial_iA_-^* + \tfrac{1}{2}iev^i\partial_iA_+^*A_+ - \tfrac{1}{2}iev^i\partial_iA_+A_+^* \qquad (44)$$

$$+\tfrac{1}{2}ev_i\bar{\psi}_-^{\dot{a}}\bar{\sigma}_{\dot{a}a}^i\psi_-^a - \tfrac{1}{2}ev_0\bar{\psi}_-^{\dot{a}}\bar{\sigma}_{\dot{a}a}^0\psi_-^a + m\epsilon_{ab}\bar{\psi}_+^{\dot{a}}\bar{\psi}_-^{\dot{b}} - \tfrac{1}{2}ev_i\bar{\psi}_+^{\dot{a}}\bar{\sigma}_{\dot{a}a}^i\psi_+^a$$

$$+\tfrac{1}{2}ev_0\bar{\psi}_+^{\dot{a}}\bar{\sigma}_{\dot{a}a}^0\psi_+^a - i\partial_i\bar{\psi}_+^{\dot{a}}\bar{\sigma}_{\dot{a}a}^i\psi_+^a + \tau\tau_1 - F_+F_+^* + \tfrac{1}{2}i\kappa_1ev^0A_-^*$$

$$+\partial_iA_+\partial_iA_+^* - \tfrac{1}{2}i\tau_1ev^0A_+^* + \tfrac{1}{2}i\tau ev^0A_+ - \tfrac{1}{2}i\kappa ev^0A_- + m\epsilon_{ab}\psi_+^a\psi_-^b - \pi_i\partial_iv^0$$

$$-m\left(A_-^*F_+^* + A_+F_- + A_-F_+ + A_+^*F_-^*\right) - \tfrac{1}{2}D^2 + \kappa\kappa_1,$$

where the convention for the indices used throughout this section is: $(i,j) : 1 \to 3$, while $(a, \dot{a}, b, \dot{b}) : 1 \to 2$, and $\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We now set up the pre-symplectic 32×32 matrix of the model[10], here denoted $\mathcal{F}^{(0)}$, by writing the corresponding Hamilton equations as described in sec. 1:

$$\int \mathcal{F}^{(0)}{}_{\lambda\tau} \, \dot{\xi}_\tau \, d^3\vec{z} \approx \frac{\partial\mathcal{H}}{\partial\xi_\lambda}, \qquad (45)$$

where ξ is the field matrix, whose transpose is given by

$$[v^0, \ v^i, \ A_+, \ A_+^*, \ A_-, \ A_-^*, \ D, \ F_+, \ F_+^*, \ F_-, \ F_-^*, \ \psi_+^c, \ \bar{\psi}_+^{\dot{c}}, \ \psi_-^c, \ \bar{\psi}_-^{\dot{c}}, \ \lambda^c, \ \bar{\lambda}^{\dot{c}}, \ \pi^i, \ \tau, \ \tau_1, \ \kappa, \ \kappa_1]$$

$$(46)$$

---

[10]This number 32 is related to the fact that v^i and π^i have three components each, and ψ_+^c, $\bar{\psi}_+^{\dot{c}}$, ψ_-^c, $\bar{\psi}_-^{\dot{c}}$, λ^c and $\bar{\lambda}^{\dot{c}}$ have two components each.

and the indices $(\lambda, \tau) : 1 \to 32$. Due to the simultaneous presence of fields having integer and half integer spin, $\mathcal{F}^{(0)}$ is symmetric with respect to some of the values of its indices and antisymmetric with respect to other ones. The elements of $\mathcal{F}^{(0)}$ for which $\mathcal{F}^{(0)}_{\sigma,\omega} = -\mathcal{F}^{(0)}_{\omega,\sigma}$ are given by

$$\mathcal{F}^{(0)}_{(2,26)} = \mathcal{F}^{(0)}_{(3,27)} = \mathcal{F}^{(0)}_{(4,28)} = \mathcal{F}^{(0)}_{(5,29)} = \mathcal{F}^{(0)}_{(6,30)} = \mathcal{F}^{(0)}_{(7,31)} = \mathcal{F}^{(0)}_{(8,32)} = -\delta^3(\vec{z} - \vec{y}) \tag{47}$$

while the elements of $\mathcal{F}^{(0)}$ for which $\mathcal{F}^{(0)}_{\sigma,\omega} = \mathcal{F}^{(0)}_{\omega,\sigma}$ are given by

$$\mathcal{F}^{(0)}_{(14,16)} = \mathcal{F}^{(0)}_{(15,17)} = \mathcal{F}^{(0)}_{(18,20)} = \mathcal{F}^{(0)}_{(19,21)} = \mathcal{F}^{(0)}_{(22,24)} = \mathcal{F}^{(0)}_{(23,25)} = i\delta^3(\vec{z} - \vec{y}) \tag{48}$$

