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THE SECOND NOETHER THEOREM FOR LOCAL FIELD THEORIES *

Colber G. Oliveira
Centro Brasileiro de Pesquisas Físicas
Rio de Janeiro, Brazil

Alberto Vidal
Facultad de Ciências
Universidad Nacional Mayor de San Marcos de Lima, Peru

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ABSTRACT

The invariance properties of localized fields under transformation groups involving arbitrary functions are studied from the point of view of the Action Principle. Such invariance implies in the existence of a set of identities at each space-time point, where the field is defined. These identities can be used as selection rules for the determination of the Lagrangian density of the field.

It shown that the invariant Lagrangian density for coupled-field is defined in a local affine connected space (the space where the interaction takes place), with a greater number of dimensions than the usual Minkowski space.

It is also studied the relations which can be obtained between an infinitesimal gauge group and other infinitesimal discrete groups of invariance, such as the Lorentz group. This is done for the case of electrodynamics.

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1. INTRODUCTION

This paper was written with the intention of being a general review of the so called gauge-like field theory.

We have stressed the uniformity of presentation of the several examples, so that the same physical situation is presented in each case under a different mathematical form. This shows that there is an uniform and general way for treating these gauge-like field theories.

The treatment is obviously based on the field theoretic principle of action, and in the conditions under which there are invariance under a given function group of transformations.

The three basic cases of functions groups in physics which deserve importance, namely the electromagnetic, the vector iso-vector theory of Yang and Mills and the gravitational gauge-like theory are discussed. We treat free fields as well as the interactions.

The first part of the work lays down the general theory of covariant fields with respect to given functions groups. The remaining sections are devoted to examples of such theories.

The section 4 treats with a slightly different approach, namely the study of the relations which may be obtained between a function groups and a discret invariance group. This is done to the first order of approximation.

2. THE SECOND NOETHER THEOREM FOR LOCAL FIELD THEORIES

Consider a system described by the functions $y_A(x)(A=1...N)$ by means of the Lagrange-density function [1]

$$L(x;\, y_A(x),\, y_{A,\rho}(x)) = L(x,\, y(x)),$$

$$x = (x^1 ... x^n);\; \rho = 1\; ...\; n,$$

$$W = \int_\Omega L\, dx\; .$$

Since any given physical situation can be described in different coordinates systems, as well as by different set of "field functions" (as for instance by different gauges) we can consider the transformations,

$$\begin{cases} y'_A(x') = f_A(x;y)\;, \\ x'^\alpha = f^\alpha(x)\;. \end{cases}$$

Which are assumed to form a continuous group with the identify transformation. These transformations defined in the Y and X spaces, can be or can be not correlated one to the other.

The infinitesimal form of these transformations will be written as,

$$\begin{cases} y'_A(x') = y_A(x) + \delta y_A(x)\;, \\ x'^\alpha = x^\alpha + \delta x^\alpha(x)\;. \end{cases} \tag{1}$$

The Noether theorem is a general staetement about the structure of the Lagrangian density of a given system under a certain group of transformations, continuous with the identity transformation.

Such groups can be divided into two main types, the first one being the conjunct of all transformations which depend on a set of given parameters: $\{G_p\}$ (for p parameters). The other part, is the conjunct of all transformations which depend on a set of given functions, they are called as function groups: $\{G_{\infty q}\}$ for q functions at each point $x \epsilon X$).[2]

The invariance of the Action integral of the system under G_p amounts to the conservation of a set $\{S_p\}$ of functionals of $y \epsilon Y$, which can be in every case identified as some physical variable associated to the system. The invariance of the Action integral under $\{G_{\infty q}\}$, amounts to a set $\{I_q\}$ of identities involving the field functions $y \epsilon Y$ on each fixed point $x \epsilon X$.

The second Noether theorem treats with the set $\{G_{\infty q}\}$ and therefore with such set $\{I_q\}$ of identities. These identities represent "selection rules" which can be used for the determination of the functional form of L. We will see that such method can be applied both, to free or to interacting fields, with the same generality.

The $G_{\infty q} \epsilon \{G_{\infty q}\}$ can be described by some set of q functions $\varepsilon^i(x)$, $i = 1 \dots q$ by means of,

$$\begin{cases} \delta x^\beta = \varepsilon^i(x)\, \xi_i^\beta(x) \ , \\ \delta y_A(x) = \varepsilon^i(x) \eta_{Ai}(x) + \varepsilon^i{}_{,\rho}(x)\, \gamma_{Ai}^\rho(x) \ . \end{cases} \tag{2}$$

We shall restrict to first order derivatives of $\varepsilon(x)$ in δy_A, since this approximation is sufficient for all known applications in physics, the introduction of higher derivatives is nevertheless a matter of straightforward procedure.

The condition that $y_A'(x')$ must satisfy in order that the field equations in the new representation be equivalent to the old ones, is:

$$\int L(x;\, y(x))dx \equiv \int L(x';\, y'(x'))dx' + \oint Q^{\rho}\, d\Sigma_{\rho}$$

or, the two Action integrals differ at most by a surface integral. The Q^{ρ} which are in general functions of $(x;\, y(x))$ are quantities of the same order as δx, δy. They are the generators of the transformation, if we treat such transformation in terms of canonical pair of variables.

Since we are not going to use the canonical functions we can make use of the arbitrarity of choosing L up to an arbitrary divergence in order to write the above relation without the generator.

It follows to first order that,

$$(\partial^{A} L)\, \bar{\delta}\, y_{A} + (\partial^{A\rho} L)\, \bar{\delta}\, y_{A,\rho} + (L\, \delta x^{\rho})_{,\rho} = 0 \,,$$

where

$$\partial^{A} L = \frac{\partial L}{\partial y_{A}} \,;\; \partial^{A\rho} L = \frac{\partial L}{\partial y_{A,\rho}} \,;\; \bar{\delta} = \delta - \delta x^{\alpha} \frac{\partial}{\partial x^{\alpha}}$$

We can as well to write this identity as,

$$L^{A}\, \bar{\delta} y_{A} + \Gamma^{\rho}_{\;,\rho} \equiv 0 \,, \qquad (3)$$

where L^{A} is the Lagrange derivative of L with respect to y_{A},

$$L^{A} = \partial^{A} L - (\partial^{A\rho} L)_{,\rho} \,,$$

and Γ^{ρ} is a short for, [3]

$$\Gamma^{\rho} = L\, \delta x^{\rho} + (\partial^{A\rho} L)\, \bar{\delta}\, y_{A} \,. \qquad (4)$$

The mathematical structure of Noether's second theorem is given by the equations (2), (3), (4) together with the property that $G_{\infty q}$ is a Lie group.

We now proceed to substitute (2) into (3),

$$\varepsilon^{i}\left[L^{A}(\eta_{Ai} - y_{A,\mu}\,\xi^{\mu}_{i}) - (L^{A}\gamma^{\rho}_{Ai})_{,\rho}\right] + \left[(\partial^{A\rho}L)\,\bar{\delta}y_{A} + L^{A}\varepsilon^{i}\gamma^{\rho}_{Ai} + L\delta x^{\rho}\right]_{,\rho} \equiv 0.$$

Since $\varepsilon^{i}(x)$ are arbitrary functions, the identity sign holds only if each of the coefficients of ε^{i}, $\varepsilon^{i}_{,\mu}$, $\varepsilon^{i}_{,\mu\sigma}$ vanish separately; these conditions can be consistently obtained by setting,

$$L^{A}(\eta_{Ai} - y_{A,\mu}\,\xi^{\mu}_{i}) - \partial_{\rho}(\gamma^{\rho}_{Ai}\,L^{A}) \equiv 0 \;, \tag{5}$$

$$\left[(\partial^{A\rho}L)\,\bar{\delta}y_{A} + L^{A}\varepsilon^{i}\gamma^{\rho}_{Ai} + L\,\delta x^{\rho}\right]_{,\rho} \equiv 0 \tag{6}$$

equation (6) is now separated into the three identities representing the null coefficients of ε^{i}, $\varepsilon^{i}_{,\alpha}$, $\varepsilon^{i}_{,\alpha\beta}$.

