SWAT/96, CBPF-NF-001/96
hep-th/9512116
December 1995

# Non-abelian duality in $N = 4$ supersymmetric gauge theories

Nicholas Dorey, Christophe Fraser, Timothy J. Hollowood

*Department of Physics, University of Wales Swansea,*
*Singleton Park, Swansea SA2 8PP, U.K.*
n.dorey, c.fraser, t.hollowood @swansea.ac.uk

and

Marco A. C. Kneipp

*Centro Brasileiro de Pesquisas Fisicas (CBPF), Rua Dr. Xavier Sigand, 150*
*22290-180 Rio de Janeiro, Brazil*
kneipp@cbpfsu1.cat.cbpf.br

## ABSTRACT

A semi-classical check of the Goddard-Nuyts-Olive (GNO) generalized duality conjecture for gauge theories with adjoint Higgs fields is performed for the case where the unbroken gauge group is non-abelian. The monopole solutions of the theory transform under the non-abelian part of the unbroken global symmetry and the associated component of the moduli space is a Lie group coset space. The well-known problems in introducing collective coordinates for these degrees-of-freedom are solved by considering suitable multi-monopole configurations in which the long-range non-abelian fields cancel. In the context of an $N = 4$ supersymmetric gauge theory, the multiplicity of BPS saturated states is given by the number of ground-states of a supersymmetric quantum mechanics on the compact internal moduli space. The resulting degeneracy is expressed as the Euler character of the coset space. In all cases the number of states is consistent with the dimensions of the multiplets of the unbroken dual gauge group, and hence the results provide strong support for the GNO conjecture.

Key-words: Supersymmetry: Yang-Mills theory; Electric-Magnetic Dualities.

## 1. Introduction

There is now strong evidence [1,2] that a version of the electromagnetic duality originally conjectured by Montonen and Olive [3] is exactly realized in $N = 4$ supersymmetric gauge theory.[1] In its simplest form, duality requires that the spectrum of BPS saturated states in the theory should be invariant under the interchange of electric and magnetic quantum numbers $m \leftrightarrow n$ (here we use the notation of [1]) together with the inversion of the gauge coupling constant $e \to 2\pi/e$. This means that a state with quantum numbers $(m, n)$ must have a distinct partner with quantum numbers $(n, m)$ whenever $m \neq n$. In principle, this condition can be checked reliably at weak coupling using standard semi-classical methods for the quantization of monopoles. Sen provided an important confirmation of duality in the case of gauge group SU(2) broken to U(1) when he demonstrated the existence of a bound-state in the two-monopole sector with quantum numbers $(1, 2)$ which is the partner of a particular Julia-Zee dyon.[2]

The Montonen-Olive duality conjecture for gauge group SU(2) was generalized to the case of an arbitrary compact Lie group by Goddard, Nuyts and Olive (GNO) in [5]. The GNO conjecture states that a gauge theory with gauge group $G$, has a dual description at strong coupling in terms of a weakly coupled gauge theory with gauge group $G^\vee$. The dual gauge group is defined by requiring that its Lie algebra $g^\vee$ has roots which are the duals of the root of $g$ defined by $\boldsymbol{\alpha}^\vee = 2\boldsymbol{\alpha}/\boldsymbol{\alpha}^2$. More precisely, we should also specify that $G$ is spontaneously broken to a subgroup $H$ by a Higgs scalar in the adjoint representation so that the theory contains magnetic monopoles. GNO duality relates the spectrum of BPS saturated monopole states in this theory to the spectrum of massive gauge bosons in the dual gauge theory where $G^\vee$ breaks to $H^\vee$. A natural question to ask is whether this generalized electromagnetic duality is an exact relation between the corresponding $N = 4$ supersymmetric gauge theories. Assuming the same states are present for all values of the coupling, such a relation places a powerful constraint on the semi-classical spectrum of BPS monopoles. This is particularly apparent when the unbroken subgroups $H$ and $H^\vee$ contain non-abelian factors. In this case the gauge bosons form multiplets of the unbroken non-abelian symmetry leading to degeneracies in the particle spectrum. In order for GNO duality to hold it is necessary that these degeneracies are precisely matched in the semi-classical spectrum of BPS monopoles in the dual theory. In this paper we will argue that this constraint is indeed satisfied.

Our aim is to determine the semi-classical spectrum of magnetic monopoles in a su-

[1] The relevance of $N = 4$ supersymmetry in this context was first suggested by Osborn [4].

[2] Actually Sen showed the existence of a whole tower of states with quantum numbers $(n, 2)$ where $n$ is any odd integer whose existence is required by the full $\mathrm{SL}(2, \mathbb{Z})$ duality. In this paper we will set $\theta = 0$ and discuss only the $\mathbb{Z}_2$ subgroup originally considered by Montonen and Olive.

persymmetric gauge theory with an unbroken non-abelian subgroup. However there are some well-known theoretical obstacles to applying the standard semi-classical reasoning in this case which we now review. In general a particular monopole solution breaks several of the global symmetries of the theory. The action of these symmetries usually gives rise to normalizable zero-modes of the Hamiltonian for small fluctuations around the monopole background. The zero-modes can be eliminated from the path integral by introducing collective coordinates which parametrize a moduli space of gauge-inequivalent solutions. A familiar example is the translational and charge rotation degrees-of-freedom of the 't Hooft-Polyakov monopole which form a moduli space isometric to $\mathbb{R}^3 \times S^1$. Semi-classical quantization of these degrees-of-freedom, yields a tower of massive particle states with integer electric charges. When the unbroken gauge group is non-abelian, it appears that the monopole can also transform under some generators of the corresponding non-abelian global symmetry. Naïvely one might expect that the resulting semi-classical spectrum would consist of multiplets of this symmetry ('chromo-dyons') analogous to the isospin multiplets which contain the nucleon and $\Delta$ states in the Skyrme model. However the action of these symmetries does *not* give rise to any additional normalizable zero-modes and collective coordinates cannot be introduced in the usual way.

The problem mentioned above has its origin in the slow fall-off of the non-abelian field of the monopole at large distance. In unitary gauge, the gauge field falls off like $P/r$, where $P$ is a constant element of the Lie algebra $h$ of the unbroken subgroup $H$. The action of the symmetry generators $Q \in h$ for which $[Q, P] \neq 0$ therefore produces a variation of the field which also falls off like $1/r$. At first sight the problem seems simple; the zero-modes corresponding to such a variation would have linearly divergent norm. Assuming this to be the case, the moment-of-inertia for motion in the symmetry directions would be infinite and the semi-classical spectrum would contain an infinite number of degenerate chromo-dyons.[3] However, Abouelsaoud showed that this scenario is incorrect [7,8]. In a gauge theory, the zero-modes must not only obey the linearized field equation but should also preserve the local gauge condition. For a zero-mode which corresponds to the non-trivial action of a global non-abelian symmetry generator on the large-distance field of the monopole, these two conditions cannot be satisfied simultaneously. In fact, in the case at hand, the magnetic monopole *does* have extra non-normalizable zero-modes which satisfy the gauge condition. However these modes fall off much slower than $1/r$ at large distance, and so they cannot be associated to the action of the non-abelian symmetry generators. Weinberg suggested that these modes should be thought of instead as the bottom of the continuum of scattering eigenstates [9]. The occurrence of these pathologies reflects a deeper problem with the very notion of global non-abelian symmetry in the monopole sector which was uncovered by Nelson and Manohar [10] (see also [11]) and by Balachandran et al [12]. These authors proved that it is impossible to find a set of generators which represent the unbroken global

[3] A similar situation is known to occur for solitons of the O(3) non-linear $\sigma$-model in (2+1)-dimensions [6].

subgroup and are well-defined at every point on a large sphere containing the monopole. The obstruction to finding such a set is topological in nature and the result is somewhat analogous to the 'hairy ball' theorem which states that there are no smooth, non-vanishing vector fields on the 2-sphere.

In the light of the above, it is not clear how to apply the semi-classical method to find the spectrum of states associated with a single monopole in isolation. A simple way to bypass these problems, which was suggested by Coleman and Nelson [13], is to consider instead a chromo-magnetic dipole; a non-abelian monopole ($M$) and anti-monopole ($\bar{M}$), whose long-range gauge fields exactly cancel.[4] If the separation, $R$, between $M$ and $\bar{M}$ is much greater than the core-size, this configuration is well approximated by a linear superposition of the corresponding gauge fields. In the absence of long-range fields, there is no obstacle to defining the action of the global symmetry generators on the field configuration. Importantly, Coleman and Nelson prove that the corresponding zero-modes now have finite norm, proportional to $R$. These modes can now be eliminated by introducing collective coordinates in the usual way and the resulting moduli space is a coset space. The finite norms of the zero-modes means that the moments-of-inertia for motion on this space are also finite. The semi-classical spectrum that arises from quantizing this motion includes a tower of chromo-dyon states which carry non-abelian charge. However these states are not localized around either the monopole or the anti-monopole, rather they are to be thought of as excitations of the colour flux tube joining the two sources, and they become increasingly hard to distinguish from the ground-state as $R \to \infty$. Unfortunately in the BPS limit the configuration of Coleman and Nelson is not appropriate because the Higgs field now has a long-range component in the $M - \bar{M}$ system which cannot be cancelled.

In the following we will pursue a simple variant of Coleman and Nelson's idea. As the problem lies in the component of the long-range field which fails to commute with the generators of $h$, we must certainly consider a configuration in which this is cancelled at order $1/r$. However, if one does not also require that the other long-range components of the field cancel, then this condition can be satisfied by considering a configuration which contains *only* monopoles and no anti-monopoles. Such a configuration has the advantage that it saturates the Bogomol'nyi bound and therefore corresponds to an exact solution of the field equations. However, when the unbroken gauge group has complex representations, there is a price which one must pay for this simplification. For example, in the case where the unbroken gauge group contains a factor $K = \mathrm{SU}(N)$, there are $\mathrm{rank}[K] = N - 1$ independent non-abelian components or hypercharges which must be cancelled and one is forced to consider a configuration of $N$ BPS monopoles. Despite this complication we will be able to obtain the semi-classical spectrum of the system by arguments very similar to those used by Coleman and Nelson and generalize the analysis to the supersymmetric case.

The aim of our analysis is to provide a semi-classical test of GNO duality. With

[4] In the following, the term 'long-range' always indicates the term of order $1/r$ in the large-distance expansion of the gauge and Higgs fields.

this in mind, it is useful to consider two slightly different physical interpretations of the semiclassical spectrum associated with the multi-monopole configuration described above. The first point of view, which is the closer in spirit to [13], is to focus on a single monopole and to regard the additional monopoles, which are necessary to cancel the long-range non-abelian fields, as a regulator which can be removed to infinity at the end of the calculation. From this point of view, it is convenient (but not necessary) to think of the multi-monopole solution as a single monopole separated by a distance $R$ from a conglomerate consisting of all the other monopoles sitting at the same point. The geometry of such a configuration is then identical to that of the monopole and anti-monopole of Coleman and Nelson. The global colour transformations which affect the first monopole lead to a moduli space which has the form of a coset. Our aim is to quantize the collective coordinate motion on this space and deduce the spectrum of quantum states of the monopole in the presence of the monopole conglomerate which cancels the long-range non-abelian fields.[5] From the degeneracy of these states in the limit $R \to \infty$ we can then infer the degeneracy of BPS saturated states of a single monopole which, according to the GNO conjecture, must match those of the gauge bosons in the dual theory.