All the elements of $\mathcal{F}^{(0)}$ not mentioned above are identically zero. Calculating the zero-modes of $\mathcal{F}^{(0)}$ and using them as explained in sec. 1, we obtain the following constraints

$$\frac{\partial \mathcal{H}}{\partial v^0} \approx \frac{\partial \mathcal{H}}{\partial F_+} \approx \frac{\partial \mathcal{H}}{\partial F_+^*} \approx \frac{\partial \mathcal{H}}{\partial F_-} \approx \frac{\partial \mathcal{H}}{\partial F_-^*} \approx \frac{\partial \mathcal{H}}{\partial D} \approx 0 \tag{49}$$

The time derivatives of these constraints are used to enlarge $\mathcal{F}^{(0)}$ as shown in sec. 1, leading to a 38×38 matrix, $\mathcal{F}^{(1)}$, whose elements are such that, for $(\lambda, \tau) : 1 \to 32$, $\mathcal{F}^{(1)}_{\lambda\tau} = \mathcal{F}^{(0)}{}_{\lambda\tau}$, and the antisymmetric elements related to the new columns and rows are given by

$$\mathcal{F}^{(1)}_{(5,33)} = \frac{1}{2} ie\tau\delta^3(\vec{z} - \vec{y}), \qquad \mathcal{F}^{(1)}_{(6,33)} = -\frac{1}{2} ie\tau_1\delta^3(\vec{z} - \vec{y}),$$

$$\mathcal{F}^{(1)}_{(7,33)} = -\frac{1}{2} ie\kappa\delta^3(\vec{z} - \vec{y}), \qquad \mathcal{F}^{(1)}_{(8,33)} = \frac{1}{2} ie\kappa_1\delta^3(\vec{z} - \vec{y}),$$

$$\mathcal{F}^{(1)}_{(9,34)} = \mathcal{F}^{(1)}_{(10,36)} = \mathcal{F}^{(1)}_{(11,35)} = \mathcal{F}^{(1)}_{(12,38)} = \mathcal{F}^{(1)}_{(13,37)} = -\delta^3(\vec{z} - \vec{y}),$$

$$\mathcal{F}^{(1)}_{(5,37)} = \mathcal{F}^{(1)}_{(6,38)} = \mathcal{F}^{(1)}_{(7,35)} = \mathcal{F}^{(1)}_{(8,36)} = -m\,\delta^3(\vec{z} - \vec{y}), \tag{50}$$

$$\mathcal{F}^{(1)}_{(5,34)}{}^* = \mathcal{F}^{(1)}_{(6,34)} = i\,\mathcal{F}^{(1)}_{(29,33)} = i\,\mathcal{F}^{(1)}_{(30,33)}{}^* = -\frac{1}{2} eA_+\delta^3(\vec{z} - \vec{y}),$$

$$\mathcal{F}^{(1)}_{(7,34)}{}^* = \mathcal{F}^{(1)}_{(8,34)} = i\,\mathcal{F}^{(1)}_{(31,33)} = i\,\mathcal{F}^{(1)}_{(32,33)}{}^* = \frac{1}{2} eA_-\delta^3(\vec{z} - \vec{y}),$$

$$\mathcal{F}^{(1)}_{(26,33)} = \frac{\partial\delta^3(\vec{z} - \vec{y})}{\partial y^{(1)}}, \quad \mathcal{F}^{(1)}_{(27,33)} = \frac{\partial\delta^3(\vec{z} - \vec{y})}{\partial y^{(2)}}, \quad \mathcal{F}^{(1)}_{(28,33)} = \frac{\partial\delta^3(\vec{z} - \vec{y})}{\partial y^{(3)}},$$