$$(L^{A}\gamma^{\rho}_{Ai} + \eta_{Ai}\,\partial^{A\rho}L - T^{\rho}_{\beta}\,\xi^{\beta}_{i})_{,\rho} \equiv 0 \;, \tag{7}$$

$$L^{A}\gamma^{o}_{Ai} + \eta_{Ai}\,\partial^{A\rho}L - T^{\rho}_{\beta}\;\xi^{\beta}_{i} + (\gamma^{\rho}_{Ai}\,\partial^{A\beta}L)_{,\beta} \equiv 0, \tag{8}$$

$$\gamma^{\rho}_{Ai}\,\partial^{A\beta}L + \gamma^{\beta}_{Ai}\,\partial^{A\rho}L \equiv 0 \;, \tag{9}$$

where

$$T^{o}_{\beta} = (\partial^{A\rho}L)y_{A,\beta} - \delta^{o}_{\beta}\;L \;.$$

The equations (5), (7), (8) and (9) represent the mathematical content of Noether's second theorem in this approximation of considering only first derivatives of the $\varepsilon^{i}(x)$ in the transformation law for the field variables $y_{A}(x)$. [4]

The identity (5) represents q relations among the N Euler Lagrange equations for the field, they are sometimes called in the literature as the Bianchi Identities [5], a name borrowed Relativity, which is one of the domains of applicability of this theorem.

3. APPLICATION FOR FREE FIELDS

3.1 The arbitrary group of transformations of coordinates in a four dimensional Ricmann space

The field variables are the ten components of the metric tensor in four dimensions. The permisible group is the one correspondent to general coordinate transformations,

$$\delta x^{\alpha} = \varepsilon^{\alpha}(x) \ .$$

Then,

$$\xi^{\beta}_{\nu} = \delta^{\beta}_{\nu} \ ; \ q = 4 \ .$$

The transformation on $y_A = g_{\mu\nu}$ are those induced by the transformation on the x^{α};

$$g'_{\mu\nu}(x') = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta} \simeq g_{\mu\nu}(x) - (g_{\lambda\mu}\delta^{\alpha}_{\nu} + g_{\lambda\nu}\delta^{\alpha}_{\mu})\varepsilon^{\lambda}_{,\alpha} \ ,$$

$$\delta g_{\mu\nu} = g'_{\mu\nu}(x) - g_{\mu\nu}(x) = (g_{\lambda\mu}\delta^{\alpha}_{\nu} + g_{\lambda\nu}\delta^{\alpha}_{\mu})\varepsilon^{\lambda}_{,\alpha} - \varepsilon^{\alpha} g_{\mu\nu,\alpha} \ .$$

Therefore, the symbols in equation (2) are here,

$$\eta_{Ai} \longrightarrow \eta_{\rho\sigma}, = 0$$

$$\gamma^{\rho}_{Ai} \longrightarrow \gamma^{\rho}_{\mu\nu,\lambda} = -(g_{\mu\lambda}\ \delta^{\rho}_{\nu} + g_{\lambda\nu}\ \delta^{\rho}_{\mu})$$

The Action integral for gravitation has a Lagrangian density which contains up to the second derivatives of the field variables, but is linear in these derivatives, a fact that allows us to separate a divergence in L,

$$L = L' + K^{\sigma}_{,\sigma}$$

where L' depends only on the first derivatives of $g_{\mu\nu}$.

Our results should apply to L' but not to L. Nevertheless, the Action W' associated to L' is a function only of $g_{\mu\nu}$ and $\Gamma^{\rho}_{\mu\nu}$, and therefore is not invariant under $G_{\infty 4}$. This amounts to the necessity of the consideration of a sub-group which satisfy the boundary conditions,

$$\tilde{G}_{\infty 4} \quad \begin{cases} \varepsilon_{\alpha}(x) \longrightarrow 0 \\ \varepsilon_{\alpha,\mu}(x) \longrightarrow 0 \end{cases} \quad ; \ |x| \longrightarrow \infty$$

Then,

$$\delta W = \delta W' \ .$$

Under this $G_{\infty 4}$ $\delta W'$ is permissible. Now, the identities (7) to (9) do not hold anymore (since they came from a divergence containing ε_{α} and $\varepsilon_{\alpha,\mu}$), and we have only the Bianchi identity (5), which is,

$$- L'^{\mu\nu} g_{\mu\nu,\rho} + \partial_{\sigma} \left[L'^{\mu\nu} (g_{\mu\rho} \delta^{\sigma}_{\nu} + g_{\nu\rho} \delta^{\sigma}_{\mu}) \right] \equiv 0 \ , \qquad (10)$$

where,

$$L'^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} L'_{\alpha\beta} \, .$$

The identity (10) can be brought to a familiar form if we write explicitly the expression for $L'_{\mu\nu}$,

$$L'_{\mu\nu} = \sqrt{-g} \ \ G_{\mu\nu} .$$

$G_{\mu\nu}$ being the Einstein's tensor,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R ,$$

$$R_{\mu\nu} = g^{\beta} R_{\mu\beta\nu\delta} .$$

Equation (10) is,

$$G^{\mu}_{\nu;\mu} \equiv 0$$

which is the well known property of the Einstein's tensor. We have proved that it comes due to the invariance of the Action integral under $\tilde{G}_{\infty 4}$.

3.2 <u>The first kind gauge transformation in Minkowski space</u>

We consider a vector field A_μ in the four dimensional Minkowski space, and we look for the form of an invariant $L \equiv L(A_\mu, A_{\mu,\nu})$, under the transformation,

$$A'_\mu(y) = A_\mu(y) + \varepsilon_{,\mu} ,$$
$$\delta x^\mu = 0 ,$$

which we call first kind gauge transformation. We are going to prove that we can reobtain by means of the previous identities, the correct Maxwell Lagrange density.

Here, q = 1, and the ξ, η, γ symbols have the rather simple form,

$$\xi^\beta = 0 ,$$
$$\eta_\mu = 0 ,$$
$$\gamma^\rho_\mu = \delta^\rho_\mu .$$

Also,

$$\bar{\delta} A_\mu = \delta A_\mu$$

since no coordinate transformations are involved.

Therefore the identities (5), (7), (8) and (9) are,

$$L^{\mu}_{,\mu} \equiv 0 \,, \tag{11}$$

$$L^{\mu} + \left(\frac{\partial L}{\partial A_{\mu,\rho}} \right) \equiv 0 \,, \tag{12}$$

$$\frac{\partial L}{\partial A_{\mu,\nu}} + \frac{\partial L}{\partial A_{\nu,\mu}} \equiv 0 \,. \tag{13}$$

Note that (5) and (7) are presently the same identity.

The symbol L^{μ} is the same as L^{A} for $A = \mu$.