Alternatively, one may take the point of view that the problems discussed above for states with long-range non-abelian fields are fundamental in nature and we should only consider the sector of the Hilbert space in which these fields are absent. This means restricting our attention to multi-particle states which carry zero total hypercharge both with respect to the unbroken non-abelian gauge group $K$ and its dual $K^\vee$. By acting on the multi-monopole configuration described above with the global symmetry generators we can construct a moduli space of solutions whose long-range magnetic fields are purely abelian. Quantizing motion on this moduli space we will find a spectrum of BPS-saturated states, consisting of $N$ fundamental monopoles, which carry zero hypercharge with respect to the unbroken chromo-magnetic gauge group $K^\vee$. These states are dual to states, containing the same number of gauge bosons which carry zero total hypercharge with respect to the unbroken chromo-electric gauge group, $K$. We should stress that the states we are considering comprise of monopoles or of gauge bosons at arbitrarily large separations: they are multi-particle states and have nothing to do with the stable bound states of the multi-monopole system discussed in [1]. Hence our proposed test of GNO duality can be interpreted in two slightly different ways. The first approach, in which the additional monopoles are regarded as a regulator, allows us to define the semiclassical spectrum associated with a single monopole while in the second we formulate duality directly between composite states with zero non-abelian hypercharge. In fact the degeneracies required for GNO duality can be demonstrated equally well in either approach. However, for simplicity of presentation, we will concentrate on the former approach in cases where more than two monopoles are required.

[5] In this paper we will explicitly consider only electrically neutral states of the individual monopoles and postpone the analysis of electric dyon states for future work.

In this expanded introductory section we sketch these ideas in the simplest case; a theory with gauge group $G = \mathrm{SU}(3)$ which is broken down to $H = \mathrm{SU}(2) \times \mathrm{U}(1)$ by an adjoint Higgs mechanism. In the remainder of the paper, we consider the general case of an arbitrary compact gauge group with arbitrary pattern of symmetry breaking. The full details of our conventions for the general case are given in Section 2 below. However, to make this section as self-contained as possible, we will introduce the relevant notation for the minimal case below.

We consider an SU(3) gauge group generated by Cartan elements $H_i$, $i = 1, 2$, and step generators $E_{\pm\boldsymbol{\alpha}}$, $E_{\pm\boldsymbol{\beta}}$ and $E_{\pm\boldsymbol{\gamma}}$. The roots $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma} = \boldsymbol{\alpha} + \boldsymbol{\beta}$ satisfy $\boldsymbol{\alpha}^2 = \boldsymbol{\beta}^2 = \boldsymbol{\gamma}^2 = 2$ with $\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = -1$. An unbroken $\mathrm{SU}(2) \times \mathrm{U}(1)$ generated by $E_{\pm\boldsymbol{\beta}}$ together with the Cartan elements is selected by chosing a vacuum expectation value for the Higgs, $\phi_0 = i\boldsymbol{v} \cdot \boldsymbol{H}$ where $\boldsymbol{v} \cdot \boldsymbol{\beta} = 0$. Spherically symmetric BPS monopole solutions of this theory were classified by Weinberg [9,14] and are obtained by embedding the standard SU(2) monopole into the gauge group SU(3). Weinberg identified a set of fundamental monopole solutions which cannot be decomposed into constituents of lower mass. The other spherically symmetric solutions consist of a number of fundamental monopoles all sitting at the same point in space. Weinberg performed an analysis of the spectrum of small fluctuations around these configurations and showed that the number of zero frequency modes was consistent with precisely this interpretation. In the following we will assume that this picture is correct and, more generally, that there exist exact solutions of the Bogomol'nyi equation describing fundamental monopoles at arbitrary separations. For an arbitrary multi-monopole solution the long-range behaviour of the fields in unitary gauge is just,

$$\begin{aligned} B_i &= \frac{x_i}{r^3}\mathcal{G} + \mathcal{O}(1/r^3), \\ \phi &= i\boldsymbol{v} \cdot \boldsymbol{H} - \frac{\mathcal{G}}{r} + \mathcal{O}(1/r^2). \end{aligned} \tag{1.1}$$

By a global gauge transformation the non-abelian charge $\mathcal{G}$ can be chosen to lie in the Cartan subalgebra, $\mathcal{G} = i\boldsymbol{g} \cdot \boldsymbol{H}$. As we review in Section 4, the vector $\boldsymbol{g}$ is then constrained to lie in an integer lattice by the Dirac quantization condition. In the current case this implies that,

$$\boldsymbol{g} = \frac{1}{2e}\left(n_{\boldsymbol{\alpha}}\boldsymbol{\alpha} + n_{\boldsymbol{\beta}}\boldsymbol{\beta}\right) \tag{1.2}$$

where $n_{\boldsymbol{\alpha}}$ and $n_{\boldsymbol{\beta}}$ are non-negative integers which we will call magnetic weights.

In our case, the fundamental monopoles have magnetic weights $\{n_{\boldsymbol{\alpha}}, n_{\boldsymbol{\beta}}\}$ given by $\{1, 0\}$ and $\{1, 1\}$. These configurations are degenerate in mass and are related by a global gauge transformation which lies in the Weyl subgroup of the unbroken SU(2). In general each of these monopoles will have a long-range field which fails to commute with the step generators $E_{\pm\boldsymbol{\beta}}$. However its easy to see that if we consider a field configuration which consists of two well-separated monopoles, one of each type, the resulting long-range fields $\propto (2\boldsymbol{\alpha} + \boldsymbol{\beta}) \cdot \boldsymbol{H}/r$ are now invariant under all the generators of the unbroken subgroup $H$. Dipole terms which do not commute with $E_{\pm\boldsymbol{\beta}}$ arise at $\mathcal{O}(1/r^2)$ and the action of these generators would now appear to give zero-modes which have the same fall off and are therefore normalizable. In the case of the monopole-anti-monopole, system Coleman and Nelson [13] were able to prove that this naïve conclusion is in fact correct. In Section 6 we will briefly review their arguments and claim that the proof given in [13] should apply equally to our multi-monopole configurations. For the moment we will assume that this is the case.

The action of the generators $\{H_1, H_2, E_{\boldsymbol{\beta}}, E_{-\boldsymbol{\beta}}\}$ of the unbroken global subgroup on the two-monopole configuration naturally gives rise to a continuous manifold of solutions. The analysis for the case at hand is simple; each monopole transforms under ordinary electric charge rotations which are generated by $i\boldsymbol{v}\cdot\boldsymbol{H}$, however it is easy to show that each monopole is invariant under the action of a linearly independent Cartan generator. This means that the only action of the Cartan subalgebra is to generate independent charge rotations of each of the two monopoles. Quantizing these degrees-of-freedom gives rise to a tower of electrically charged states or dyons associated with either monopole. For the simplest check of GNO duality, we only need to count the number of electrically neutral BPS saturated states of the system and, for this reason we will ignore these directions in the moduli space. Similarly, the six coordinates which describe the centre-of-mass positions of the two monopoles can be thought of as fixed for our purposes. Our manifold of solutions then is generated by the action of the two step generators $E_{\pm\boldsymbol{\beta}}$ and can therefore be identified with the coset space $\mathcal{M} = \mathrm{SU}(2)/\mathrm{U}(1)$.[6]

The occurrence of a moduli space of dimension two seems to present something of a puzzle. The linearized Bogomol'nyi equation has a well known symmetry under the action of the three inequivalent almost complex structures on four dimensional Euclidean space.[7] (See for example [15]). Under the action of this discrete symmetry, zero-modes naturally come in multiplets of four. Familiar examples are the three translational and single charge rotational mode of the monopole which naturally form such a set. In the SU(2) case, the almost complex structures on $\mathbb{R}^4$ descend to give three inequivalent almost complex structures on the moduli space which can then be shown to be a hyper-Kähler manifold. We shall argue that the two zero-modes generated by the action of $E_{\pm\boldsymbol{\beta}}$ are related to each other by a single complex structure on $\mathbb{R}^4$ which endows the coset space SU(2)/U(1) with its canonical Kähler form. We will find that the other two independent complex structures on $\mathbb{R}^4$ generate two further normalizable zero-modes of the two monopole system. At the level of a linearized analysis, it is not clear whether or not new collective coordinates are required for these modes. The only reasonable candidates for such coordinates would be the relative non-abelian group orientation between the two monopoles. However these directions in function space would clearly take us outside the subspace of configurations for which the long-range non-abelian fields cancel, which appears to contradict the fact that

[6] In fact there is also a discrete subgroup which must be divided out [13]. However, as all the states which we consider are SU(2) singlets, this will play no role in our discussion.

[7] As we will review in Section 7, the monopole can be thought of as configuration in a four dimensional Euclidean space where the bosonic fields form a four-vector $W_m = (A_i, \phi)$.

the corresponding modes are normalizable. In any case, these modes do not correspond to the action of any unbroken global symmetry generator on the two-monopole system and we will assume that their presence does not affect our conclusions.[8]

As usual the semi-classical spectrum is obtained by allowing the collective coordinates which live on $\mathcal{M} = \mathrm{SU}(2)/\mathrm{U}(1)$ to become time dependent [16]. As discussed above, variations of these coordinates give rise to normalizable zero-modes of the configuration and the corresponding moments-of-inertia define a Riemannian metric on $\mathcal{M}$. For a coset space $K/L$, the metric is determined, up to a scale factor, by invariance under natural action of $K$. The relevant scale factor is given by the norm of the zero-modes which grows linearly with the separation $R$ between the two monopoles. Quantizing this system is straightforward, the resulting (electrically neutral) states transform in representations of SU(2) labelled by integral spin $j$. The mass spectrum is $E = M_{\rm cl} + j(j+1)/2I$ where $M_{\rm cl}$ is the classical mass of the two monopole system and $I \sim R/e^2$ is the moment-of-inertia. The states with non-zero $j$ are analogous to the chromo-dyons identified by Coleman and Nelson. Because of the non-zero rotational contribution to their energy, these states do not saturate the Bogomol'nyi bound and hence they are not relevant to our proposed test of duality. The main result of this analysis therefore is that the two monopole system in this model has a unique BPS-saturated ground-state. In Section 4, we will argue that this result holds for an arbitrary gauge group and therefore the degeneracies in the semi-classical spectrum required for GNO duality are not present in ordinary (non-supersymmetric) gauge theory. Of course supersymmetry is certainly necessary anyway for exact GNO duality for many other reasons such as the question of monopole spin [4]. In the following we will show that it also provides the necessary degeneracies of states to fill out the appropriate multiplets of the dual gauge group.