The new symmetric elements of $\mathcal{F}^{(1)}$, in turn, are given by

$$
\mathcal{F}^{(1)}_{(14,33)} = \tfrac{1}{2}e\bar{\psi}^{(1)}_+\delta^3(\vec{z}-\vec{y}), \qquad
\mathcal{F}^{(1)}_{(15,33)} = \tfrac{1}{2}e\bar{\psi}^{(2)}_+\delta^3(\vec{z}-\vec{y}), \qquad
\mathcal{F}^{(1)}_{(16,33)} = -\tfrac{1}{2}e\psi^{(1)}_+\delta^3(\vec{z}-\vec{y}),
$$

$$
\mathcal{F}^{(1)}_{(17,33)} = -\tfrac{1}{2}e\psi^{(2)}_+\delta^3(\vec{z}-\vec{y}), \qquad
\mathcal{F}^{(1)}_{(18,33)} = -\tfrac{1}{2}e\bar{\psi}^{(1)}_-\delta^3(\vec{z}-\vec{y}), \qquad
\mathcal{F}^{(1)}_{(19,33)} = -\tfrac{1}{2}e\bar{\psi}^{(2)}_-\delta^3(\vec{z}-\vec{y}),
$$

$$
\mathcal{F}^{(1)}_{(20,33)} = \tfrac{1}{2}e\psi^{(1)}_-\delta^3(\vec{z}-\vec{y}), \qquad
\mathcal{F}^{(1)}_{(21,33)} = \tfrac{1}{2}e\psi^{(2)}_-\delta^3(\vec{z}-\vec{y})
$$

$$(51)$$

All the elements of $\mathcal{F}^{(1)}$ not mentioned above are identically zero. $\mathcal{F}^{(1)}$ has a zero-mode, M, given by

$$
M = [N,\ \partial_i U,\ -\tfrac{1}{2}iUeA_+,\ \tfrac{1}{2}iUeA_+^*,\ \tfrac{1}{2}iUeA_-,\ -\tfrac{1}{2}iUeA_-^*,\ 0,\ \tfrac{1}{2}iUemA_-^*,\ -\tfrac{1}{2}iUemA_-,\ -\tfrac{1}{2}iUemA_+^*,
$$

$$
\tfrac{1}{2}iUemA_+,\ \tfrac{1}{2}iUe\psi^c_+,\ \tfrac{1}{2}iUe\bar{\psi}^{\dot{c}}_+,\ \tfrac{1}{2}iUe\psi^c_-,\ -\tfrac{1}{2}iUe\bar{\psi}^{\dot{c}}_-,\ 0,\ 0,\ 0,\ \tfrac{1}{2}iUe\tau,\ -\tfrac{1}{2}iUe\tau_1,\ -\tfrac{1}{2}iUe\kappa,\ \tfrac{1}{2}iUe\kappa_1,
$$

$$
U,\ 0,\ 0,\ 0,\ 0,\ 0]
$$

$$(52)$$

where U and N are arbitrary functions of the space-time variables. This zero mode does not lead to new constraints, and, as shown in sec. 2, in such a case the elements of M can be taken as the infinitesimal gauge transformations ($\delta\rho_\kappa = \varsigma M_\kappa$ - ς is a infinitesimal parameter) leaving the Lagrangean invariant. So (52) implies

$$
\begin{aligned}
\delta(v^i) &= \varsigma\partial_i U, & \delta(A_+) &= -\tfrac{1}{2}\varsigma iUeA_+, & \delta(A_-) &= \tfrac{1}{2}\varsigma iUeA_-, \\
\delta(v^0) &= \varsigma N & \delta(F_+) &= -\tfrac{1}{2}\varsigma iUeF_+, & \delta(F_-) &= \tfrac{1}{2}\varsigma iUeF_-, \\
\delta(D) &= 0 & \delta(\psi^c_+) &= -\tfrac{1}{2}\varsigma iUe\psi^c_+, & \delta(\psi^c_-) &= \tfrac{1}{2}\varsigma iUe\psi^c_-, \\
\delta(\lambda^c) &= 0 & \delta(\pi_i) &= 0
\end{aligned}
$$

$$(53)$$

Regarding the infinitesimal transformation $\delta(v^0) = \varsigma N$, (42) together with $\delta(\pi_i) = 0$ imply $N = -\partial_0 U$, so that the infinitesimal transformation rule for v^0 is in fact given by $\delta(v^0) = -\varsigma\partial_0 U$. These results are in agreement with the gauge transformations admitted in super-QED shown in [6].