Identity (12) implies that,

$$\frac{\partial L}{\partial A^{\beta}} = 0$$

and so,

$$L^{\beta} = -\left(\frac{\partial L}{\partial A_{\beta,\rho}} \right)_{,\rho}$$

The identity (13) implies that L depends on the derivatives of A_{β} only through the antisymmetric combination,

$$A_{\mu,\nu} - A_{\nu,\mu} = F_{\mu\nu}$$

which together with (14) gives

$$L = L(F_{\mu\nu}) \,.$$

Finally the Bianchi identity (11) by using (15) can be written as

$$L^{\mu}_{,\mu} = -\partial_{\rho\mu}\left(\frac{\partial L}{\partial A_{\mu,\rho}} \right) = -\frac{1}{2}\partial_{\rho\mu} \frac{\partial L}{\partial F_{\mu\rho}} \equiv 0 \,.$$

These are the steps on which we can fix the form of L, and we must only to add to this scheme, the requirement of Lorentz covariance; which gives along with (17),

$$L = \frac{k}{2} F_{\mu\nu} F^{\rho\mu}, \tag{18}$$

and by (15) and (17),

$$L^{\mu} = k\, \partial_{\rho}\, F^{\rho\mu} \,. \tag{19}$$

The constant k depends on the system of units we choose. The Bianchi identity is then,

$$\partial_{\mu\nu} F^{\mu\nu} \equiv 0 \,.$$

The field equations are therefore,

$$L^{\mu} = k\, \partial_{\rho}\, F^{\rho\mu} = 0 \,,$$

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} \,.$$

Which are the Maxwell's equations (for a free massless vector field). We have seen therefore that the field must be massless for the invariance under the gauge transformation be maintained; a condition which comes directly from the identity (14). Such identity will be violated whenever the vector field has a rest mass different from zero; a conclusion which comes immediately from the Lorentz invariance of L,

$$L = \frac{k}{2} F_{\mu\nu} F^{\mu\nu} + m^2 A_{\mu} A^{\mu}$$

which we see directly to be non-gauge invariant.

This is a first example of the applicability of this theorem; nevertheless, we did not obtain any now result, since the Maxwell equations are a more familiar result to the physicists then an involved theorem as the present one; and they are constructed in such a way that we see immediately the gauge invariance, without the need to go through elaborate methods to prove it. Nevertheless, this is a trivial example of the range of applicability of this theorem; we have worked out a general prescription for obtaining the form of L, given any function group. In principle this is a straightforward procedure, which can be or can be not handled, depending on the particular difficulties of each given transformation group.

4. THE GAUGE GROUP AND THE DISCRETE INVARIANCE GROUP OF ELECTRODYNAMICS

The gauge group of electrodynamics thought as a function group contains an infinite number of discrete parameters. These parameters may be represented by the coefficients of the arbitrary function $\Lambda(x)$ when this function is expanded as a Taylor's powers series of x around the origin.

$$\Lambda(\ ) = \Lambda(0) + x^{\alpha}\left(\frac{\partial\Lambda}{\partial x^{\alpha}}\right)_{o} + \frac{1}{2}\, x^{\alpha}\, x^{\beta}\left(\frac{\partial\ \Lambda}{\partial x^{\alpha}\partial x}\right)_{o} + \ldots \tag{20}$$

The variation in the four-potential of the field being given by the gradient of Λ

$$A'_{\mu}(x) = A_{\mu}(x) + \Lambda_{,\mu}(x) \ . \tag{21}$$

In order to preserve the Lorentz condition on the four-potential we impose that $\Lambda(x)$ be a solution of the wave equation $\Box\Lambda = 0$, over all the infinite demain of x. From equation (20) it follows that

$$\Box\Lambda(x) = (\Box\Lambda(x))_{o} + x^{\alpha}\left(\frac{\partial}{\partial x^{\alpha}}\Box\Lambda\right)_{o} + \frac{1}{2}\, x^{\alpha}\, x^{\beta}\left(\frac{\partial}{\partial x^{\alpha}\partial x^{\beta}}\Box\Lambda\right)_{o} + \ldots$$

Thus, $\Box\Lambda$ must vanish at the origin simultaneously with all its derivatives.

$$(\Box\Lambda)_0 = 0$$

$$\left(\frac{\partial}{\partial x^\alpha}\Box\Lambda\right)_0 = 0$$

$$\left(\frac{\partial}{\partial x^\alpha \partial x^\beta}\Box\Lambda\right)_0 = 0$$

$$\cdots\cdots\cdots\cdots\cdots$$

From here on we consider the situation where Λ is an infinitesimal function of x. We want to discuss the relations which may exist between a function group such as the gauge group and the discrete invariance groups of the theory. We consider as the discrete invariance group, the homogenous infinitesimal Lorentz group, which is given by the transformation matrix,

$$x'^\mu = L^\mu{}_\nu\, \alpha^\nu$$

$$L^\mu{}_\nu = \delta^\mu_\nu + \varepsilon^\mu{}_\nu$$

$$\varepsilon^{\mu\nu} = g^{\lambda\nu}\,\varepsilon^\mu{}_\lambda = -\,\varepsilon^{\nu\mu}$$

where the $\varepsilon^{\mu\nu}$ are six first order infinitesimal parameters. We are going to prove that a gauge transformation on the potentials of a free field, satisfying the condition $\Box\Lambda = 0$, can be directly related to a Lorentz transformation. To prove this we consider the transformation on the four-potentials generated by the Lorentz transformation matrix

$$A'_\mu(x') = \frac{\partial^{\ \alpha}}{\partial x'^\mu}\, A_\alpha(x) = (\delta^\alpha_\mu - \varepsilon^\alpha_\mu)\, A_\alpha(x)$$

$$= A_\mu(x) - \varepsilon^\alpha{}_\mu\, A_\alpha(x) \,.$$

We can write to the first order in ε^{μ}

$$A'_{\mu}(x') = A'_{\mu}(x^{\alpha} + \varepsilon^{\alpha}{}_{\nu}\, x^{\nu}) = A'_{\mu}(x) + \varepsilon^{\alpha}{}_{\nu}\, x^{\nu}\left(\frac{\partial A_{\mu}}{\partial x^{\alpha}}\right)_{\varepsilon=0} .$$

Thus

$$A'_{\mu}(x) - A_{\mu}(x) = -\varepsilon^{\alpha}{}_{\nu}\, x^{\nu}\left(\frac{\partial A_{\mu}}{\partial x^{\alpha}}\right)_{\varepsilon=0} - \varepsilon^{\alpha}{}_{\mu}\, A_{\alpha}(x) .$$

In order to write this variation in a form similar to equations (20) and (21), we expand $A_{\alpha}(x)$ and $\left(\frac{\partial A_{\mu}}{\partial x^{\alpha}}\right)_{\varepsilon=0}$ in Taylor's series about the origin and replace this expansion into relation (22). Since this expansion is multiplied by $\varepsilon^{\alpha}{}_{\mu}$ in (22), all its terms will be of first order, but nevertheless we will have an infinite number of terms

$$\begin{aligned} A'_{\mu}(x) - A_{\mu}(x) = {} & -\varepsilon^{\alpha}{}_{\mu} A_{\alpha}(0) - x^{\nu}\left\{\varepsilon^{\alpha}{}_{\nu}\left(\frac{\partial A_{\mu}}{\partial x^{\alpha}}\right)_{0} + \varepsilon^{\alpha}{}_{\mu}\left(\frac{\partial A_{\alpha}}{\partial x^{\nu}}\right)_{0}\right\} \\ & - x^{\nu} x^{\lambda}\left\{\varepsilon^{\alpha}{}_{\nu}\left(\frac{\partial^{2} A_{\mu}}{\partial x^{\alpha}\partial x^{\lambda}}\right)_{0} + \frac{1}{2}\varepsilon^{\alpha}{}_{\mu}\left(\frac{\partial^{2} A_{\alpha}}{\partial x^{\nu}\partial x^{\lambda}}\right)_{0}\right\} \\ & - x^{\nu} x^{\lambda} x^{\rho}\left\{\frac{1}{2}\varepsilon^{\alpha}{}_{\nu}\left(\frac{\partial^{3} A_{\mu}}{\partial x^{\alpha}\partial x^{\lambda}\partial x^{\rho}}\right)_{0} + \frac{1}{3!}\varepsilon^{\alpha}{}_{\mu}\left(\frac{\partial^{3} A_{\alpha}}{\partial x^{\nu}\partial x^{\lambda}\partial x^{\rho}}\right)_{0}\right\} + \dots \end{aligned}$$