In a gauge theory with extended supersymmetry, BPS monopoles are coupled to massless Dirac fermions. The Callias index theorem [17] states that the fermion fields will have exact zero-modes in the monopole background. The number of these modes is determined by the topological charge, as well as the number of massless fermions in the theory and their transformation properties under the SU(2) subgroup in which the monopole is embedded. Like their bosonic counterparts, these modes can be eliminated from the path integral by introducing collective coordinates. For fermion zero-modes these coordinates are time-dependent Grassman numbers and the resulting semi-classical description now involves the quantum mechanics of these degrees-of-freedom coupled to the bosonic coordinates on $\mathcal{M}$.

In the case of extended supersymmetry, BPS monopole configurations have the important property that they are invariant under the action of half the supersymmetry generators. These unbroken supersymmetries provide a natural pairing between bosonic and fermionic zero-modes. In the case of SU(2) gauge theory with $N = 2$ supersymmetry, Gauntlett [15] demonstrated that the unbroken supersymmetries in spacetime correspond directly to unbroken supersymmetries in the collective coordinate quantum mechanics on

[8] See Section 9 for further discussion of this issue.

the moduli space. These results were generalized to the $N = 4$ case by Blum [18]. We will see below that a similar phenomenon occurs for the semi-classical quantization of the global non-abelian degrees-of-freedom of the monopole described above.

The compact moduli spaces $\mathcal{M}$ which arise from the action of the global symmetry generators in the general case are always a product of Lie group coset spaces. Each of these factors has a natural complex structure which descends from a particular complex structure on $\mathbb{R}^4$ which we discuss in Section 7. This complex structure is covariantly constant and therefore each factor can be thought of independently as a Kähler manifold. Quantum mechanics on a Kähler manifold can be endowed with up to four one-component supersymmetries ($N = 4 \times \frac{1}{2}$) and our main result is that all four of these worldline supersymmetries are realized precisely when the overlying gauge theory has $N = 4$ spacetime supersymmetry. These supersymmetries determine the exact form of the collective coordinate Lagrangian. The Hilbert space of supersymmetric ground-states of this system corresponds to the de Rham complex of harmonic forms on $\mathcal{M}$ [19]. As these ground-states necessarily have zero energy, they correspond to inequivalent, electrically neutral BPS-saturated states of the multi-monopole system. Importantly, because all the zero-modes associated with the action of the unbroken non-abelian gauge group, transform in the fundamental representation of the embedding SU(2), the states we consider all carry zero angular momentum.[9]

It turns out that in all cases of interest, the index of the de Rham complex is saturated by forms of even degree. This means that all the resulting ground-states states are bosonic. Hence, the multiplicity of these ground-states is simply equal to the Euler character, $\chi(\mathcal{M})$, of the internal moduli space. Using this result we can now check the GNO conjecture in the minimal case described above. As the gauge group is simply-laced, the theory should be self-dual. The spectrum of massive gauge bosons in this model consists of an SU(2) doublet with electric charge $q = e\boldsymbol{\alpha} \cdot \hat{\boldsymbol{v}}$ and their anti-particles. Clearly the number of BPS-saturated states of electric charge $2q$ which carry zero SU(2) hypercharge is just two. Thus for exact duality we should find a two-fold degeneracy in the semi-classical spectrum of our two monopole system at large separation. The relevant moduli space is just the coset SU(2)/U(1) which is homeomorphic to $S^2$. As $\chi(S^2) = 2$, the result of our calculation is consistent with duality. The wavefunction of the additional ground-state, which is not present in the bosonic theory, is just the usual volume form on the coset space.

In the remainder of the paper we show that a similar agreement holds in the case of an arbitrary compact Lie group with arbitrary symmetry breaking. The only possible exceptions to this agreement are the so-called degenerate cases (see [14,20]) where additional collective coordinates are known to arise. However, even in these cases, a naïve application of our results produces a spectrum which is consistent with duality. The results provide compelling evidence for GNO duality in $N = 4$ supersymmetric gauge theory. The paper

9 More precisely, each of these ground-states will separately give rise to a small representation of supersymmetry after the space-time degrees-of-freedom of the monopole are quantized.

is organized as follows: in Section 2 we discuss the possible patterns of symmetry breaking in the general case. Section 3 is devoted to the spectrum of massive gauge bosons while Section 4 reviews the construction and classification of BPS monopoles in these theories. In Section 5 we review the pathological situation for a single monopole in isolation and in Section 6 we discuss how this can be solved by considering the multi-monopole configuration described above. Section 7 contains the discussion of supersymmetry and fermion zero-modes. Finally in Section 8 we illustrate the consistency the semi-classical spectrum of monopoles with the expectations of the GNO conjecture in a range of examples.

## 2. Symmetry breaking patterns

In this section we consider the various symmetry breaking patterns that are possible in a gauge theory with a single Higgs field in the adjoint representation of the gauge group. For this purpose, we need only consider a bosonic gauge theory, with a compact simple Lie group $G$, and a single Higgs scalar field in the adjoint representation. We choose a unitary gauge in which the Higgs field on a large sphere at infinity is a constant $\phi_0$. We can always choose a Cartan subalgebra of the Lie algebra $g$ of $G$ so that[10]

$$\phi_0 = i\boldsymbol{v}\cdot\boldsymbol{H}, \tag{2.1}$$

where $\boldsymbol{H}$ are the Cartan elements of $g$ considered as an $r=\mathrm{rank}(g)$ vector. The Higgs field breaks the symmetry to a subgroup $H\subset G$ which consists of group elements which commute with the Higgs vacuum expectation value:

$$H = \left\{ U\in G |\; U\phi_0 U^{-1} = \phi_0 \right\}. \tag{2.2}$$

The Lie algebra $h$ of $H$ consists of the generators of $g$ commuting with $\phi_0$. Introducing the usual Cartan-Weyl basis for the complexification of $g$, consisting of Cartan elements $\boldsymbol{H}$ and step generators $E_{\boldsymbol{\alpha}}$, where $\boldsymbol{\alpha}$ is a root of $g$, then the elements of the complexification of $h$ are the Cartan elements $\boldsymbol{H}$ and the step generators $E_{\boldsymbol{\alpha}}$ with $\boldsymbol{\alpha}\cdot\boldsymbol{v}=0$.[11] Generically, the unbroken gauge group will be the maximal torus $\mathrm{U}(1)^r\subset G$; however, if $\boldsymbol{v}$ is orthogonal to any root of $g$ (i.e. $\phi_0$ is not *regular*) the unbroken gauge group will be non-abelian.

Let $\boldsymbol{\alpha}_i$, for $i=1,\ldots,r$, be a set of simple roots of $g$, chosen so that $\boldsymbol{\alpha}_i\cdot\boldsymbol{v}\geq 0$. We can visualize the symmetry breaking by means of the Dynkin diagram of $g$. The vector $\boldsymbol{v}$ can be expanded in terms of the fundamental weights[12]

$$\boldsymbol{v} = \sum_{i=1}^{r} v_i\boldsymbol{\omega}_i. \tag{2.3}$$

[10] Notice that if the unbroken gauge group $H$ is non-abelian then the choice of the Cartan subalgebra $\boldsymbol{H}$ is not unique.

[11] Our Lie algebra conventions are $[\boldsymbol{H},E_{\boldsymbol{\alpha}}]=\boldsymbol{\alpha}E_{\boldsymbol{\alpha}}$, $[E_{\boldsymbol{\alpha}},E_{-\boldsymbol{\alpha}}]=2\boldsymbol{\alpha}\cdot\boldsymbol{H}/\boldsymbol{\alpha}^2$ and $\boldsymbol{H}^\dagger=\boldsymbol{H}$ and $E_{\boldsymbol{\alpha}}^\dagger=E_{-\boldsymbol{\alpha}}$. The elements of the real form of the Lie algebra will be chosen to be anti-hermitian.

[12] These are the $r$ vectors $\boldsymbol{\omega}_i$ such that $\boldsymbol{\omega}_i\cdot\boldsymbol{\alpha}_j=(\boldsymbol{\alpha}_j^2/2)\delta_{ij}$.

The unbroken gauge group has the general form

$$H = \frac{\mathrm{U}(1)^{r'} \times K}{Z} \tag{2.4}$$

where $K$ is semi-simple and the Dynkin diagram of $K$ is obtained from that of $g$ by 'knocking-out' spots corresponding to each non-zero $v_i$. So the simple roots of $k$, the Lie algebra of $K$, are the subset of the simple roots of $g$ orthogonal to $\boldsymbol{v}$. $Z$ is a finite group which specifies the global structure of $H$ and is isomorphic to a subgroup of the centre of $K$ [5].

## 3. The spectrum of gauge bosons

The classical spectrum of gauge bosons in such a model can be calculated by diagonalizing the mass term induced by the Higgs vacuum expectation value in the Lagrangian [21]. This term is

$$-\mathrm{Tr}\left([\phi_0, A_\mu]^2\right). \tag{3.1}$$

Associating the states with elements of the Lie algebra $g$ the states corresponding to the Cartan elements are massless while the states associated to the step generators $E_{\boldsymbol{\alpha}}$ have a mass

$$M_{\boldsymbol{\alpha}} = e|\boldsymbol{v} \cdot \boldsymbol{\alpha}|, \tag{3.2}$$

which saturates the Bogomol'nyi bound since these states carry electric charge $Q_e = e\hat{\boldsymbol{v}} \cdot \boldsymbol{\alpha}$, where $\hat{\boldsymbol{v}} = \boldsymbol{v}/|\boldsymbol{v}|$. Notice that the states corresponding to a root and its negative are degenerate.

The massless gauge bosons correspond to the unbroken gauge group $H$ and the massive gauge bosons form multiplets of $H$. However, we expect that in the quantum theory some of the massive states are unstable; namely the ones for which $M^{\pm}_{\boldsymbol{\gamma}} \geq M^{\pm}_{\boldsymbol{\alpha}} + M^{\pm}_{\boldsymbol{\beta}}$, with $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ all having non-zero inner product with $\boldsymbol{v}$, and for which there exists a coupling in the Lagrangian which can mediate the decay. Such a coupling exists whenever $\boldsymbol{\gamma} = \boldsymbol{\alpha} + \boldsymbol{\beta}$, and so the state associated to $\boldsymbol{\gamma}$ is at the threshold for decay if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are either *both* positive roots or *both* negative roots.

The stable massive gauge bosons can be associated to sets of roots of $G$, containing a simple root $\boldsymbol{\alpha}_i$ with a non-zero inner product with $\boldsymbol{v}$, which form representations of $H$ and whose lowest weight is the simple root $\boldsymbol{\alpha}_i$. The negative simple root $-\boldsymbol{\alpha}_i$ is associated in the same way with the complex conjugate representation. In this way we associate stable massive gauge multiplets with spots on the Dynkin diagram of $g$ which have been 'knocked-out'.[13]

[13] It has very recently been shown [22,23], that the gauge bosons at threshold do actually form bound-states which match bound-states of monopoles in the dual picture—see the Addendum. This fact does not directly affect our conclusions.

## 4. Classical monopole solutions

In this section we explain how BPS monopole solutions can be constructed for the theories that we are considering and how the the Dirac quantization condition is applied.