2.3   Dirac Brackets

In order to proceed further and determine the Dirac brackets of the model, we need to fix the gauge. We choose to work in the Feynman gauge, adding to the Lagrangean (41)

the term $-\frac{1}{2}(\partial_n v^n)^2$. After fixing the gauge, the momentum π_0 becomes

$$\pi_0 \approx \frac{\partial \mathcal{L}}{\partial(\partial_0 v^0)} = -\partial_0 v_0 - \partial_i v_i. \tag{54}$$

The new Hamiltonian is thus obtained by adding to (44) the term $-\frac{1}{2}\pi_0^2 - \pi_0 \partial_i v^i + \frac{1}{4}(\partial_i v^i)^2$. This addition introduces a few changes in the calculations performed in the previous section, leading to a non singular matrix $\mathcal{F}^{(2)}$ - the symplectic matrix for the model. These changes can be summarized as follows.

1. Due to the fixing of the gauge, $\frac{\partial \mathcal{H}}{\partial v^0}$, which was before taken weakly null (49), is now not null, so that we don't need to enlarge $\mathcal{F}^{(0)}$ with the derivative of this constraint as was done in the previous section. Hence, all the elements of $\mathcal{F}^{(0)}$ in equations (50) or (51) related to $\frac{\partial \mathcal{H}}{\partial v^0} \approx 0$ (i.e., those with line or column number 33) are just not present in $\mathcal{F}^{(2)}$.

2. The fixing of the gauge also turns v^0 *invertible*, so that it is necessary to incorporate π^0 to $\mathcal{F}^{(2)}$ and to the field matrix[11] (46); we inserted π^0 in position 26, before the π^i. The introduction of π^0 in turn leads to a new element different from zero, $\mathcal{F}^{(2)}{}_{(1,26)} = -\delta^3(\vec{z} - \vec{y})$. Now, the gauge fixing process led to the addition of one line and column at position 26 and the removal of one line and column at position 33. Hence, all the elements of $\mathcal{F}^{(0)}$ having column number between 26 and 32 appear in $\mathcal{F}^{(2)}$ with this number incremented by one (e.g., $\mathcal{F}^{(2)}_{(2,28)} = \mathcal{F}^{(0)}_{(2,27)}$); this is the case of all the elements of $\mathcal{F}^{(0)}$ entering (47).

3. For the same reasons, all the elements of $\mathcal{F}^{(0)}$ shown in (48) are present in $\mathcal{F}^{(2)}$ in the same position (all of them have number of line or column less than 26), and the same happens with all the elements of $\mathcal{F}^{(0)}$ appearing in (50) not having line or column number 33.

With these changes, the matrix $\mathcal{F}^{(2)}$ is not singular anymore, from where the elements of its inverse are the Dirac Brackets of the model,

$$\{\varphi_\theta(t, \vec{x}), \varphi_\eta(t, \vec{y})\}_D = (\mathcal{F}^{(2)})^{-1}{}_{\theta\eta}. \tag{55}$$

---

[11] We recall from (13) that the momenta entering the field matrix are those associated to the invertible velocities.

The expression of these DB in terms of the fields of the supermultiplet and the momenta is given by[12]

$$\{v^m, \pi_m\}_D = \delta^3(\vec{x} - \vec{y}),$$

$$\{A_+, \tau\}_D = \{A_-, \kappa\}_D = \delta^3(\vec{x} - \vec{y}),$$

$$\{F_+, \kappa_1\}_D = \{F_-, \tau_1\}_D = -m\delta^3(\vec{x} - \vec{y}),$$

$$\{\psi^c_+, \bar{\psi}^{\dot{c}}_+\}_D = \{\psi^c_-, \bar{\psi}^{\dot{c}}_-\}_D = i\delta^3(\vec{x} - \vec{y})\,\delta_{c\dot{c}}, \qquad (56)$$

$$\{\lambda^c, \bar{\lambda}^{\dot{c}}\}_D = i\delta^3(\vec{x} - \vec{y})\,\delta_{c\dot{c}},$$

$$\{D, \tau_1\}_D = \{D, \tau\}^*_D = -\frac{e}{2}A_+\delta^3(\vec{x} - \vec{y}),$$

$$\{D, \kappa_1\}_D = \{D, \kappa\}^*_D = \frac{e}{2}A_-\delta^3(\vec{x} - \vec{y}),$$

All the DB not shown above are identically zero.

3   Linearization of Lagrangeans

When using the FJ method, the starting point is a description of the system using a Lagrangean linear in the velocities [2]. Such a linear Lagrangean is built from the standard Lagrangean by introducing auxiliary coordinates. In the case of a quantum theory, both Lagrangeans will be equivalent if the standard one can be obtained from the linear one by integrating all the auxiliary coordinates in the functional generator of the latter. In the case of a classical theory, that equivalence is assured if, after removing the auxiliary coordinates using their equations of motion, the resulting equations for the physical fields are those that can be derived from the standard Lagrangean.