This equation now posess a similar form to that of the variation given by (20) and (21). This means, both variations are entirely written as power series expansion of x. By equating the coefficients of the terms with the same power in those equations, we obtain

$$\left(\frac{\partial \Lambda}{\partial x^{\mu}}\right)_0 = - \varepsilon^{\alpha}{}_{\mu}\, A_{\alpha}(0)$$

$$\left(\frac{\partial^2 \Lambda}{\partial x^{\mu}\partial x^{\nu}}\right)_0 = - \varepsilon^{\alpha}{}_{\nu}\left(\frac{\partial A_{\mu}}{\partial x^{\alpha}}\right)_0 - \varepsilon^{\alpha}{}_{\mu}\left(\frac{\partial A_{\alpha}}{\partial x^{\nu}}\right)_0$$

$$\left(\frac{\partial^2 \Lambda}{\partial x^{\alpha}\partial x^{\beta}\partial x^{\rho}}\right)_0 = - 2\,\varepsilon^{\lambda}{}_{\alpha}\left(\frac{\partial^2 A_{\mu}}{\partial x^{\lambda}\partial x^{\beta}}\right)_0 - \varepsilon^{\lambda}{}_{\mu}\left(\frac{\partial^2 A_{\lambda}}{\partial x^{\lambda}\partial x^{\beta}}\right)_0$$

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However, the last terms on the left are three symmetric terms over all indices where the R.H.S. terms are not symmetric. Thus we need to symmetrize this terms. We obtain

$$\left(\frac{\partial \Lambda}{\partial x^{\mu}}\right)_0 = - \varepsilon^{\alpha}{}_{\mu}\, A_{\alpha}(0) \tag{24.1}$$

$$\left(\frac{\partial^2 \Lambda}{\partial x^{\mu}\partial x^{\mu}}\right)_0 = - \frac{1}{2}\varepsilon^{\alpha}{}_{\nu}\, S^{(o)}_{\mu\alpha} - \frac{1}{2}\varepsilon^{\alpha}{}_{\mu}\, S^{(o)}_{\alpha} \tag{24.2}$$

$$\left(\frac{\partial^3 \Lambda}{\partial x^{\alpha}\partial x^{\beta}\partial x^{\mu}}\right)_0 = - \frac{1}{3}\varepsilon^{\lambda}{}_{\alpha}\, S^{(o)}_{\mu\lambda\beta} - \frac{1}{3}\varepsilon^{\lambda}{}_{\beta}\, S^{(o)}_{\mu\lambda\alpha} - \frac{1}{3}\varepsilon^{\lambda}{}_{\mu} S^{(o)}_{\alpha\lambda\beta} \tag{24.3}$$

$$\left(\frac{\partial^4 \Lambda}{\partial x^{\alpha}\partial x^{\beta}\partial x\ \partial x}\right)_0 = - \frac{1}{4}\varepsilon^{\lambda}{}_{\alpha}\, S^{(o)}_{\mu\lambda\beta\gamma} - \frac{1}{4}\varepsilon^{\lambda}\, S^{(o)}_{\mu\lambda\alpha\gamma} - \frac{1}{4}\varepsilon^{\lambda}{}_{\gamma}\, S^{(o)}_{\mu\lambda\beta\alpha} - \frac{1}{4}\varepsilon^{\lambda}{}_{\mu}\, S^{(o)}_{\alpha\lambda\beta\gamma} \tag{24.4}$$

$$\left(\frac{\partial^{p} \Lambda}{\partial x^{\mu}\cdots\, \partial x^{\mu_p}}\right)_0 = - \frac{1}{}\left\{\varepsilon^{\lambda}{}_{\mu}\, S^{(o)}_{\lambda\mu\cdots\mu} + \cdots \varepsilon^{\lambda}_{\mu}\, S^{(o)}_{\lambda\mu\cdots\mu_{p-1}}\right\} \tag{24}$$

where the $S^{(o)}_{\mu,\dots\mu_k}$ are fully symmetric expressions involving the derivatives of $A_\mu(x)$ at the origin,

$$S^{(o)}_{\mu\alpha} = \left(\frac{\partial A_\mu}{\partial x^\alpha}\right)_o + \left(\frac{\partial A_\alpha}{\partial x^\mu}\right)_o \tag{25.1}$$

$$S^{(o)}_{\mu\lambda\gamma} = \left(\frac{\partial^2 A_\mu}{\partial x^\lambda \partial x^\gamma}\right)_o + \left(\frac{\partial^2 A_\gamma}{\partial x^\lambda \partial x^\mu}\right)_o + \left(\frac{\partial^2 A_\lambda}{\partial x^\gamma \partial x^\mu}\right)_o \tag{25.2}$$

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$$S^{(o)}_{\mu,\dots\mu_k} = \sum_{\mu,\dots\mu_k} \left(\frac{\partial^{k-1} A_{\mu_k}}{\partial x^\mu \dots \partial x^{\mu_{k-1}}}\right)_o . \tag{25.3}$$

Here $\sum_{\mu_1,\dots\mu_k}$ means sum over all the permutations of $\mu_1,\dots\mu_k$. Now, it is easy verify that the expansion represented by expansion (20) where the coefficients are given by the relations (24) is a solution of the wave equation over all the infinite domain of x. Indeed, from (24.2) we obtain:

$$(\Box\Lambda)_o = 0 \tag{26}$$

as consequence of the fact that $S^{(o)}_{\mu\alpha}$ is symmetric. From (24.3) we get

$$(\partial_\mu \Box\Lambda)_o = -\frac{1}{3}\varepsilon^\lambda{}_\mu \; S^{(o)}_{\lambda\alpha}{}^{\alpha} \tag{27}$$

and from (24.4),

$$(\partial^2_{\mu\gamma}\Box\Lambda)_o = -\frac{1}{4}\varepsilon^\lambda{}_\gamma \; S^{(o)}_{\mu\lambda\alpha}{}^{\alpha} - \frac{1}{4}\varepsilon^\lambda{}_\mu \; S^{(o)}_{\gamma\lambda\alpha}{}^{\alpha} \tag{28}$$

in general, we will get

$$(\partial^n_{\mu,\dots\mu_n}\Box\Lambda)_o = -\left(\frac{1}{n+2}\right)\Big\{\varepsilon^\lambda{}_{\mu_1} S^{(o)}_{\mu_2\dots\mu_n\lambda\alpha}{}^{\alpha} + \varepsilon^\lambda{}_{\mu_2} S^{(o)}_{\mu_1\mu\dots\mu_n}{}^{\alpha} + \dots \varepsilon^\lambda{}_{\mu_n} S^{(o)}_{\mu_1\dots\mu_{n-1}\lambda\alpha}{}^{\alpha}\Big\} \tag{29}$$

It is simply to verify that $S^{(o)}_{\mu_1 \cdots \mu_k \alpha}{}^{\alpha}$ vanish as a consequence of the field equations

$$\Box A_{\mu}(x) = 0$$
$$\partial_{\mu} A^{\mu}(x) = 0$$

which hold for all values of x. Therefore, all derivatives of $\Box\Lambda$ at the origin will vanish. This along with equation (22) shows that $\Box\Lambda$ vanishes for all value of x. Thus for free fields, we can relate a gauge transformation to a Lorentz transformation.

5. INTERACTION BETWEEN FIELDS

5.1 - The second kind gauge transformation

Consider a complex matter field [6] with arbitrary spin, in the Minkowski space, this amountsto leave N arbitrary and to set n = 4.

The Lagrangian density is invariant under the one-parameter phase transformation,

$$\psi' - \psi\, e^{i\varepsilon}$$
$$\varepsilon = \text{constant}$$

Such trivial invariance property gives rise to the conservation of the "current of probability",

$$j^{\mu} = \frac{\partial L}{\partial \psi_{,\mu}}\, \psi - \psi^{*} \frac{\partial L}{\partial \psi^{*}_{,\mu}}$$

(for spinor field * means adjoint conjugate).