It is convenient to work in a unitary gauge in which the Higgs field field is constant on the sphere at infinity. Such a gauge can be achieved in a non-singular way by working in the two-patch formalism of [24]. In this gauge there is a canonical definition of the unbroken gauge group $H$, unlike in the radial gauge of [25], where the unbroken gauge group lies inside $G$ in a position dependent way.

For a multi-monopole configuration the asymptotic form of the magnetic field $B_i$ and the Higgs field, in this gauge are

$$\begin{aligned} B_i &= \frac{x_i}{r^3}\mathcal{G} + \mathcal{O}\left(\frac{1}{r^3}\right), \\ \phi &= i\boldsymbol{v}\cdot\boldsymbol{H} - \frac{\mathcal{G}}{r} + \mathcal{O}\left(\frac{1}{r^2}\right), \end{aligned} \tag{4.1}$$

where $\mathcal{G}$ is a constant element of the Lie algebra $h$ and so commutes with the asymptotic value of the Higgs field. Notice that in the BPS limit the Higgs field has an $\mathcal{O}(1/r)$ component. We can use the freedom present in the definition of the Cartan subalgebra of $g$ (see section 2) to have

$$\mathcal{G} = i\boldsymbol{g}\cdot\boldsymbol{H}, \tag{4.2}$$

for a real vector $\boldsymbol{g}$. The Dirac quantization can now be stated in a simple way [5,26]. The vector $\boldsymbol{g}$, up to a normalizing factor, must lie in the co-root lattice of $g$ denoted $\Lambda_R^\vee$. This is the lattice defined as the integer span of the simple co-roots

$$\boldsymbol{\alpha}_i^\vee = \frac{2}{\boldsymbol{\alpha}_i^2}\boldsymbol{\alpha}_i. \tag{4.3}$$

With the correct normalization

$$\boldsymbol{g} = \frac{1}{2e}\boldsymbol{\alpha}^\vee, \qquad \boldsymbol{\alpha}^\vee \in \Lambda_R^\vee. \tag{4.4}$$

The vectors $\lambda\boldsymbol{\alpha}_i^\vee$, where $\lambda$ is a possible normalizing factor, are the simple roots of a Lie algebra $g^\vee$, corresponding to a group $G^\vee$ [5], the 'dual group' to $G$. All the simply-laced and exceptional groups are self-dual. The only non-self dual groups are $\mathrm{SO}(2n+1) \leftrightarrow \mathrm{Sp}(n)$.

The magnetic charge of such a solution is defined by

$$Q_m = -|\boldsymbol{v}|^{-1}\int dS_i\,\mathrm{Tr}\,(B_i\phi) = 4\pi\boldsymbol{g}\cdot\hat{\boldsymbol{v}}, \tag{4.5}$$

where the integration is over the surface of the sphere at infinity.

In [9] Weinberg, following Bais [21], constructs a number of BPS monopole solutions which are spherically symmetric in the radial gauge.[14] Let $\phi^k(x;\lambda)$ and $A_i^k(x;\lambda)$ be the SU(2) Prasad-Sommerfield monopole or anti-monopole solution [28] corresponding to the Higgs expectation value at infinity being $\lambda$.[15]

To construct Weinberg's spherically symmetric solutions take a homomorphism of SU(2) into $G$ defined by a root $\boldsymbol{\alpha}$, with $\boldsymbol{\alpha}\cdot\boldsymbol{v}\neq 0$ (i.e. $\boldsymbol{\alpha}$ is *not* a root of $k$):

$$\begin{aligned} t^1(\boldsymbol{\alpha}) &= (i/2)(E_{\boldsymbol{\alpha}}+E_{-\boldsymbol{\alpha}}), \\ t^2(\boldsymbol{\alpha}) &= (1/2)(E_{\boldsymbol{\alpha}}-E_{-\boldsymbol{\alpha}}), \\ t^3(\boldsymbol{\alpha}) &= i\frac{\boldsymbol{\alpha}\cdot\boldsymbol{H}}{\boldsymbol{\alpha}^2}. \end{aligned} \tag{4.6}$$

The monopole (or anti-monopole) solution is then

$$\begin{aligned} \phi(x) &= \sum_{k=1}^{3}\phi^k(x;\boldsymbol{v}\cdot\boldsymbol{\alpha})t^k(\boldsymbol{\alpha}) + i\left(\boldsymbol{v}-(\boldsymbol{\alpha}\cdot\boldsymbol{v}/\boldsymbol{\alpha}^2)\boldsymbol{\alpha}\right)\cdot\boldsymbol{H}, \\ A_i(x) &= \sum_{k=1}^{3}A_i^k(x;\boldsymbol{v}\cdot\boldsymbol{\alpha})t^k(\boldsymbol{\alpha}). \end{aligned} \tag{4.7}$$

By expanding these solutions in the asymptotic regime we find they have $\boldsymbol{g}=1/(2e)\boldsymbol{\alpha}^\vee$ and so magnetic charge $Q_m=(2\pi/e)\boldsymbol{\alpha}^\vee\cdot\hat{\boldsymbol{v}}$. Their masses saturate the Bogomol'nyi bound:

$$\tilde{M}_{\boldsymbol{\alpha}} = \frac{2\pi}{e}\left|\boldsymbol{\alpha}^\vee\cdot\boldsymbol{v}\right|. \tag{4.8}$$

By this construction, it appears that fundamental monopole or anti-monopole solutions are associated to any root of the algebra $g$ with non-zero inner product with $\boldsymbol{v}$. This is not true for three reasons. Firstly, some of the solutions may actually consist of superpositions of several stationary fundamental monopoles at the same point. Secondly, there are more SU(2) homomorphisms into $G$ than suggested by (4.6). In fact we shall find that they sweep out moduli spaces of which (4.6) is just a point. Finally, it is known that for non-simply-laced groups not all the monopole solutions can be understood in terms of embeddings of the SU(2) monopole [14,20].

Turning to the first point, the question as to whether the solutions above are fundamental can be answered by calculating the number of zero-modes around a solution. If a solution is not fundamental then there will exist additional zero-modes which describe the freedom to alter the relative separation of the fundamental monopoles. Using an index theory calculation, Weinberg [9,14,29] argued convincingly that non-fundamental solutions are associated to those roots $\boldsymbol{\gamma}$ such that $\boldsymbol{\gamma}^\vee=\boldsymbol{\alpha}^\vee+\boldsymbol{\beta}^\vee$ where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are either both positive

[14] The construction of monopole solutions for arbitrary gauge groups has also been considered in [27].

[15] With respect to an anti-hermitian basis $t^k$, $k=1,2,3$, with $[t^j,t^k]=-\epsilon_{jkl}t^l$.

roots or negative roots of $g$, i.e. $\tilde{M}_{\boldsymbol{\gamma}} = \tilde{M}_{\boldsymbol{\alpha}} + \tilde{M}_{\boldsymbol{\beta}}$. In other words these non-fundamental solutions are simply superpositions of stationary fundamental monopoles.

We now investigate the second point relating to the SU(2) homomorphisms into $G$. Notice, that $\boldsymbol{t}(\boldsymbol{\alpha}) = (t^1(\boldsymbol{\alpha}), t^2(\boldsymbol{\alpha}), t^3(\boldsymbol{\alpha}))$, the SU(2) related to the root $\boldsymbol{\alpha}$ in (4.6), can be deformed by conjugation with constant elements of $H$, the unbroken gauge group, to give a continuous family of degenerate monopole solutions:

$$\boldsymbol{t} \to [\mathrm{Ad}\, U]\, \boldsymbol{t}, \quad U \in H, \tag{4.9}$$

where $[\mathrm{Ad}\, U]x = UxU^{-1}$. Let us identify this family of solutions. A certain subgroup $H_0 \subset H$ will leave the SU(2) invariant, and so the space of SU(2) homomorphisms connected to $\boldsymbol{t}(\boldsymbol{\alpha})$ will be a quotient space $H/H_0$ defined by identifying elements of $H$ under right action by elements of $H_0$. The stabilizer of $\boldsymbol{t}(\boldsymbol{\alpha})$, $H_0$ has the form $H_0 = \mathrm{U}(1)^{r'} \times K_0$. The $K_0$ part of $H_0$ acts only on the non-abelian part of $H$ and $r'-1$ of the abelian parts of $H_0$ act only on the abelian parts of $H$. Precisely one of the U(1) factors of $H_0$ acts non-trivially on both the abelian part and non-abelian parts of $H$. Hence, up to a finite subgroup, the quotient $H/H_0$ has the form $[\mathrm{U}(1) \times (K/K_0)]/\mathrm{U}(1)$ which can be described locally as the product

$$\mathrm{U}(1) \times [K/K_0 \times \mathrm{U}(1)]. \tag{4.10}$$

Globally the quotient is a fibre bundle with base space $K/K_0 \times \mathrm{U}(1)$ and fibre U(1). In general, if the finite subgroup $Z$ is not contained within $\mathrm{U}(1) \times K_0$ there will be a residual $Z$ action which must be divided out.

The U(1) action given by the fibre corresponds to transformations of the form

$$t^3(\boldsymbol{\alpha}) \to t^3(\boldsymbol{\alpha}), \quad E_{\pm\boldsymbol{\alpha}} \to \exp(\pm i\theta) E_{\pm\boldsymbol{\alpha}}, \tag{4.11}$$

generated by $i\boldsymbol{v} \cdot \boldsymbol{H}$. These are the charge rotations familiar from the SU(2) monopole [30]. The additional factor $\mathcal{M} = K/K_0 \times \mathrm{U}(1)$, forming the base space of the bundle, is the space swept out by transformations generated by step operators $E_{\boldsymbol{\beta}}$ of $k$ with $\boldsymbol{\beta} \cdot \boldsymbol{\alpha} \neq 0$. In general, this part which arises from the non-abelian part of $H$, is the product of several Lie group coset spaces, a fact which will play an important role in our subsequent analysis.

The family of solutions swept out from an SU(2) homomorphism defined by the root $\boldsymbol{\alpha}$ can contain the solutions defined by other roots which differ from $\boldsymbol{\alpha}$ by roots of the Lie algebra of $K$. These solutions are connected as in (4.9) by elements of the Weyl group of $K$ [9]. From our perspective, there is actually nothing special about the solutions corresponding to a particular root, any homomorphism of SU(2) into $G$ specified by a point in the space (4.10) is equally as good.

These results suggest that the moduli space of a fundamental monopole will be of the form which is locally $\mathbb{R}^3 \times \mathrm{U}(1) \times \mathcal{M}$ where the first factor represents the centre-of-mass position of the monopole and the second factor corresponds to U(1) charge rotations. The additional factor $\mathcal{M}$, which arises when the unbroken gauge group is non-abelian, reflects the coset degeneracy of solutions described above. However, as we discussed in the introduction, there are certain well-known problems with this picture for the case of a single monopole in isolation. Before we review these problems and our proposed solution, we return to the question of whether all monopole solutions can be understood in terms of embeddings of the SU(2) monopole. In fact it is known that in certain cases with non-simply-laced gauge groups, when the monopoles are constructed from short roots, this is not the case. In these situations, an 'accidental' degeneracy can occur with two monopole solutions corresponding to two different length roots have the same mass but whose associated SU(2) homomorphisms are not connected by conjugation with an element of the Weyl group of $K$. Weinberg showed for the case when $G = \mathrm{SO}(5)$ [20] (see also [31]) that there is a whole family of interpolating solutions between these degenerate monopoles. These additional solutions are manifested by the existence of zero-modes and collective coordinates which are apparently not related to any symmetry.