The method usually suggested in the literature for setting up a linear Lagrangean (see for instance [3]) works well with Lagrangeans quadratic in the velocities, and basically consists of replacing the quadratic terms as in

$$\dot{q}^2 \rightarrow 2\dot{q}\tau - \tau^2 \qquad (57)$$

where τ is an auxiliary coordinate introduced in the process. We note however that there is another possible prescription for linearizing singular Lagrangeans which is independent

---

[12]In all the DB of (56), $\{\varphi_\theta(t, \vec{x}), \varphi_\eta(t, \vec{y})\}_D$ is represented by $\{\varphi_\theta, \varphi_\eta\}_D$.

of the degree in the velocities, and also seems to us more convenient for dealing with su-
persymmetric models. In this case, (57) may not be enough and additional considerations
may be required. Our idea is based on the observation that, for a system of N degrees of
freedom described by a singular Lagrangean L with rank of the Hessian matrix $R < N$,
the corresponding Routh function (here denoted by G) satisfies[13]

$$
\begin{aligned}
G(p_a, q_i, \dot{q}_\rho) &= p_a f_a(q_i, p_b, \dot{q}_\rho) - L(q_i, f_a(q_i, p_b, \dot{q}_\rho), \dot{q}_\rho) \\
&= H(p_b, q_i) - g_\rho(q_i, p_b)\dot{q}_\rho
\end{aligned}
\tag{58}
$$

where, as in equation (2) $i : 1 \to N$, $(a, b) : 1 \to R$ and $\rho : R + 1 \to N$. From this
definition, and no matter the degree of L in the velocities, G will be linear in the $\dot{q}_\rho$;
hence a possible linear Lagrangean for the model in terms of the canonical Hamiltonian
is given by

$$
\begin{aligned}
L_f(p_a, q_i, \dot{q}_i) &= p_a \dot{q}_a - G(p_a, q_i, \dot{q}_\rho) \\
&= p_a \dot{q}_a - H(p_b, q_i) + g_\rho(q_i, p_b)\dot{q}_\rho
\end{aligned}
\tag{59}
$$

This prescription generalizes in some sense what we usually do in the case of non-singular
systems, where the linear Lagrangean can be written directly as $L_f(p_i, q_i, \dot{q}_i) = p_i \dot{q}_i - H(p_i, q_i)$, with the p_i playing the role of auxiliary coordinates.

Examples

We illustrate here the use of the prescription (59) to construct linear Lagrangeans in two
examples in which (57) may be of no help or require additional considerations. As the
first example, consider the singular Lagrangean of arbitrary degree N in the velocities

$$
L = \frac{a}{2} \dot{q}_1^2 + \frac{b}{2} \dot{q}_2^2 + c \dot{q}_1 \dot{q}_2 + h \dot{q}_3^N,
\tag{60}
$$

where $a b = c^2$ and a, b and h are functions of (q_1, q_2, q_3). The momenta for this model
are given by

$$
p_1 \approx a \dot{q}_1 + c \dot{q}_2
$$

---

[13]For the definition of the Routh function see for instance [9].

$$p_2 \approx c\,\dot{q}_1 + b\,\dot{q}_2 \tag{61}$$

$$p_3 \approx N\,h\,\dot{q}_3^{\,N-1}$$

Here the rank of the Hessian matrix is $R = 2$ (either $\dot{q}_1$ or $\dot{q}_2$ cannot be expressed in terms of (q_i, p_i)) so that equations (1) and (2), when applied to this problem, render

$$\dot{q}_1 \approx \frac{(p_1 - c\,\dot{q}_2)}{a}$$

$$\dot{q}_3 \approx \left(\frac{p_3}{hN}\right)^{\frac{1}{N-1}} \tag{62}$$

$$p_1 \approx \frac{a}{c}\,p_2$$

From the above and (58), the Routh function for this example is then given by

$$G = \frac{1}{2a}p_1{}^2 - \frac{c}{a}p_1\dot{q}_2 + N^{-\frac{1}{N-1}}\left(\frac{N-1}{N}\right)\left(\frac{p_3^N}{c}\right)^{\frac{1}{N-1}} \tag{63}$$

and hence, for arbitrary N, a linear Lagrangean L_f equivalent to L is given by

$$L_f = p_1\dot{q}_1 + p_3\dot{q}_3 - \frac{1}{2a}p_1{}^2 + \frac{c}{a}p_1\dot{q}_2 - N^{-\frac{1}{N-1}}\left(\frac{N-1}{N}\right)\left(c^{-1}p_3{}^N\right)^{\frac{1}{N-1}}. \tag{64}$$