However, this invariance breaksdown if $\varepsilon = \varepsilon_{(x)}$ [7]; but since this arbitrary choice of phase at different points, has no physical significance we must look for some new Lagrangian who could be able to maintain the invariance under such transformation. Indeed, the introduction of a new field in L, such that its variation cancels out all terms proportional to $\varepsilon_{,\mu}$, can reduce this problem to the case where ε = constant [8]. This procedure being the converse of what we have done up to here, since we have postulated a given transformation and tried to find out the invariant Lagrangian density. Following this line, we postulate a coupled transformation between the field A_μ [9] and the matter field, and look for the invariant L, this will be the Lagrangian for the complex system.

Physically we may say that only the coupled system as a whole is able to evertaken the arbitrary choise of phases $\varepsilon = \varepsilon(x)$.

The transformations are,

$$\psi' = \psi e^{i\varepsilon(x)}$$
$$\psi'^* = \psi^* e^{-i\varepsilon(x)}$$
$$A'_\mu = A_\mu + \varepsilon_{,\mu}$$

which we call by second kind gauge transformation. Then,

$$y_i = \{\psi, \psi^*, A_\mu\}$$
$$\delta x^\beta = 0$$
$$\delta\psi^* = -i\varepsilon\psi^*$$
$$\delta\psi = i\varepsilon\psi$$

The symbols ξ, η, γ are

$$\eta_\psi = i\psi , \quad \eta_\mu = 0$$
$$\eta_{\psi^*} = - i \psi^* , \gamma^\rho_\mu = \delta^\rho_\mu$$
$$\gamma^\rho_\psi = \gamma^\rho_{\psi*} = 0 , \quad \xi^\beta = 0$$

The equations (5), (7), (8) and (9) are

$$\left(L^\beta + i \frac{\partial L}{\partial \psi_{,\beta}} \psi - i \psi^* \frac{\partial L}{\partial \psi^*_{,\beta}} \right)_{,\beta} = 0 , \tag{30}$$

$$\partial_\beta \left(\frac{\partial L}{\partial A_{\mu,\beta}} \right) + L^\mu + i \frac{\partial L}{\partial \psi_{,\mu}} \psi - i \psi^* \frac{\partial L}{\partial \psi^*_{,\mu}} \equiv 0 \tag{31}$$

$$\frac{\partial L}{\partial A_{\mu,\beta}} + \frac{\partial L}{\partial A_{\beta,\mu}} \equiv 0 \tag{32}$$

$$i \, L_\psi \psi = L \, \psi^* \, L_{\psi *} - L^\mu_{,\mu} = 0 \tag{33}$$

Here, L_ψ is a short for,

$$L_\psi = \frac{\partial L}{\partial \psi^*} - \partial_\sigma \left(\frac{\partial L}{\partial \psi_{,\sigma}} \right) ,$$

with a similar expression for $L_{\psi*}$.

Presently the identity (33) is the Branch identity from (31) we get

$$j + \frac{\partial L}{\partial A_\mu} \equiv 0 \tag{34}$$

where

$$j^\mu = i \frac{\partial L}{\partial \psi_{,\mu}} \psi - i \psi^* \frac{\partial L}{\partial \psi^*_{,\mu}} . \tag{35}$$

The identity (32) which is formally the same for the free case implies that,

$$L = L(\psi, \psi^*, \psi_{,\mu}, \psi^*_{,\mu}, A_\mu, F_{\mu\nu}) \qquad (36)$$

Using (35) we rewrite (30) as,

$$L^{\beta}_{,\beta} + \partial_\beta j^\beta \equiv 0 \qquad (37)$$

But,

$$L^\beta = \frac{\partial L}{\partial A_\beta} - \partial_\sigma\left(\frac{\partial L}{\partial A_{\beta,\sigma}}\right) = \frac{\partial L}{\partial A_\beta} - \frac{1}{2}\partial_\sigma\left(\frac{\partial L}{\partial F_{\beta\sigma}}\right).$$

Then,

$$L^{\beta}_{,\beta} = \partial_\beta\left(\frac{\partial L}{\partial A_\beta}\right).$$

And the identity (37) is,

$$\partial_\beta\left(\frac{\partial L}{\partial A_\beta}\right) + \partial_\beta j^\beta \equiv 0 . \qquad (37')$$

For the Bianchi identity we have

$$ie\,(L_\psi \psi - \psi^* L_{\psi^*}) - \partial_\rho\left(\frac{\partial L}{\partial A_\rho}\right) \equiv 0 \qquad (33')$$

Therefore we have reduced the identities to only three independent ones,

$$j^\mu + \frac{\partial L}{\partial A_\mu} \equiv 0 ,$$

$$L = L(\psi; \psi^*; \psi_{,\mu}, \psi^*_{,\mu}; A_\mu; F_{\mu\nu}) , \qquad (36')$$

$$ie\,(L_\psi \psi - \psi^* L_{\psi^*}) - \partial_\rho\left(\frac{\partial L}{\partial A_\rho}\right) \equiv 0 ,$$

the identity (34) means that A_μ appears always pair-wise with j^μ.

Now we incorporate in this scheme the requirement of Lorentz covariance, this allows us to separate explicity the dependence of $F_{\mu\nu}$ in L,

$$L = L_1(\psi; \psi^*, \psi_{,\mu}; \psi^*_{,\mu}; A_\mu) + f(F_{\mu\nu})$$

$$f(F_{\mu\nu}) = \frac{k}{2} F_{\mu\nu} F^{\mu\nu} + \omega \Omega_{\mu\nu}(\psi, \psi^*) F^{\mu\nu}.$$

The first part of $f(F_{\mu\nu})$ being just the free massless Lagrangian density for the spin 1 fields. The second part represent a coupl ing between $F_{\mu\nu}$ and the antisymmetric second rank tensor $\Omega_{\mu\nu}(\psi, \psi^*)$ build up with the field variables of the matter [10].

Examples being as follows,

Spin 1/2	$\Omega_{\mu\nu} = \bar{\psi}\, \sigma_{\mu\nu}\, \psi$
Spin 3/2	$\Omega_{\mu\nu} = \bar{\psi}_\alpha\, \sigma_{\mu\nu}\, \psi^\alpha$
Spin 1 (with mass)	$\Omega_{\mu\nu} = \mathcal{F}_{\mu\nu}$
Spin 2 (without mass)	$\Omega_{\mu\nu} = 0$, we have only $\Omega_{\mu\nu\rho\sigma} \neq 0$

Such interactions in the non-relativistic limit give rise to anomalous magnetic moments to these particles.

The identity (34) can be interpreted as giving an interaction $j_\mu A^\mu$, but we can do a little more of algebrism in order to bring this identity to a more familiar form:

Define,

$$\begin{cases} \psi_{;\mu} = \psi_{,\mu} - i\, A_\mu \psi, \\ \psi^*_{;\mu} = \psi^*_{,\mu} + i\, A_\mu \psi^* \end{cases} \tag{38}$$

Then,

$$\frac{\partial L}{\partial \psi_{,\mu}} = \frac{\partial L}{\partial \psi_{;\alpha}} \frac{\partial \psi_{;\alpha}}{\partial \psi_{,\mu}} = \frac{\partial L}{\partial \psi_{;\mu}} ,$$

$$\frac{\partial L}{\partial \psi^*_{,\mu}} = \frac{\partial L}{\partial \psi^*_{;\mu}} ,$$

$$\frac{\partial L}{\partial A_\mu} = \frac{\partial L}{\partial \psi_{;\tau}} \frac{\partial \psi_{;\tau}}{\partial A_\mu} + \frac{\partial L}{\partial \psi^*_{;\tau}} \frac{\partial \psi^*_{;\tau}}{\partial A_\mu} = - i \frac{\partial L}{\partial \psi_{,\mu}} \psi + i\, \psi^* \frac{\partial}{\partial \psi^*_{;\mu}} .$$

By using the definition of j^μ, and the two first relations written previously

$$j^\mu = i \frac{\partial}{\partial \psi_{;\mu}} \psi - i\, \psi^* \frac{\partial L}{\partial \psi^*_{,\mu}} .$$

Combining this with the $\frac{\partial L}{\partial A_\mu}$ calculated above, we see that the identity (34) is satisfied.