## 5. Monopole zero-modes

In this section we consider the various zero-modes of a single monopole in a non-degenerate situation (see the remark at the end of the last section). To each degree-of-freedom of a monopole solution we can associate a collective coordinate $X^a$. Variations of the collective coordinates lead to small fluctuations of the fields $\delta_a A_i$ and $\delta_a \phi$ which automatically satisfy the linearized equations-of-motion. However, in general they do not satisfy the back-ground gauge condition:

$$D_i \delta_a A_i + [\mathrm{ad}\,\phi]\delta_a \phi = 0, \tag{5.1}$$

where $[\mathrm{ad}\,\phi]\psi = [\phi, \psi]$, which ensures that they are orthogonal to local gauge transformations. The physical zero-modes are equal to the variations of the solutions by the collective coordinates and a compensating gauge transformation which ensures that the gauge condition (5.1) is satisfied:

$$\begin{aligned} \delta_a A_i &= \frac{\delta A_i(X)}{\delta X^a} + D_i \epsilon_a, \\ \delta_a \phi &= \frac{\delta \phi(X)}{\delta X^a} + [\mathrm{ad}\,\phi]\epsilon_a. \end{aligned} \tag{5.2}$$

Thus $\epsilon_a$ is the generator of a local gauge transformation which must tend to zero at infinity.

To perform a semi-classical quantization one needs to calculate the moments-of-inertia tensor corresponding to the zero-modes which defines a Riemannian metric on the moduli space:

$$\mathcal{G}_{ab} = -\int d^3x \, \left(\mathrm{Tr}\,[\delta_a A_i \delta_b A_i] + \mathrm{Tr}\,[\delta_a \phi \delta_b \phi]\right). \tag{5.3}$$

We are interested in finding zero-modes associated with variations of the coordinates which parametrize the space of SU(2) homomorphisms. Such zero-modes will be generated by gauge transformations with parameters $\Omega_a$ which must approach a constant at infinity [7,8]:

$$\delta_a A_i = D_i \Omega_a, \quad \delta_a \phi = [\phi, \Omega_a], \tag{5.4}$$

which in unitary gauge outside the monopole core take values in the Lie algebra $h$ of the unbroken gauge group $H$. In order that they be zero-modes we require that they satisfy the gauge condition (5.1) which implies the following Laplace-like equation for the gauge parameters:

$$D_i D_i \Omega_a + [\mathrm{ad}\, \phi]^2 \Omega_a = 0. \tag{5.5}$$

In order to determine which gauge transformations lead to zero-modes consider the solution of (5.5). The only non-trivial solutions correspond to generators of $h$ which fail to commute with $\boldsymbol{t}(\boldsymbol{\alpha})$. These are $t^3(\boldsymbol{\alpha})$ itself and the step generators $E_{\boldsymbol{\beta}}$ of the Lie algebra $k$ with $\boldsymbol{\beta} \cdot \boldsymbol{\alpha} \neq 0$. For non-degenerate cases $\boldsymbol{\alpha}$ is a long root and the inner product can be $\boldsymbol{\beta} \cdot \boldsymbol{\alpha} = \pm \boldsymbol{\alpha}^2/2$ only. The element $t^3(\boldsymbol{\alpha})$ generates the U(1) charge rotations whilst the step generators of $k$ can be thought of as being associated to tangent vectors of the coset space $\mathcal{M}$. Outside the monopole core, we can expand the gauge parameter in terms of these generators:

$$\Omega(x_i) = \sum_{\boldsymbol{\beta}} \Omega_{\boldsymbol{\beta}}(x_i) E_{\boldsymbol{\beta}} + \Omega_0(x_i) t^3(\boldsymbol{\alpha}). \tag{5.6}$$

The possible solutions for each of the components $\Omega_{\boldsymbol{\beta}}(x_i)$ are written in terms of monopole harmonics [24]:

$$\Omega_{\boldsymbol{\beta}}(r, \theta, \phi) = r^{\alpha} Y_{qlm}(\theta, \phi), \tag{5.7}$$

where $q = -\boldsymbol{\alpha} \cdot \boldsymbol{\beta}/\boldsymbol{\alpha}^2 = \pm 1/2$ is the eigenvalue of $[\mathrm{ad}\, i t^3(\boldsymbol{\alpha})]$ on the generator $E_{\boldsymbol{\beta}}$. The quantum numbers $l$ amd $m$ are the total angular momentum and its $x_3$ component, so that $l = 0, 1/2, 1, \ldots$ and $-l \leq m \leq l$. The allowed values of $l$ are $|q|,\ |q|+1, \ldots$. The parameter $\alpha$ is determined from the radial equation: $\alpha(\alpha+1) = l(l+1)$.

The gauge parameter corresponding to the abelian subgroup of $H$ generated by $t^3(\boldsymbol{\alpha})$ is simply a constant in the region outside the monopole core. This transformation leads to a normalizable zero-mode of the monopole which corresponds to the freedom to perform U(1) charge rotations. In fact taking into account the fields in the core, it actually dies off exponentially outside the core and so its associated moment-of-inertia $\sim R_{\mathrm{core}}$.

The non-abelian part of the unbroken gauge group leads to gauge transformations $\Omega_{\boldsymbol{\beta}}(x_i)$ for each $\boldsymbol{\beta}$ with $\boldsymbol{\beta} \cdot \boldsymbol{\alpha} \neq 0$. In Appendix A, we analyse the corresponding Laplace equation in the region outside the core. The results specialize those of Abouelsaood [7,8] to the case of BPS monopoles. We find four solutions with quantum numbers $(q, l, m) = (\pm\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2})$. For each of these solutions the radial equation dictates that $\alpha = 1/2$. The gauge parameters $\Omega_{\boldsymbol{\beta}}$ therefore grow like $\mathcal{O}(r^{1/2})$ which clearly contradicts our condition that the resulting modes should correspond to a global gauge rotation of the monopole at large distance. Thus the monopole has bosonic zero-modes which die off like $\mathcal{O}(r^{-1/2})$ at large distance, rather than the expected $\mathcal{O}(1/r)$, and have no obvious relation to the action of the global non-abelian symmetry generators. For the considerations of the next section it is important that all of these modes are well-behaved in the core of the monopole. To determine this we should solve the linearized Bogomol'nyi equation and gauge condition at short distance also. Fortunately this has already been accomplished by Weinberg in Appendix C of [9] who solved an equivalent Dirac equation. Weinberg's analysis reveals the same number of bosonic zero-modes which go like $\mathcal{O}(r^{-1/2})$ at large distance. Each of these modes is well-behaved at $r = 0$ as required.[16]

## 6. Multi-monopole solutions and the non-abelian zero-modes

In the last section we have seen that there is a problem in proceeding to a semi-classical quantization of a monopole when it transforms under the action of the non-abelian unbroken symmetry group. The usual relation between continuous degeneracies of classical solutions and normalizable zero-modes of the small fluctuation operator breaks down. This relation is the cornerstone of the usual method of collective coordinates and without it, it is not clear how to proceed. Before we can make progress, it is clear that we need some way of regulating the pathological behaviour of the zero-modes identified above. Fortunately such a scheme is available. Coleman and Nelson [13] consider a chromo-magnetic dipole, which consists of a separated monopole ($M$) and an anti-monopole ($\bar{M}$). They work in an SU(3) theory with a non-zero scalar potential and therefore outside the BPS limit. The fields of $M$ and $\bar{M}$ are chosen so that at long distance the gauge potential falls off faster than the $\mathcal{O}(1/r)$ of a single monopole. This requires that $M$ and $\bar{M}$ are characterized by elements $\mathcal{G}_\pm$ with $\mathcal{G}_+ + \mathcal{G}_- = 0$, see eqn (4.1). Away from the BPS limit, the Higgs fields fall off to their vacuum expectation value exponentially and can be ignored at long distance. In the dipole system, there *are* now zero-modes which correspond to global colour rotations which have a finite norm growing like $R$, the separation of the sources. These modes correspond to gauge transformations which approach a constant at distances much greater than $R$. The important point is that they are not localized around the cores, since the gauge parameters fall off as $(r/R)^{\sqrt{3/4}-1/2}$ in the vicinity of $M$ or $\bar{M}$, which vanishes for any fixed $r$ as $R \to \infty$. This result is slightly different from the $r^{1/2}$ behaviour found in the last section; this is because in the BPS limit the Higgs field contributes to the behaviour of the modes as indicated in (5.5). The non-abelian global colour modes are therefore interpreted as being a property of the non-abelian flux tube joining the two sources. A semi-classical quantization can now be performed. The non-abelian degrees-of-

[16] Explicitly, the bosonic modes which we have found are related by his equation (3.5) to the Dirac zero-modes implied by his equation (C.14) where we set $a_1 = 0$. Note that Weinberg works in radial gauge where the monopole has spherical symmetry.

freedom lead to quantum mechanics on the coset space SU(2)/U(1).[17] The states of this quantum mechanics are are chromo-dyons associated to representations of SU(2) and have energies $\sim e^2/R$. Coleman and Nelson go on to argue that these diffuse excitations of the colour flux tube effectively disappear as $R \to \infty$ and one is left with only the ground-state of the quantum mechanics.

We would like to understand the issue of global colour transformations for a theory with a more general gauge group and specifically in the BPS limit. Unfortunately, we immediately run into a problem because although we can arrange for the long-range parts of the gauge fields of $M$ and $\bar{M}$ to cancel, the $\mathcal{O}(1/r)$ parts of their Higgs fields would add. When we introduce extended supersymmetry the situation is even more inconvenient because the $M - \bar{M}$ configuration is not invariant under any of the supersymmetry generators since it does not saturate the Bogomol'nyi bound. A moment's thought shows that the problem of the non-abelian modes is intimately related with the fact that the corresponding gauge transformations do not commute with the long-range $\mathcal{O}(1/r)$ part of the non-abelian gauge field and Higgs field; the part lying inside the Lie algebra $k$. So rather than than consider a configuration where all the $\mathcal{O}(1/r)$ parts of the gauge field and Higgs field are cancelled, we can construct a configuration consisting purely of monopoles (i.e. no anti-monopoles), where only the non-abelian $\mathcal{O}(1/r)$ parts of the fields are cancelled.

