NOTAS DE FÍSICA
VOLUME XX
Nº 2

AN APPROXIMATE SOLUTION FOR THE CYLINDRICALLY SYMMETRIC MAGNETIC FIELD WITH SOURCE

by
A.F. da F. Teixeira and M.M. Som

CENTRO BRASILEIRO DE PESQUISAS FÍSICAS
Av. Wenceslau Braz, 71 - Botafogo - ZC-82
RIO DE JANEIRO, BRAZIL
1973

Notas de Física - Volume XX - Nº 2

AN APPROXIMATE SOLUTION FOR THE CYLINDRICALLY SYMMETRIC MAGNETIC FIELD WITH SOURCE

A.F. da F. Teixeira and M.M. Som
Centro Brasileiro de Pesquisas Físicas
Rio de Janeiro, Brazil

(Received 8th June, 1973)

ABSTRACT

An approximate solution of the field equations of Einstein-Maxwell's theory is obtained for cylindrically symmetric metric due to oppositely charged clusters of particles moving in circles in counter directions.

I. INTRODUCTION

After Melvin[1] discovered his magnetic universe as a singularity-free solution to the Einstein-Maxwell equations in general relativity, Thorne[2] has considered the physical structure of such a universe having no matter anywhere. He showed that Melvin universe is an absolutely stable universe under radial perturbation. Som[3] has obtained explicit solutions where matter coexists with cylindrically symmetric axial magnetic field. But the magnetic field is source-free in this case too. Later Banerji[4] introduced source in the form of a conduction current in the azimuthal direction inside a perfectly conducting dust. Though the solution can be matched with an outside magnetic field solution due to Bonnor[5], the Melvin magnetic universe cannot be fitted with dust distribution in this way.

In the present paper, we propose to study the case where the cylindrically symmetric axial magnetic field arises solely due to steady motion of charged particles. The distribution considered here is in the form of two clusters of oppositely charged particles moving in circles in counter directions, so that the net angular momentum is zero and the system as a whole is electrically neutral. In view of difficulty in finding an exact solution, we have found a approximate solution. For a particular choice of azimuthal current, the approximate solution describes a cylindrically symmetric axial magnetic field wholly within the bounded distribution and outside exterior solution is again that of Marder[6].

II. BASIC EQUATIONS

For regions in which there are both matter and electromagnetic field, the Einstein-Maxwell equations are

$$R^{\mu}_{\nu} - \delta^{\mu}_{\nu} R/2 = - 8\pi G(T^{\mu}_{\nu} + E^{\mu}_{\nu})/c^4 , \qquad (2.1)$$

with

$$T^{\mu\nu} = c^2 \sum_i \rho_{(i)} u^{\mu}_{(i)} u^{\nu}_{(i)} \qquad (2.2)$$

and

$$E^{\lambda}_{\nu} = (F^{\lambda}{}_{\alpha} F^{\alpha}{}_{\nu} - \delta^{\lambda}_{\nu} F^{\alpha}{}_{\beta} F^{\beta}{}_{\alpha}/4) / (\mu_o c^2) . \qquad (2.3)$$

where $\rho_{(i)}$ is the matter density of $i^{\underline{th}}$ group of charged particles having velocity $u^{\mu}_{(i)}$ and μ_o is magnetic permeability. In the present case there are only two groups of oppositely charged particles, so that $i = 1,2$. The Maxwell equations are

$$(\sqrt{-g}\ F^{\lambda\nu})_{;\lambda} = \mu_o c^2 \sqrt{-g}\ j^{\nu} \qquad (2.4)$$

and

$$F_{[\mu\nu;\rho]} = 0 \quad ; \qquad (2.5)$$

we are using

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} .$$

For the cylindrically symmetric system, we number the coordinates ct, r, z, ϕ as $0, 1, 2, 3$ respectively. So for a purely axial magnetic field, the only surviving components of $F_{\mu\nu}$ are $F_{31} = - F_{13}$. Equation