Therefore, we have proved that A_μ appears in L only through the combination (38); which is the usual gauge invariant prescription of doing,

$$\partial_\mu \psi \rightarrow (\partial_\mu - i\, A_\mu)\, \psi ,$$
$$\partial_\mu \psi^* \rightarrow (\partial_\mu + i\, A_\mu)\, \psi^* .$$

Finally, the Bianchi identity has the meaning of the conservation of the total charge of the matter field if the equations of these fields are satisfied,

$$L_\psi = L_{\psi^*} = 0 \;, \quad \partial_\rho\, j^\rho = 0 .$$

Note that such conservation law does not call for $L^{\mu} = 0$.

The correct factor involving the charge e is obtained when we take,

$$\varepsilon(x) = e\,\chi(x)$$

which is equivalent to replace i by ie in all previous relations.

Another interesting property, comes from the fact that the electromagnetic interaction is obtained by the replacement of $\frac{\partial \psi}{\partial x^{\mu}}$ by $\psi_{;\mu}$, which looks like some kind of "covariant derivative", in the same sense that we need to replace $\frac{\partial \psi}{\partial x^{\mu}}$ (if ψ is not a scalar) by the covariant $\psi_{;\mu}$ when we go to consider gravitational interactions. The same analogy will be present in the case of the Yang Mills interaction.

Again we have obtained very well known results, which are used frequently without the necessity of going through elaborated processes than guessing that the replacement (38) in L, is sufficient for taking care of the invariance of L under the coupled transformation of ψ, ψ^* and A_{μ}:

$$g^{\mu\nu}\,\psi^*_{;\mu}\,\psi_{;\nu} = g^{\mu\nu}\,\psi'^*_{,\mu}\,\psi'_{;\nu}\,.$$

This fact is by no means a draw back to the method presently reported, the reason being that such method has the real advantage of its own generality, which allow us to get a more profound insight into the process; for instance it allows to see that apparently all boson fields are coupled to the matter field by the similar introduction of "covariant derivatives", where the affine connection is given by the strength of the boson field.

5.2 - The isotopic gauge of Yang and Mills

As is known, the fact that the proton and the neutron may be considered two different charge states (positive and neutral) of what is known as the nucleon, suggests the introduction of an "isotopic spin" quantum number to characterize these states. Furthermore, the strongly interacting particles has been grouped in isotopic multiplets, where the components take values of T_3, the 3-component of isotopic spin or isospin T, ranging from -T to T.

The conservation of the isotopic spin in strong interactions holds so far the interactions are invariant under rotation in an "internal space" called "isospin space".

In what follows, we shall work in the domain of this type of interactions. Electromagnetic fields will not enter in the discussion.

Let's begin by noting that the analog of the second kind gauge transformation can be here obtained as follows: the Lagrangian density for the free nucleon field is invariant under rotations of the isotopic axis. This implies that any orientation at a given point x^μ is physically similar to any other, and is therefore unobservable.

The transformation law of the nucleon field under infinitesimal rotations of isotopic axis is:

$$\delta \psi^A = i \sum_{C=1}^{3} \epsilon^o \, \tau^A_{(c)B} \, \Psi^B$$

where $\tau_{(c)}$ are the isotopic spin matrices for isospin 1/2 . The rotation have been made around the iso-vector $\vec{\epsilon}$ by an infinitesimal amount ϵ. Since $\vec{\epsilon}$ is independent of x^{μ} , at all points the isotopic axis will be turned around the direction of $\vec{\epsilon}$ in the same way.

Therefore, the rotational symmetry in the iso-space, implies that the Lagrangian of the nuelon field has an extended or non-local kind of invariance law.

Since one of the most common concepts of field theory is the local character of the system, it appears that such discrimination of the fixation of the components of Ψ for all x^{μ} is too strong.

We should like to have a formulation where the discrimination of proton and neutron is arbitrary at each point, or equivalently: a local isotopic spin rotation law, under which the Lagrangian is invariant.

Mathematically this amount to consider the direction of rotation as function of (x^{μ}), and therefore the underlying isotopic spin space of this local (p-n) symmetric theory is deviated from its previous euclidian structure.

Similarly to the case of electromagnetic interactions in the coordinate space, some sort of field must be present in order to counteract the terms in $\epsilon_{,\mu}$[11]. As we are going to see, such field gives a curvature (locally) to this space.

The complexity of this local theory is comparison with the previous situation, is the price we pay in order to avoid unphysical terms as $\vec{\epsilon}_{,\mu}$ in the Lagrangian . Such complexity is just the same which exists between any Lorentz-covariant theory and a general covariant theory, where we have the necessity of introducing non-linear field equations in order to maintain the covariance (in General Relativity which is an example of a general covariant field theory, we need to introduce the scalar curvature-which is quadratic in the first derivatives of the field variables, as the Lagrangian). In spite of the mathematical complexity, these general covariant fields theories have more physical insight, since they are local in structure (nevertheless, we are vary for from having a reasonable mastering of the properties of these theories), we will return to this point later on.

Following our line of approach we look for the possible transformation law of this new field. First of all a term $\vec{\epsilon}_{,\mu}$ need to be present (as $\epsilon_{,\mu}$ is for electromagnetic gauges) which implies that the field has isotopic spin 1 and spin 1, this term is independent of any rotation of axes; another term giving account of the rotations must be present, since we have $\vec{\epsilon}$ for the rotation of iso-spinors, we shall have $2\vec{\epsilon}$ for rotation of iso-vectors; therefore this terms is,

$$2\,\vec{B}_{\mu}\times\vec{\epsilon}$$

if we call the field by $\vec{B}_{\mu}$ (the arrow means as before, the vector character in the iso-space). Then,

$$\delta\vec{B}_{\mu}=\vec{\epsilon}_{,\mu}+2\,\vec{B}_{\mu}\times\vec{\epsilon}\,.$$

We want to find out the invariant Lagrangian under the coupled transformations,

$$\delta\Psi = i\,\vec{\epsilon}\cdot\vec{\tau}\,\Psi,$$

$$\delta\bar{\Psi} = -i\,\bar{\Psi}\,\vec{\epsilon}\cdot\vec{\tau}\,,$$

$$\delta\vec{B}_\mu = \vec{\epsilon}_{,\mu} + 2\,\vec{B}_\mu \times \vec{\epsilon}\,,$$

$$\delta x^\mu = 0\,,$$

the symbols ξ, η, γ are presently,

$$\xi^\beta_i = 0\ ,\ \Psi\gamma^\rho_i = \bar{\Psi}\gamma^\rho_i = 0\ ,$$

$$\eta^{\bar{\Psi}}_i = -\,i\,\bar{\Psi}\,\tau_i,\ \eta_{\mu k}^{\ i} = \epsilon^i_{\ jk}\,B^j_\mu\ ,$$

$$\eta^{\Psi}_i = i\,\tau_i\Psi\ ,\ \gamma^\rho_{\mu k} = \delta^\rho_\mu\,\delta^i_k\ ,$$

the symbol _ means (T,*) in the iso-space and T in spin times $\beta = \gamma^0$, we also note that the $\vec{\tau}$ matrices are hermitian in the iso-space; latin indices indicate iso-spin variables which run from 1 to 3. The symbol $\epsilon^i_{\ jk}$ is the permutation symbol for $\epsilon^1_{23} = 1$, in this order.