To implement this idea, consider a configuration of monopoles defined by elements $\mathcal{G}_i$. In order to superimpose these solutions to get a multi-monopole solution we have to ensure that $[\mathcal{G}_i, \mathcal{G}_j] = 0$. We then choose the first monopole to have arbitrary magnetic charge. For each possible choice of the first monopole it is always possible to find a set monopoles so that the $\mathcal{O}(1/r)$ non-abelian parts of the long range field are cancelled. This requires that $\sum_i \mathcal{G}_i$ lies entirely within the abelian part of $h$. In cases involving groups with real representations a single extra monopole will suffice to cancel the long-range non-abelian field, however, in the cases with complex representations it will be necessary to have more than one additional monopole. It will be convenient for the following analysis to have the additional monopole(s) $\mathcal{G}_i$, for $i > 1$, all located at the same point a distance $R$ from the first monopole. The conglomerate consisting of the monopoles with $i > 1$ will have its own set of zero-modes, for instance in the case when it does consist of more than one fundamental monopole there will be zero-modes corresponding to the freedom to separate the monopoles. However, we are interested in the additional zero-modes that arise in the presence of the first monopole.

As we have explained in the last section there are no zero-modes corresponding to non-abelian global colour transformations for a single monopole. However, if we have cancelled the $\mathcal{O}(1/r)$ non-abelian parts of the field by adding a set of compensating monopoles some distance away, then there are normalizable zero-modes corresponding to global colour rotations *of the system as a whole*. In such a background field configuration there exist

[17] This space is even simpler than the one considered in [13] because we are suppressing additional U(1) factors which correspond to independent electric charge rotations of $M$ and $\bar{M}$.

solutions of equation (5.5) which asymptote to a constant as required. The argument is directly analogous to the argument of Coleman and Nelson for the monopole anti-monopole system (see the Appendix of [13]). The problem of solving the linear partial differential equation (5.5) with constant boundary conditions is equivalent to the problem of inverting the differential operator which occurs on the LHS of the equation in an appropriate space of functions. These authors show that this problem is in turn equivalent to the problem of minimizing a differentiable, convex, function which is bounded below. The existence of a solution to this latter problem is then guaranteed by standard theorems from the calculus of variations. The whole argument depends only on the large distance fall-off of the fields which contribute to the LHS of (5.5) . In the present case, there is a power-law contribution to the Laplace equation from the Higgs field. However, its fall-off far away from the sources is of the same form as the gauge field $\mathcal{O}(1/r^2)$. In addition, the fact that there are still long-range abelian fields in our system does not affect the result, because the non-abelian generators of $k$ commute with these abelian fields. Each pair of generators $E_{\pm\boldsymbol{\beta}}$ of $k$ such that $\boldsymbol{\beta}\cdot\boldsymbol{\alpha}\neq 0$ will lead to two non-abelian modes of the combined system. In the vicinity of the monopole, the two normalizable modes must be a certain linear combinations of the four modes that we discussed in the previous section. In Appendix A we determine the unique linear combinations with the correct angular behaviour. On dimensional grounds the modes are generated by gauge parameters which behave as $(r/R)^{1/2}$ near the monopole. As pointed out by Coleman and Nelson as $R\to\infty$ they are 'expelled' from the core.

To summarize, in the combined system, there will be an additional normalizable non-abelian zero-modes which correspond to global colour transformations. There will be one such mode for each direction in the coset $\mathcal{M}$. By a straightforward extension of the arguments given in [13] these modes have a norm which grows linearly with $R$, the separation of the monopole and the conglomerate.

If we now perform a semi-classical quantization for the degrees-of-freedom associated to the non-abelian zero-modes of the combined system then we will be led to consider quantum mechanics on the coset space $\mathcal{M}$. States will be associated to representations of $K$ and have energies determined by the scale $e^2/R$. Our interpretation of these chromo-dyon states is exactly the same as that of Coleman and Nelson. They correspond to excitations of the non-abelian flux tube joining the two sources. The ground-state of the quantum mechanics is simply the constant wavefunction which is a singlet under the non-abelian group $K$.

## 7. Supersymmetry and evidence for GNO duality

In this section we consider the generalization of our analysis of the multi-monopole system to theories with extended supersymmetry. The bosonic zero-modes of a multi-monopole configuration come in multiples of four. The reason is that the Bogomol'nyi equations can be formulated as the self-dual Yang-Mills equations for a time-independent Euclidean gauge field with components $W_m = (A_i, \phi)$, with $m = 1, 2, 3, 4$. This auxiliary Euclidean space has three inequivalent almost complex structure $J^{(i)}_{mn}$, $i = 1, 2, 3$, since it is a hyper-Kähler manifold. If we have one bosonic mode $\delta W_m$ which is a solution of the linearized self-dual Yang-Mills equations then by acting with the three almost complex structures we can generate three other modes $J^{(i)}_{mn}\delta W_n$ (see for example [15]). What is not clear from this analysis is the exact relation between these zero-modes and corresponding infinitesimal changes of the collective coordinates of the background field configuration. For the charge rotation zero-mode of a single monopole, there is no difficulty in interpreting the other three zero-modes. They correspond to moving the centre-of-mass of the monopole. For the non-abelian global colour modes of the multi-monopole solution the situation is more subtle. In this case the non-abelian modes come in multiples of two for each positive root of the Lie algebra $k$ with $\boldsymbol{\beta} \cdot \boldsymbol{\alpha} \neq 0$. We have argued in Appendix A that these two modes are related by just one of the almost complex structures which we denote as $J_{mn}$. A specific almost complex structure in spacetime is picked out simply because our system has a preferred direction, the axis between the monopole and the conglomerate. Its explicit form is given in the Appendix but plays no role in the following discussion. The interpretation of the other two zero-modes generated by the action of the two other almost complex structures is not so clear. The question is whether they correspond to variations of some collective coordinates of the configuration? As we have mentioned in the introduction, the only reasonable candidates for these coordinates are the relative non-abelian group orientation of the monopole and the conglomerate. These correspond to functional directions which take us outside the space of configurations for which the long-range non-abelian fields cancel. They certainly do not in any case correspond to global gauge transformations. We we will assume in the rest of our analysis that these modes do not affect our arguments regarding the global colour zero-modes.

Let us choose some basis for the global colour zero-modes $\delta_a W_m$, where $a = 1, \ldots, \dim(\mathcal{M})$. The single almost complex structure $J_{mn}$ induces an action on these zero-modes

$$\mathcal{J}_{ab}\delta_b W_m = -J_{mn}\delta_a W_n, \tag{7.1}$$

which naturally interchanges the pair of modes associated to the generators $E_{\pm\boldsymbol{\beta}}$. The moments-of-inertia tensor of the non-abelian modes defines a metric on $\mathcal{M}$

$$\mathcal{G}_{ab} = -\frac{1}{R}\int d^3x \,\mathrm{Tr}\left(\delta_a W_n \delta_b W_n\right), \tag{7.2}$$

where we have separated out the overall scale $R$ to make the metric dimensionless. This metric is the natural $K$-invariant metric on the coset space $\mathcal{M}$. Furthermore, the almost complex structure defined previously descends to the almost complex structure $\mathcal{J}_{ab}$ on $\mathcal{M}$. (Recall that $\mathcal{M}$ is a Kähler manifold and so admits one almost complex structure.)

Consider to begin with an $N = 2$ supersymmetric gauge theory.[18] Our discussion will be brief because the necessary details of the semi-classical quantization in such theories has appeared elsewhere. Our approach follows very closely the approach adopted in [15,18,32]. For an $N = 2$ theory there is single Dirac spinor which is the super-partner of the Euclidean gauge field. We now have to consider possible fermionic zero-modes that can arise. In the background of a set of monopoles which saturate the Bogomol'nyi bound half the supersymmetries of the theory are broken. These are the anti-chiral supersymmetry transformations in the Euclidean space. The zero-modes of the configuration naturally form multiplets of the unbroken supersymmetry generated by chiral spinors. Usually this pairs the four bosonic zero-modes related by the three almost complex structures with two fermionic zero-modes. For the non-abelian zero-modes this is no longer the case. Instead two non-abelian bosonic zero-modes are naturally paired with one fermion zero-mode by half of the unbroken supersymmetries. To make this explicit, we define following [15,32] a $c$-number chiral (Dirac) spinor satisfying

$$\epsilon_+^\dagger \epsilon_+ = 1, \quad J_{mn}\Gamma_n\epsilon_+ = i\Gamma_m\epsilon_+. \tag{7.3}$$

where the $\Gamma_n$ are hermitian, Euclidean gamma matrices. The fermionic modes pair with the normalizable bosonic modes $\delta_a W_m$ as

$$\chi_a = \delta_a W_m \Gamma_m \epsilon_+. \tag{7.4}$$

However, by construction, only half of these fermion zero-modes are independent since

$$\mathcal{J}_{ab}\chi_b = i\chi_a. \tag{7.5}$$

One important property of the fermion zero-modes associated to the non-abelian bosonic zero-modes is that they carry zero space-time spin. This follows from the fact that, unlike the more familiar fermion zero-modes which are paired with the spacetime and charge degrees-of-freedom of the monopole, these modes transform in the fundamental representation of the SU(2) subgroup in which the monopole is embedded.

The next stage in the semi-classical quantization program is to perform an expansion of the action in terms of collective coordinates [15,32]. First of all, we introduce a set of time-dependent coordinates $X^a$ on the coset space $\mathcal{M}$. For the $N = 2$ theory, each bosonic coordinate $X^a$ of the coset space is accompanied by a fermionic coordinate $\lambda^a$. Because of (7.5) the fermionic coordinates are not independent. They are related by the complex structure $\mathcal{J}_{ab}$ of $\mathcal{M}$:

$$-i\lambda^a \mathcal{J}_{ab} = \lambda^b. \tag{7.6}$$

The resulting quantum mechanics that arises from the non-abelian degrees-of-freedom is described by the action

$$S_{\text{eff}} = \frac{R}{e^2}\int dt\, \mathcal{G}_{ab}\left[\dot{X}^a\dot{X}^b + 4i\lambda^{\dagger a}D_t\lambda^b\right]. \tag{7.7}$$

[18] Our notations follow exactly those of Gauntlett [15] except that we use $\mu$ etc. for spacetime indices and $m$ etc. as indices of the auxiliary Euclidean space.

In the above, $\mathcal{G}_{ab}$ is the $K$ invariant metric on the coset space $\mathcal{M}$ defined in (7.2). The quantum mechanics is $N = 4 \times \frac{1}{2}$ supersymmetric. The supersymmetry transformations are explicitly

$$\begin{aligned} \delta X^a &= i\beta_1 \lambda^a + i\beta_2 \lambda^b \mathcal{J}_{ba} \\ \delta\lambda^a &= -\beta_1 \dot{X}^a - \beta_2 \dot{X}^b \mathcal{J}_{ba}, \end{aligned} \tag{7.8}$$

where $\beta_1$ and $\beta_2$ are two real parameters. It is well-known [19] that states in this theory are in one-to-one correspondence with the holomorphic forms on $\mathcal{M}$. Excited states of the theory will have energies $\sim e^2/R$. Our interpretation of these states is exactly analogous to the situation in the purely bosonic theory. States with non-zero energy are identified with excitations of the non-abelian colour flux tube linking the two sources. Zero energy states, on the other hand will be interpreted as being due to a degeneracy of the monopole itself. Ground-states of the quantum mechanics correspond to the Dolbeault complex of holomorphic harmonic forms on $\mathcal{M}$, i.e. associated to elements of the cohomology $H^{p,0}(\mathcal{M})$. For the coset spaces we are consider only $b^{p,p} \neq 0$ and in addition $b^{0,0} = 1$, and so there is only a single ground-state corresponding to the constant function. Hence in the context of an $N = 2$ supersymmetric gauge theory (with no matter fields) the conclusion is the same as for a purely bosonic gauge theory: the monopoles carry no additional degrees-of-freedom.