(2.5) is then automatically satisfied, since F_{31} is a function of radial coordinate r only. Therefore $E^1_1 = -E^2_2$ and $T^1_1 = T^2_2 = 0$, so that we obtain

$$R^0_0 + R^3_3 = 0 , \tag{2.6}$$

and then the metric may be taken in Weyl's canonical form (Synge[7])

$$g_{\mu\nu} = \mathrm{diag}(e^{2\alpha} , -e^{2\beta - 2\alpha}, -e^{2\beta - 2\alpha}, -r^2 e^{-2\alpha}) , \tag{2.7}$$

where α and β are functions of r only. Let us define $u^\mu_{(i)}$ as

$$\begin{aligned} u^\mu_{(1)} &= (u^0 , 0, 0, \omega/c) , \\ u^\mu_{(2)} &= (u^0 , 0, 0, -\omega/c) , \end{aligned} \tag{2.8}$$

and $\rho_{(1)} = \rho_{(2)} = \rho/2$ and

$$F^3{}_1 = -c\, B/r . \tag{2.9}$$

Then from equations (2.2), (2.3), (2.8) and (2.9) one obtains

$$T^\mu_\nu = \mathrm{diag}\left[\rho(c^2 + \omega^2 r^2 e^{-2\alpha}), 0, 0, -\rho\omega^2 r^2 e^{-2\alpha} \right] \tag{2.10}$$

and

$$E^\mu_\nu = (2\mu_0)^{-1} e^{-2\beta} B^2 \,\mathrm{diag}(1, -1, 1, -1). \tag{2.11}$$

From (2.4) one obtains

$$j^3 = -(c\,\mu_o)^{-1}\ e^{2\alpha - 2\beta}\ B_1/r \quad , \qquad (2.12)$$

where the subscript 1 means d/dr. Since the axial magnetic field is considered to be only due to the circular motions of oppositely charged particles in counter directions, the azimuthal current j^3 must satisfy the condition

$$j^3 = k\omega\,\rho/c \quad , \qquad (2.13)$$

where $k = q/m$ is the specific charge of each particle.

III. APPROXIMATE SOLUTIONS OF THE FIELD EQUATIONS

From equations (2.1), (2.7), (2.10) and (2.11) one can write the field equations explicitly as

$$(-\alpha_1^2 + \beta_1/r)\ e^{2\alpha - 2\beta} = \left[8\pi G/(2\mu_o\, c^4)\right] B^2\, e^{-2\beta} \ , \qquad (3.1)$$

$$(\alpha_1^2 + \beta_{11})\ e^{2\alpha - 2\beta} = 8\pi G\left[\rho\omega^2\, r^2\, e^{-2\alpha} + B^2\ e^{-2\beta}/(2\mu_o)\right]/c^4 \text{ and} \qquad (3.2)$$

$$(-2\alpha_{11} + \alpha_1^2 - 2\alpha_1/r + \beta_{11})\ e^{2\alpha - 2\beta} = -\,8\pi G\left[p\, c^2 + \rho\omega^2\, r^2\, e^{-2\alpha} + \right.$$

$$\left. + B^2\ e^{-2\beta}/(2\mu_o)\right]/c^4 \ ; \qquad (3.3)$$

and from (2.12) and (2.13) ,

$$k\omega\rho = - e^{2\alpha - 2\beta} B_1/(\mu_o r) . \qquad (3.4)$$

Combining (3.1) and (3.3) one gets

$$\rho = (4\pi G/c^2)^{-1} (\alpha_{11} - \alpha_1^2 + \alpha_1/r - \beta_{11}) e^{2\alpha - 2\beta} , \qquad (3.5)$$

and from (3.2)

$$B^2 = \mu_o c^4(- \alpha_1^2 + \beta_1/r) e^{2\alpha} /(4\pi G) . \qquad (3.6)$$

Now eliminating ρ and β from the set (3.1) to (3.4) one obtains

$$\left[e^{2\alpha}(\alpha_{11} + \alpha_1/r) - K B^2\right] \left[\alpha_1 (e^{2\alpha} + 2\omega^2 r^2/c^2) - \omega^2 r/c^2 - k\omega B r/c^2\right] = 0 \qquad (3.7)$$

and

$$k\omega r \left[e^{2\alpha}(\alpha_{11} + \alpha_1/r) - K B^2\right] = - K c^2 (e^{2\alpha} + 2\omega^2 r^2/c^2) B_1 \qquad (3.8)$$

where $K = 4\pi G/(\mu_o c^4)$.