The identities of the Noether theorem, are presently:

$$\partial_\rho\left(L^\rho_k + i\,\frac{\partial L}{\partial\Psi_{,\rho}}\,\tau_k\Psi - i\,\bar{\Psi}\,\tau_k\,\frac{\partial L}{\partial\bar{\Psi}_{,\rho}} + \epsilon^i_{\ jk}\,B^j_\mu\,\frac{\partial L}{\partial B^i_{\mu,\rho}}\right) = 0\ , \qquad (39)$$

$$L^\rho_k + i\,\frac{\partial L}{\partial\Psi_{,\rho}}\,\tau_k\Psi - i\,\bar{\Psi}\,\tau_k\,\frac{\partial L}{\partial\bar{\Psi}_{,\rho}} + \epsilon^i_{\ jk}\,B^j_\mu\,\frac{\partial L}{\partial B^k_{\mu,\rho}} + \partial^\beta\left(\frac{\partial L}{\partial B^k_{\rho,\beta}}\right) \equiv 0\ , \qquad (40)$$

$$\frac{\partial L}{\partial B^k_{\rho,\beta}} + \frac{\partial L}{\partial B^k_{\beta,\rho}} \equiv 0 \ , \qquad (41)$$

$$i\, L_{\Psi}\, \gamma_i \Psi - i\bar{\Psi}\, \gamma_i\, L_{\bar{\Psi}} + L^{\mu}_k\, \varepsilon^k{}_{ji}\, B^j_{\mu} - (L^{\mu}_i)_{,\mu} \equiv 0 \ . \qquad (42)$$

The last one being the Bianchi identity. By (41) we see that the derivatives of B^k_ρ appear in L only through the antisymmetric combination,

$$\varphi^k_{[\rho,\beta]} = \frac{\partial B^k_\rho}{\partial x^\beta} - \frac{\partial B^k_\beta}{\partial x^\rho} + f^k_{[\rho,\beta]} \qquad (43)$$

where $f^k_{[\rho,\beta]} = -f^k_{[\beta,\rho]}$ is a function only of B^i_ν and therefore is constant through the differentiation in (41).

Therefore, the Lagrangian derivative L^ρ_k is,

$$L^\rho_k = \frac{\partial L}{\partial B^k_\rho} - \partial^\mu\left(\frac{\partial L}{\partial B^k_{\rho,\mu}}\right) \equiv \frac{\partial L}{\partial B^k_\rho} - \frac{1}{2}\partial^\mu\left(\frac{\partial L}{\partial \varphi^k_{[\rho\mu]}}\right),$$

$$L^\rho_{k,\rho} \equiv \partial^\rho\left(\frac{\partial L}{\partial B^k_\rho}\right) .$$

From these relations we verify that the divergence of the identity (40) gives (39), which means that they are not independent.

Now, from (40),

$$\frac{\partial L}{\partial B^k_\rho} + i\frac{\partial L}{\partial \Psi_{,\rho}}\gamma_k \Psi - i\bar{\Psi}\gamma_k \frac{\partial L}{\partial \bar{\Psi}_{,\rho}} + \varepsilon^i{}_{jk}\, B^j_\mu \frac{\partial L}{\partial \varphi^i_{[\mu\rho]}} \equiv 0 \ . \qquad (44)$$

Define:

$$\begin{cases} \Psi_{;\mu} = \Psi_{,\mu} - i\,\vec{\tau}\cdot\vec{B}_\mu \Psi \\ \bar{\Psi}_{;\mu} = \bar{\Psi}_{,\mu} + i\bar{\Psi}\,\vec{\tau}\cdot\vec{B}_\mu \end{cases} \tag{45}$$

Then,

$$\frac{\partial L}{\partial B^k_\rho} = \frac{\partial L}{\partial \Psi_{;\mu}} \frac{\partial \Psi_{;\mu}}{\partial B^k_\rho} + \frac{\partial L}{\partial \bar{\Psi}_{;\mu}} \frac{\partial \bar{\Psi}_{;\mu}}{\partial B^k_\rho} + \frac{1}{2} \frac{\partial L}{\partial \varphi^i_{[\mu\nu]}} \frac{\partial \varphi^i_{[\mu\nu]}}{\partial B^k_\rho} =$$

$$= - i \frac{\partial L}{\partial \Psi_{;\rho}} \tau_k \Psi + i\bar{\Psi}\tau_k \frac{\partial L}{\partial \bar{\Psi}_{;\rho}} + \frac{1}{2} \frac{\partial L}{\partial \varphi^i_{[\mu\nu]}} \frac{\partial f^i_{[\mu\nu]}}{\partial B^k_\rho}$$

substitution of this into (44) gives,

$$\frac{\partial L}{\partial \varphi^i_{[\mu\nu]}} \left(\delta^\rho_\nu\, \varepsilon^i{}_{jk}\, B^j_\mu + \frac{1}{2} \frac{\partial f^i_{[\mu\nu]}}{\partial B^k_\rho} \right) \equiv 0 \ , \tag{46}$$

which can be solved for $f^i_{[\mu\nu]}$, we find:

$$f^i_{[\mu\nu]} = - 2\, \varepsilon^i{}_{jk}\, B^j_\mu \, B^k_\nu \ .$$

Therefore (45) takes the form:

$$\vec{\varphi}_{[\mu\nu]} = \frac{\partial \vec{B}_\mu}{\partial x^\nu} - \frac{\partial \vec{B}_\nu}{\partial x^\mu} - 2\, \vec{B}_\mu \times \vec{B}_\nu \tag{47}$$

This expression is invariant under the transformation $\vec{B}_\nu \rightarrow \vec{B}_\nu + \delta\vec{B}_\nu$, similarly to what happens with the electromagnetic field strengths in relation to the gauge transformation of the potentials.

The field $\vec{B}_\mu$ is contained in L both by means of the "covariant derivatives" (45) as well as quadratically by means of $\vec{\varphi}_{[\mu\nu]}$. This means that the field equations are non-linear in $\vec{B}_\mu$.

Now we use Lorentz invariance and recall that the Lagrangian density for the free matter field is the well known expression for free nucleons; all this together allows us to write the complete Lagrangian density:

$$L = -\frac{1}{4}\vec{\varphi}_{[\mu\nu]}\cdot\vec{\varphi}^{[\mu\nu]} - \bar{\Psi}\gamma^{\mu}(\partial_{\mu} - i\vec{\tau}\cdot\vec{B}_{\mu})\Psi - m\bar{\Psi}\Psi . \qquad (48)$$

Before going on, we want to stress the analogy between $\vec{\varphi}_{[\mu\nu]}$ and the Riemann tensor, in the term $\vec{B}_{\mu} \times \vec{B}_{\nu}$ similar to the term $\Gamma\cdot\Gamma$ in $R_{\mu\nu\rho\sigma}$. Naturally the linear term $\vec{B}_{\mu,\nu} - \vec{B}_{\nu,\mu}$ has the same analogy.

Indeed it was this analogy who gave to Utiyama the first idea on his work about this subject. [8]

We see that the "generalized" space where the interaction takes place, has a curvature given by $\vec{\varphi}_{[\mu\nu]}$; such curvature comes from the local arbitrariety of defining the triplet of directions $\vec{\epsilon}(x)$.

Gravitation is not present since we have imposed Lorentz invariance (the coordinate space is a flat space).

Truly speaking, this local generalized space is seven dimensional; by the same taken the generalized space where the electromagnetic interaction is a five dimensional space $(x, \epsilon(x))$ which has a curvature $F_{\mu\nu}$ along the fifth axis.[12]

Such multi-dimensional spaces have been considered by some physicists (for instance Rayski in Poland) for the classification of the elementary particles.