That L_f above is equivalent to (60) can be verified by calculating its equations of motion:

$$\dot{p}_1 \approx 0 \tag{65}$$

$$a\dot{q}_1 - p_1 + c\dot{q}_2 \approx 0 \tag{66}$$

$$\dot{p}_3 \approx 0 \tag{67}$$

$$\dot{q}_3 - \left(\frac{p_3}{Nc}\right)^{(N-1)^{-1}} \approx 0 \tag{68}$$

Solving (66) for p_1 and substituting into (65) eliminates the "auxiliary coordinate" p_1 and leads to one of the equation of motion one can derive from L; solving (68) for p_3 and substituting into (66) eliminates p_3 and leads to the other equation of motion one can obtain from L. As a second example, consider the case of a Lagrangean density defined in superspace given by

$$\mathcal{L} = \frac{1}{2}\bar{\theta}\theta\,\ddot{\phi}^i\,\dot{\phi}^i + \frac{1}{2}i\,\left(\bar{\theta}\frac{\partial\phi^i}{\partial\bar{\theta}} - \theta\,\frac{\partial\phi^i}{\partial\theta}\right)\dot{\phi}^i + \frac{1}{2}\frac{\partial\phi^i}{\partial\bar{\theta}}\,\frac{\partial\phi^i}{\partial\theta} - V(\phi^i) \tag{69}$$

where θ e $\bar{\theta}$ are Grassmann variables and $\phi^i(t,\theta,\bar{\theta})$ are supercoodinates. The momentum here is defined by

$$\pi_i \approx \frac{\partial \mathcal{L}}{\partial \dot{\phi}^i} = \bar{\theta}\theta\dot{\phi}^i(x) + \frac{i}{2}\left(\bar{\theta}\frac{\partial \phi^i(x)}{\partial \bar{\theta}} - \theta\frac{\partial \phi^i(x)}{\partial \theta}\right) \tag{70}$$

In such a case, it is not possible to use the momenta as auxiliary superfields since it is not possible to express the velocities $\dot{\phi}^i$ in terms of the π_i [3, 4]. This obstacle is removed by introducing auxiliary superfields Υ_i satisfying $\bar{\theta}\theta\Upsilon_i = \bar{\theta}\theta\dot{\phi}^i(x)$. Proceeding with the building of the Routh function, the resulting linear Lagrangean equivalent to (69) is given by

$$\mathcal{L} = \left[\bar{\theta}\theta\Upsilon_i + \frac{i}{2}\left(\bar{\theta}\frac{\partial \phi^i(x)}{\partial \bar{\theta}} - \theta\frac{\partial \phi^i(x)}{\partial \theta}\right)\right]\dot{\phi}^i(x) - \frac{1}{2}\bar{\theta}\theta\Upsilon_i^2 + \frac{1}{2}\frac{\partial \phi^i}{\partial \bar{\theta}}\frac{\partial \phi^i}{\partial \theta} - V(\phi^i). \tag{71}$$

4   Conclusions

In this work, the DB for super-QED were calculated as the elements of the inverse of the symplectic matrix of the model, in turn calculated directly from the Hamiltonian. This model is interesting, among other things, due to the presence of gauge invariance, which was shown to be directly connected to the existence of null modes in the pre-symplectic matrix. As regards the FJ and Dirac methods, it was shown that a symplectic matrix can be set up directly from the Hamiltonian, and the same argumentation actually supports the use of Dirac's matrix of constraints (8) but in a Lagrangean framework. From all this, we conclude that the difference between these methods is somehow restricted to the choice of the matrix to work with (of constraints or symplectic) instead of to the choice of the framework (Hamiltonian or Lagrangean). More concretely, it was shown that the options are to work with all the velocities or just with the non-invertible ones, and this is actually what turns the symplectic and the Dirac approaches different.

Finally, a simple prescription for linearizing Lagrangeans, based on the setup of the Routh function, was shown; the advantage is that this prescription works correctly with Lagrangeans of arbitrary degree in the velocities and with Lagrangeans defined in super-space.

Acknowledgments

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