The situation is different in an $N = 4$ supersymmetric gauge theory. Now there are two Dirac fermion fields which means that the number of fermion zero-modes is doubled. To each bosonic coordinate of the space $\mathcal{M}$ there is an associated two component real spinor $\psi^a$. The resulting supersymmetric quantum mechanics is described by an action [18]

$$S_{\text{eff}} = \frac{R}{e^2} \int dt \left[ \mathcal{G}_{ab} \dot{X}^a \dot{X}^b + i\mathcal{G}_{ab} \bar{\psi}^a \gamma^0 D_t \psi^b + \frac{1}{6} \mathcal{R}_{abcd} \bar{\psi}^a \psi^c \bar{\psi}^b \psi^d \right]. \tag{7.9}$$

In the above $\mathcal{R}_{abcd}$ is the Riemann tensor of $\mathcal{M}$. The quantum mechanics now has an $N = 2 \times 1$ supersymmetry. It is well-known [19] that the states of the theory are in one-to-one correspondence with the forms on $\mathcal{M}$. As before, we are interested in the number of ground-states of the quantum mechanics which in this case correspond to the de Rahm complex of harmonic forms on $\mathcal{M}$. For the coset spaces we are considering the non-trivial cohomology appears in the $H^{p,p}(\mathcal{M})$. So all the ground-states are bosonic since they correspond to forms of even degree and the degeneracy is equal to the Euler character $\chi(\mathcal{M})$. Furthermore, the harmonic forms are precisely the $K$-invariant forms [33].

As described previously, the states with non-zero energy are interpreted as being excitations of the colour flux tube connecting the two sources. Our interpretation of the degenerate ground-states is rather different. Their energy, being zero, is manifestly independent of the separation of the sources so they represent a true degeneracy of the combined system. Since the combined system is constrained to have vanishing $\mathcal{O}(1/r)$ non-abelian gauge and Higgs fields this degeneracy is the same as the degeneracy of the single monopole associated to $\boldsymbol{\alpha}$. With GNO duality in mind, the degenerate ground-states have three important properties: they are all bosonic; they carry zero space-time spin; they are all singlets of the unbroken gauge group.

In the subsequent section we shall find by example (at least for non-degenerate cases) that this degeneracy is exactly what is required for the monopoles to fill out the representations of the dual gauge group. So we have found very strong evidence of GNO duality the context of $N = 4$ supersymmetry. Interestingly the dual gauge group acts on the space harmonic forms on the manifold $\mathcal{M}$ which suggest that the generators of the dual gauge group should somehow be realized as bilinears of the fermion zero-modes.

## 8. Examples

In this section we discuss a number of examples which allows us to illustrate that the multiplicity of monopoles is consistent with the GNO duality conjecture. We also discuss the cases with degenerate monopole solutions (the case when monopole solutions have the same mass and magnetic charge, but which are not related by conjugation in $H$) where the analysis is more subtle due to the existence of additional zero-modes.

The space $\mathcal{M}$ is in general the product of several Lie algebra coset spaces. For convenience we list all the spaces that appear in our construction and their Euler characters below:

$$\begin{aligned} \mathrm{SU}(n+m)/\mathrm{SU}(n)\times\mathrm{SU}(m)\times\mathrm{U}(1), &\quad (n+m)!/(n!m!),\\ \mathrm{SO}(2n)/\mathrm{U}(n), &\quad 2^{n-1},\\ \mathrm{SO}(n+2)/\mathrm{SO}(n)\times\mathrm{U}(1), &\quad n+1,\ n\in 2\mathbb{Z}+1;\ n+2,\ n\in 2\mathbb{Z},\\ \mathrm{Sp}(n)/\mathrm{U}(n), &\quad 2n,\\ E_6/\mathrm{SO}(10)\times\mathrm{U}(1), &\quad 27,\\ E_7/E_6\times\mathrm{U}(1), &\quad 56,\\ \mathrm{Sp}(n)/\mathrm{Sp}(n-1)\times\mathrm{U}(1), &\quad 2n. \end{aligned} \tag{8.1}$$

All these spaces except the last are hermitian symmetric spaces. For the simply-laced cases the Euler character is simply equal to the dimension of the representation associated in the normal way to spot of the Dynkin diagram of the group in the numerator of the coset that must be removed to give the Dynkin diagram of the group in the denominator of the coset.[19]

(i) $G = \mathrm{SU}(7)$. Choose the Higgs field so that $v_1$, $v_2$, and $v_5$ are non zero. The unbroken gauge group is $[\mathrm{U}(1)^3\times\mathrm{SU}(3)\times\mathrm{SU}(2)]/Z$. The stable massive gauge bosons form multiplets of $K$ with lowest weights given by the simple roots $\boldsymbol{\alpha}_i$, $i = 1, 2, 5$. So

[19] For a discussion of the cohomology of these spaces in the context of conformal field theory see [34]. For a discussion in the mathematical literature see [35].

under $H$ the quantum numbes of the representations are

$$\boldsymbol{\alpha}_1 : \quad (0,0)_{v_1}, \quad \boldsymbol{\alpha}_2 : \quad (\bar{3},0)_{v_2}, \quad \boldsymbol{\alpha}_5 : \quad (3,2)_{v_5}, \tag{8.2}$$

where the subscript is the electric charge of the multiplet in units of $e$. The negatives of these roots give the complex conjugate representations.

Fundamental monopole solutions can be associated to the roots $\boldsymbol{\alpha}_i$, $i = 1, 2, 5$. The non-abelian modes are associated to the coset spaces

$$\begin{aligned} \boldsymbol{\alpha}_1 :& \quad 1, \\ \boldsymbol{\alpha}_2 :& \quad \mathrm{SU}(3)/\mathrm{SU}(2) \times \mathrm{U}(1), \\ \boldsymbol{\alpha}_5 :& \quad [\mathrm{SU}(3)/\mathrm{SU}(2) \times \mathrm{U}(1)] \times [\mathrm{SU}(2)/\mathrm{U}(1)], \end{aligned} \tag{8.3}$$

Notice that the factors in these moduli spaces are hermitian symmetric spaces: $\mathrm{SU}(3)/\mathrm{SU}(2) \times \mathrm{U}(1) \simeq \mathbb{C}P^2$ and $\mathrm{SU}(2)/\mathrm{U}(1) \simeq \mathbb{C}P^1$. Following the discussion in section 7 the degeneracy of these monopole solutions is given by the Euler characteristic of the coset manifolds and so we find multiplicities and magnetic charges (in units of $2\pi/e$)

$$\boldsymbol{\alpha}_1 : \quad 1_{v_1}, \quad \boldsymbol{\alpha}_2 : \quad 3_{v_2}, \quad \boldsymbol{\alpha}_5 : \quad (3 \times 2)_{v_5}, \tag{8.4}$$

These multiplicities match exactly those in (8.2) and are consistent with the GNO conjecture since in this case the group is self-dual.

The generalization to arbitrary $\mathrm{SU}(n)$ with arbitrary symmetry breaking is now obvious. If the unbroken gauge group is $H = \mathrm{S}\left(\mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_p)\right)/Z$ then the stable massive gauge bosons come in multiplets with lowest weights $\boldsymbol{\alpha}_i$ with $i = 1 + \sum_{j=1}^{k-1} n_j$ and $n_k = 1$, so that $\boldsymbol{\alpha}_i \cdot \boldsymbol{v} \neq 0$. The gauge bosons associated to this root transforms as a $(n_{k-1}, \bar{n}_{k+1})$ of $\mathrm{SU}(n_{k-1}) \times \mathrm{SU}(n_{k+1})$, with electric charge $\boldsymbol{\alpha}_i \cdot \boldsymbol{v}$ and is a singlet of all the other factors in $H$. The negative simple root $-\boldsymbol{\alpha}_i$ gives a multiplet transforming in the complex conjugate representation. Monopole solutions are associated to $\boldsymbol{\alpha}_i$ and the associated non-abelian degrees-of-freedom are described by the coset space

$$[\mathrm{SU}(n_{k-1})/\mathrm{SU}(n_{k-1}-1) \times \mathrm{U}(1)] \times [\mathrm{SU}(n_{k+1})/\mathrm{SU}(n_{k+1}-1) \times \mathrm{U}(1)]. \tag{8.5}$$

A factor of the form $\mathrm{SU}(n)/\mathrm{SU}(n-1) \times \mathrm{U}(1) \simeq \mathbb{C}P^{n-1}$ has Euler character $n$. So the multiplicity and magnetic charge of the monopole and anti-monopole are $(n_{k-1} \times n_{k+1})_{\pm \boldsymbol{\alpha}_i \cdot \boldsymbol{v}}$. It is apparent that the multiplicity of monopole solutions is consistent with the fact that the gauge group is self-dual in this case.

(ii) $G = E_6$. Our labelling of the roots of $E_6$ matches that in [36]. We choose the Higgs vacuum expectation value to have $v_2 \neq 0$. The unbroken gauge group is $H = [\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(5)]/Z$. The stable massive gauge bosons come in multiplets with lowest weights $\pm\boldsymbol{\alpha}_2$ giving representations $(2, 10)_{\pm v_2}$ of $H$.

The monopole solutions are associated to the roots $\pm\boldsymbol{\alpha}_2$ and the non-abelian degrees-of-freedom are described by the coset space

$$[\mathrm{SU}(2)/\mathrm{U}(1)] \times [\mathrm{SU}(5)/\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)]. \tag{8.6}$$

In this case the second factor is an example of a complex Grassmanian with Euler character 10. Hence the monopole and anti-monopole have a multiplicity $(2 \times 10)_{\pm v_1}$ matching the gauge bosons.

(iii) $G = \mathrm{Sp}(r)$.[20] Consider the case when only $v_1 \neq 0$. In this case the unbroken gauge group is $H = [\mathrm{U}(1) \times \mathrm{Sp}(r-1)]/Z$. The stable massive gauge bosons come in multiplets with lowest weight $\boldsymbol{\alpha}_1$, i.e. a $(2r-2)_{v_1}$ of $H$. The negative simple root $-\boldsymbol{\alpha}_1$ corresponds to the complex conjugate representation $(2r-2)_{-v_1}$.