From (48) the field equations are obtained as follows: [13]

$$\frac{\partial \vec{\varphi}_{[\mu\nu]}}{\partial x^{\nu}} + 2\,(\vec{B}_{\nu} \times \vec{\varphi}_{[\mu\rho]})\, g^{\rho\nu} + \vec{J}_{\mu} = \vec{L}_{\mu} = 0 \;,$$

$$\gamma_{\mu}\left(\partial^{\mu} - i\,\vec{\tau}\cdot\vec{B}^{\mu}\right)\Psi + m\Psi = L_{\Psi} = 0 \;,$$

$$\vec{J}_{\mu} = i\,\vec{\bar{\Psi}}\gamma_{\mu}\vec{\tau}\,\Psi.$$

If they are satisfied, the Bianchi identity (42) implies in the conservation of the total isotopic spin of the system: [14]

$$\partial_{\mu}\vec{L}^{\mu} = 0 \;,$$

or,

$$\partial_{\mu}(\vec{J}^{\mu} + 2\,\vec{B}_{\nu} \times \vec{\varphi}^{[\mu\nu]}) = 0 \;,$$

if we call,

$$\vec{J}^{\mu} + 2\,\vec{B}_{\nu} \times \vec{\varphi}^{[\mu\nu]} = \vec{K}^{\mu},$$

(note that, $\vec{L}^{\mu} = \vec{K}^{\mu} + \partial_{\nu}\,\vec{\varphi}^{[\mu\nu]}$) then $\int d_3 \times \vec{K}$ [4,5,8] is the isotopic spin current for the total system; and $\int d_3 \times \vec{K}$ [13] is the total isotopic spin.

Yang and Mills [13] impose the subsidiary condition,

$$\frac{\partial \vec{B}^{\mu}}{\partial x^{\mu}} = 0 \;,$$

in order to drop out the scalar part of each one of the three iso-vectors-mesons (T = 1, S = 1) which are coupled singly to the nucleons by means of (48),

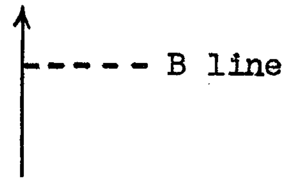
$$L_{int} = \vec{J}_{\mu}\cdot\vec{B}^{\mu},$$

they are the analogues of the condition,

$$\frac{\partial A^{\mu}}{\partial x^{\mu}} = 0$$

of the electromagnetic gauge.

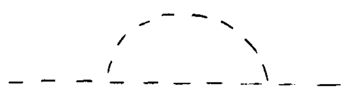
Then, they make the quantization, and conclude that these three isovector vector mesons can have all the three states of charge: (+ 1, 0, -1) which will be observable when we introduce external electromagnetic interactions on the system. We shall have the vertices of Feynman diagrams.

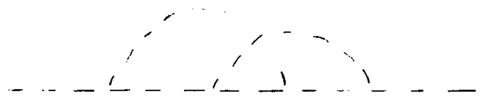


Similarly to the vertices in quantum electrodynamics. But this is not the whole history since the $\bar{B}$ field is non-linear, we can have interactions corresponding to the diagrams

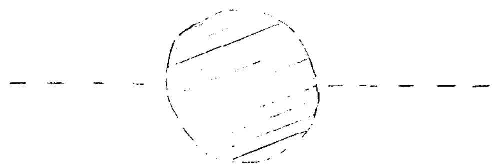
which are absent in the electromagnetic situation.

Nevertheless, the mass of the $\vec{B}_\mu$ field is not well defined. From the equations of motion (page (29)) we could say that in absence of nucleon currents $\vec{J}_\mu = 0$, the mass of the $\vec{B}_\mu$ is zero: this is not true in general, since $\vec{B}_\mu$ interacts with itself (scond term in the equation). Therefore, a line - - - indicating the propagator for the $\bar{B}$ field, can have all kinds of internal lines depending on the order of expansion of the scattering matrix elements in the Interaction Representation; the Feynman digram being, for instance:





In general,



These diagram give divergent matrix elements, therefore the determination of δm_B is not well defined.

5.3 - Interaction between the matter field and an internal gravitational field

In our language, matter field is any field in the framework of the first quantization and therefore nothing has to do with the sources of gravitation which are in this case the matter in bulk, we can formally write down the interaction between gravitation and the matter field however, as it is clear the gravitational field will be always an external field, and therefore a classical field.

This interaction is obtained by taking the transition,

Lorentz invariance ⟶ Generalized invariance

which is equivalent to say that the $a^{\mu}{}_{\rho}$ coefficients of the Lorentz matrix must go over arbitrary functions,

$$x'^{\mu} = a^{\mu}{}_{\rho}\, x^{\rho} \longrightarrow a^{\mu}{}_{\rho}\, x^{\rho} = f^{\mu}(x)$$

this is just the statement that locally we never will be able to determine a inercial frame of reference (as we did in restrict relativity) which is one of the basic ideas of Einstein's 1916 theory.

Since the choice of reference system turns out to be an irrestrict matter of choice, all the theory would be without any significance unless we have some agent able to overtaken such arbitrariety. This comes from the requirement of general covariance, which says that the field equations are to be symmetric with respect to the arbitrary coice of coordinates.

In our language used up to here, this is the statement that gravitational fields are the agent who takes care of disguising the non-physical coordinates, in the same very that A_μ takes care of the local arbitrariety in the choice of phases in Ψ, and $\vec{B}_\mu$ in the arbitrariety in the orientations $\vec{\epsilon}$.

REFERENCES:

1. We shall not treat more than first derivatives of $y_A(x)$ in L, since there is no case of interest with such behaviour (non-local field theories will not be treated here), even the gravitational Einstein's Lagrangian density can be brought to this form by neglecting a divergence.

2. Since all known function groups in physics are also Lie groups, we shall restric our discussion to this kind of groups.

3. We mention that Γ^{ρ} is determined up to the curl of an arbitrary second rank antisymmetric tensor. Presently this is not relevant, since we always have the divergence of Γ^{ρ} and not Γ^{ρ} itself.

4. R. Utiyama - Supp. Prog. Th. Phys., number 9, 19 (1959).

5. P. G. Bergmann - Problems of quantization, Note of the Institute of Mathematics of the University of Rome (1958).

6. We call by this name any field, with exception of the electromagnetic and gravitational fields.

7. Since then, the terms $\psi^{*}_{,\mu}\ \psi^{,\mu}$ in L are not invariant.

8. R. Utiyama - Phys. Rev. 101, 1957 (1956).

9. We are guided by some amount of intuition, since it could be possible A_{μ} were not sufficient for the determination of an invariant L in this situation we must look for another field. In a quite general situation it could happen that it does not exist L for any field.

10. We observe that in this coupling derivatives or ψ will give nothing, since terms as,

$$\int d_4x\ (\psi^{*}_{,\mu}\,\psi_{,\nu} - \psi^{*}_{,\nu}\,\psi_{,\mu})\ F^{\mu\nu}(x)$$

have a null Fourier transform. This implies that spin 0 fields cannot have an anomalous magnetic moment since this is the unique way that such fields could couple to $F_{\mu\nu}$.

11. Such terms have no physical significance, in the same way that the choice of coordinates in a generally covariant field theory is a matter of irrestricted arbitrariety. This new isotopic space of all the directions $\vec{\epsilon}(x)$ is a sort of general covariant theory, where the "coordinates" are the directions $\vec{\epsilon}(x)$.

12. Which the tensor $F_{\mu\nu}$ plays the formal role of a curvature, was demonstrated by S. Mandelstam [15] – Quantum Electrodynamics without potentials; and Quantization of the gravitational field. (See reference 15).

13. C. N. Yang, R. L. Mills - Phys. Rev. 96, 191 (1954).

14. Note that here we need to impose at the same time: $L_{\psi} = L_{\bar{\psi}} = \vec{L}^{\mu} = 0$, whereas for charge conservation, only $L_{\psi} = L_{\bar{\psi}} = 0$ is necessary. This comes from the fact that A_{μ} has no charge.

15. S. Mandelstam, Ann. Phys. 19, 1, 25 (1962).