The system of equations (3.7) - (3.8) gives rise to two possible cases:

A) $$\begin{cases} e^{2\alpha} (\alpha_{11} \quad \alpha_1/r) - K B^2 = 0 \\ B_1 = 0 \end{cases}$$

and

$$\text{B)}\left\{\begin{array}{l} \alpha_1(e^{2\alpha} + 2\omega^2 r^2/c^2) - \omega^2 r/c^2 - k\omega Br/c^2 = 0 \\ k\omega r\left[e^{2\alpha}(\alpha_{11} + \alpha_1/r) - KB^2\right] = -Kc^2(e^{2\alpha} + 2\omega^2 r^2/c^2)B_1 \ . \end{array}\right.$$

<u>Case A</u>) $e^{2\alpha}(\alpha_{11} + \alpha_1/r) = K\,B^2$

where $B = \text{const}$.

Hence $e^{2\alpha} = (K\,B^2/4)(r/\ell)^2\left[(r/r_1)^\ell + (r_1/r)^\ell\right]^2$, (A.1)

and from (3.6) and (3.5) we get

$$e^{2\beta} = (r/r_2)^{2(1+\ell^2)}\left[(r/r_1)^\ell + (r_1/r)^\ell\right]^4 \quad \text{(A.2)}$$

and

$$\rho = 0 \quad , \quad \text{(A.3)}$$

where r_1, ℓ , r_2 are constants of integration. The solution corresponds to exterior field previously obtained by Ghosh and Sengupta(8).

<u>Case B</u> represents the axial magnetic field due to the distribution we are considering. Now the system of equations B reduces to, for $\omega = \text{const}$,

$$\begin{aligned}(e^{2\alpha} + 2\omega^2 r^2/c^2)&\left[(e^{2\alpha} + 2\omega^2 r^2/c^2)\,\alpha_1/r\right]_1 - \\ -r&\left[\omega^2/c^2 - (e^{2\alpha} + 2\omega^2 r^2/c^2)\,\alpha_1/r\right]^2 \\ &+ \kappa\omega^2(r\alpha_{11} + \alpha_1)e^{2\alpha}/c^2 = 0 \quad , \quad \text{(B.1)}\end{aligned}$$

where $\kappa = k^2/(K c^2)$.

In view of diffuculty in finding exact solution, we try an approximate solution in powers of $r\omega/c \ll 1$ which implies that $r=r_0$ defines the range of the distribution in such a way that $\rho(r > r_0) = 0$. Let us put $y = \omega^2 r^2/c^2$. Then (B.1) reduces to

$$2(e^{2\alpha} + 2y) \left[2\alpha_1 (e^{2\alpha} + 2y)\right]_1 - \left[1 - 2\alpha_1 (e^{2\alpha} + 2y)\right]^2 +$$

$$+ 2\kappa (2\alpha_1 + 2\alpha_{11} y) e^{2\alpha} = 0 \qquad \text{(B.2)}$$

where the subscript 1 denotes differentiation with respect to y. Now let us suppose

$$2\alpha = a\, y + b\, y^2 + O(y^3) \quad ; \qquad \text{(B.3)}$$

a constant additive term is unnecessary since for ξ = const we find that $\xi e^{2\alpha}$ is also a solution of (B.2) if we substitute y by $z = \xi y$ and reinterprete the subscript 1 as d/dz.

Now substituting (B.3) in (B.1) and collecting terms independent of y one obtains for ω = const

$$a(a + 6 + 2\kappa) + 4b = 1 \ . \qquad \text{(B.4)}$$

The first equation of the system (B) now reduces to

$$B = \omega\left[a - 1 + y(a^2 + 2a + 2b)\right]/k + O(y^2) \ . \qquad \text{(B.5)}$$

Since our solution should correspond to a solution $B = 0$, when $k = 0$

i.e. when the particles are uncharged, we put

$$a - 1 = e\kappa + O(\kappa^2) \tag{B.6}$$

and

$$a^2 + 2a + ab = g\kappa + O(\kappa^2) \quad , \tag{B.7}$$

where e and g are constants independent of κ . Then (B.5) reduces to

$$B = k\,\mu_o\, c^2\, \omega(e + gy)/(4\pi G) + O(y^2) + O(k^3) \quad . \tag{B.8}$$

Now from (3.1) one gets

$$d\beta/dy = e^2\kappa/2 + (1 + 2\,e\kappa)y/2 + O(y^2) + O(\kappa^2) \quad ,$$

which on integration gives

$$e^{2\beta} = h\left[1 + y^2/2 + \kappa e\, y(e + y)\right] + O(y^3) + O(\kappa^2) \quad . \tag{B.9}$$

We take the constant h of interation as 1 in order to have $g_{11} = -1$ on the axis.

An expression for ρ is obtained by adding (3.2) and (3.3),

$$\rho = \omega^2 \left[1 - 7y + \kappa(e + 2gy)\right] /(2\pi G) + O(y^2) + O(\kappa^2) \quad . \tag{B.10}$$

For $r \to r_o$ one must have

$$B \to \mu_o \tfrac{1}{2}\, \rho\, \omega\, r_o(r_o - r) \simeq k\,\mu_o\, c^2\, \omega(r_o^2\, \omega^2/c^2 - r^2\, \omega^2/c^2)/(4\pi G), \tag{B.11}$$

so by comparison with (B.8) we deduce

$$e = r_o^2 \ \omega^2/c^2 \qquad \text{and} \qquad g = -1 \quad ; \qquad \text{(B.12)}$$

the corresponding values of a and b satisfy (B.4) .

The expressions for g_{oo} , g_{11} , ρ and B correct up to the order of r /c which appears in the lowest order in κ are thus

$$g_{oo} = 1 + (r\omega/c)^2 - (r\omega/c)^4 + \kappa r^2(r_o^2 - r^2/2)(\omega/c)^4 \quad , \qquad \text{(B.13)}$$

$$g_{11} = -1 + (r\omega/c)^2 - 5(r\omega/c)^4/2 + \kappa r^2(r_o^2 - r^2/2)(\omega/c)^4 \quad , \qquad \text{(B.14)}$$

$$\rho = \left[1 - 7(r\omega/c)^2 + \kappa(r_o^2 - 2r^2)(\omega/c)^2\right]\omega^2/(2\pi G) \quad , \qquad \text{(B.15)}$$

$$B = k\,\mu_o\ \omega^3(r_o^2 - r^2)/(4\pi G) \quad . \qquad \text{(B.16)}$$

For k = 0, the solution goes over to that of Teixeira and Som (9) for uncharged distribution.

ACKNOWLEDGEMENTS

The authors wish to express their appreciation to Idel Wolk for helpful discussion and thank to the Conselho Nacional de Pesquisas for a grant awarded.

REFERENCE:

1. Melvin, M.A., 1964 Phys. Lett. 8, 65

2. Thorne, K.S., 1965 Phys. Rev. 139, B 244

3. Som, M.M., 1968 J.Phys. A 1, 108

4. Banerji, S. 1968 J. Phys. A 1, 520

5. Bonnor, W.B., 1954 Proc.Phys.Soc. A 67, 225

6. Marder, L. 1958 Proc.Roy.Soc. 244A 524

7. Synge, J.L., 1960 Relativity, The General Theory (Amsterdam: North Holland)

8. Ghosh, R. and Sengupta, R., 1965 Nuovo Cim. 38, 1579

9. Teixeira, A.F. da F. and Som, M.M. 1973 - Notas de Física, Vol. XIX 12, 318.

* * *