In this case the gauge group is not self-dual, rather $\mathrm{Sp}(r)^\vee = \mathrm{SO}(2r+1)$. So we need to enumerate the multiplicities of monopole solutions in a $\mathrm{SO}(2r+1)$ theory with a Higgs field which leads to an unbroken gauge group $H^\vee = [\mathrm{U}(1) \times \mathrm{SO}(2r-1)]/Z^\vee$. The monopole solutions are associated to the simple root $\pm\boldsymbol{\alpha}_1$ and the non-abelian degrees-of-freedom are described by the coset space

$$\mathrm{SO}(2r-1)/\mathrm{SO}(2r-3) \times \mathrm{U}(1). \tag{8.7}$$

This manifold has Euler character $2r-2$. So the multiplicities and magnetic charges of the monopoles are $(2r-2)_{\pm v_1}$. Again we see that the multiplicities are exactly what is required by the GNO conjecture.

Unfortunately it is not possible to verify the conjecture in the reverse direction, i.e. comparing the gauge bosons of the $\mathrm{SO}(2r+1)$ with the monopoles of the $\mathrm{Sp}(r)$ theory, since this is a case with degenerate monopole solutions (see the remark at the end of this section).

For all the cases without degenerate monopole solutions the following general picture emerges. Let the gauge group be $G$ with simple roots $\boldsymbol{\alpha}_i$ and the dual gauge group be $G^\vee$ with simple roots $\lambda\boldsymbol{\alpha}_i^\vee$. The Higgs vacuum expectation values in the two dual pictures are specified by the vectors $\boldsymbol{v}$ and $\boldsymbol{v}^\vee$, respectively. In order that the spectrum of the gauge bosons can be equal to the spectrum of monopoles in the dual picture, and vice-versa, requires that the coupling constant of the dual theory is

$$e^\vee = \frac{2\pi}{e\lambda} \tag{8.8}$$

and $\boldsymbol{v}^\vee = \boldsymbol{v}$. The stable massive gauge bosons in the $G$ theory are associated to the simple roots $\pm\boldsymbol{\alpha}_i$ with $\boldsymbol{\alpha}_i \cdot \boldsymbol{v} \neq 0$. If the unbroken gauge group is $H = [\mathrm{U}(1)^{r'} \times K_1 \times \cdots \times K_p]/Z$, where each of the factors is simple, then the states transform in representations $(R_1, \ldots, R_p)_{v_i}$, where $R_a$ is the representation of $K_a$ with lowest weight given

[20] In our conventions the Lie algebra of $\mathrm{Sp}(r)$ has rank $r$.

by $\boldsymbol{\alpha}_i$ projected into the root space of $K_a$. The root $-\boldsymbol{\alpha}_i$ gives the complex conjugate representations.

In the dual description, the gauge group breaks to $H^\vee = [\mathrm{U}(1)^{r'} \times K_1^\vee \times \cdots \times K_p^\vee]/Z^\vee$. The monopoles of the dual description are associated to the simple roots $\lambda\boldsymbol{\alpha}_i^\vee$ and the non-abelian degrees-of-freedom are described by the product of coset spaces

$$[K_1^\vee/L_1] \times \cdots \times [K_p^\vee/L_p]. \tag{8.9}$$

Each factor $L_a$ is either $K_a^\vee$, itself, in which case the factor is trivial, or $L_a$ has the form $L_a = \mathrm{U}(1) \times L'_a$, in which case the factor is a hermitian symmetric space. The multiplicity of the monopole solution is then the product of the Euler characters of these spaces. The fact that the multiplicities of the gauge bosons of the $G$ theory associated to $\boldsymbol{\alpha}_i$ and the monopoles of the $G^\vee$ theory associated to $\lambda\boldsymbol{\alpha}_i^\vee$ match follows from the fact that in non-degenerate cases, and so when $\lambda\boldsymbol{\alpha}_i^\vee$ is a long root,

$$\dim(R_a) = \chi(K_a^\vee/L_a). \tag{8.10}$$

Even in cases with degenerate monopole solutions the naïve counting of states is also consistent with the GNO conjecture. The point is that as it stands the relation (8.10) no longer holds; however there now exist monopoles of the dual theory which are degenerate with the $\lambda\boldsymbol{\alpha}_i^\vee$ monopole but which are related to roots of $G^\vee$ which are not simple. For example, consider the $\mathrm{SO}(2r+1) \leftrightarrow \mathrm{Sp}(r)$ example discussed above. The monopole solutions of the $\mathrm{Sp}(r)$ theory associated to the short root $\boldsymbol{\alpha}_1$ have a multiplicity given by the Euler character of the coset space

$$\mathrm{Sp}(r-1)/\mathrm{Sp}(r-2) \times \mathrm{U}(1), \tag{8.11}$$

which is $2r-2$, and this is not enough states to fill out the vector representation of the dual unbroken gauge group $[\mathrm{U}(1) \times \mathrm{SO}(2r-1)]/Z$. However, there is a degenerate monopole solution corresponding to the long root $2\boldsymbol{\alpha}_1 + 2\boldsymbol{\alpha}_2 + \cdots + 2\boldsymbol{\alpha}_{r-1} + \boldsymbol{\alpha}_r$ of $\mathrm{Sp}(r)$. This solution is invariant under conjugation by the unbroken gauge group has so (naïvely) contributes one additional state that makes the multiplicity of monopole states $2r-1$, exactly the dimension of the vector representation of $\mathrm{SO}(2r-1)$. In spite of this, it is not clear whether this simple analysis is valid because, as we have previously noted, the monopole solutions in this case have additional normalizable zero-modes with associated collective coordinates which can connect the two degenerate monopole solutions [20].

## 9. Discussion

In this paper we have obtained the semi-classical spectrum of non-abelian magnetic monopoles in $N=4$ supersymmetric gauge theory with a general gauge group. Our results indicate that, at least for the purposes of counting BPS-saturated states, non-abelian monopoles can be thought of as having a compact internal moduli space $\mathcal{M}$ which is generated by the action of the unbroken non-abelian symmetry group. Although this picture is hard to substantiate for a single monopole in isolation, we have shown that the usual problems associated with long-range non-abelian fields can be by-passed by considering suitable multi-monopole configurations. Standard semi-classical arguments then lead one to consider a maximally supersymmetric quantum mechanics on $\mathcal{M}$. In this framework, the wavefunctions of BPS-saturated states correspond to harmonic forms on $\mathcal{M}$ and the multiplicity of these states is given by the Euler character, $\chi(\mathcal{M})$. In all cases this simple result yields exactly the degeneracy of states predicted by GNO duality, although the analysis may not be reliable in certain special cases where it is known that additional continuous degeneracies of BPS monopole solutions exist [20].

One of the remaining puzzles is the occurrence of additional normalizable zero modes of the multi-monopole configurations discussed in this paper. In fact, the presence of these zero-modes is in agreement with the usual expectation that the moduli space should be a hyper-Kähler manifold. This is in contrast to the coset spaces considered above which are, in general, merely Kähler. We have explicitly assumed that the presence of additional normalizable zero modes does not alter our conclusions. In this respect we must place a caveat on the results described above. We cannot rule out the possibility that there are new collective coordinates of the multi-monopole configurations which correspond directly to the additional zero modes.[21] The fact that the modes are normalizable suggests that the additional directions in the moduli space would correspond to new configurations with vanishing long-range non-abelian fields. This would imply some hitherto undiscovered continuous degeneracy of the BPS multi-monopole system, *even for monopoles at arbitrarily large separation.* If this were the case then each of the coset spaces we have identified would be a subspace of a hyper-Kähler moduli space of larger dimension and a complete analysis would lead one to consider the de Rham cohomology of this larger space. In this case we must be content to assume that the harmonic forms on the coset sub-space are in one to one correspondence with the harmonic forms on the full moduli space. However, there are two other possibilities which we consider more likely. The first is that the additional collective coordinates are simply not present. This would mean that the extra zero modes occur only at the level of a linearized analysis and do not correspond to any finite variations of the fields. The second possibility is that the additional collective coordinates are present but correspond to directions in function space which take us outside the subspace of configurations for which the long-range non-abelian fields cancel. In either of these two cases our semiclassical results should be reliable without further assumptions.

Several features of our results are worthy of comment. Firstly we expect the degeneracy of states discovered here to persist in other theories where Montonen-Olive duality is

[21] In fact, as mentioned above, this possibility is realized in the degenerate cases described in [20].

expected to be exact for gauge group SU(2) [37]. An example is $N = 2$ super Yang-Mills theory coupled to $N_f$ matter hypermultiplets in the fundamental representation, where $N_f$ is chosen so that the perturbative $\beta$-function vanishes. For $N_f$ less than this critical value, the $N = 2$ theories are not in a non-abelian coulomb phase and there is no non-abelian gauge symmetry in the infra-red. Theories of this type with general gauge groups have been considered (see for example [38]). It seems possible that the form of exact GNO duality which appears to hold in the $N = 4$ case is, however, relevant to the understanding of the effective duality in the non-abelian coulomb phase proposed by Seiberg in $N = 1$ theories [39].

Finally, our analysis provides an explicit semi-classical description of the states which make up the multiplets of the dual gauge group. They consist of monopoles with different (even) numbers of fermion zero modes excited. This suggests that it may be possible to find an explicit realization for the generators of the dual gauge group in terms of the fermionic fields in the original theory.

The results given in this paper are incomplete in a number of respects. In our analysis, we have not considered the spectrum of dyons that would result from exciting the abelian charge degrees-of-freedom of the monopoles. In addition we have set the theta angle to be zero. It would be interesting to see how the extended $\mathrm{SL}(2, \mathbb{Z})$ duality, which arises from considering non-zero theta angle in the SU(2) theory, appears in the case of a general group. These questions are under active investigation.

We would like to thank Ed Corrigan for useful discussions. ND and TJH are supported by PPARC Advanced Fellowships.

**Addendum:** Since completing this work we have recieved two preprints [22,23] which analyse the spectrum of monopoles and dyons in N=4 supersymmetric gauge theory in the case of maximal symmetry breaking. Specifically these authors consider the case of gauge group $G = \mathrm{SU}(3)$ breaking down to $H = \mathrm{U}(1) \times \mathrm{U}(1)$. As the results pertain to a theory where only an abelian subgroup is left unbroken they do not overlap directly with ours. However it is interesting to consider the spectrum of monopole states in their model in the limit in which the non-abelian symmetry $\mathrm{SU}(2) \times \mathrm{U}(1)$ is recovered and compare with our results. Specifically we can compare to the analysis of the case $\mathrm{SU}(3) \to \mathrm{SU}(2) \times \mathrm{U}(1)$ given in Section 1 above. Using the notation of Section 1, maximal symmetry breaking is obtained by changing the choice of the vector $\boldsymbol{v}$ such that $\boldsymbol{v} \cdot \boldsymbol{\beta} > 0$. Now there are massive fundamental monopole states associated with the two positive simple roots $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. In addition to this, the authors of [22,23] demonstrate the existence of a threshold bound-state of two fundamental monopoles associated with the non-simple root $\boldsymbol{\alpha} + \boldsymbol{\beta}$. If we now take the limit $\boldsymbol{v} \cdot \boldsymbol{\beta} \to 0$ the $\boldsymbol{\beta}$ monopole becomes massless and the bound-state becomes exactly degenerate with the $\boldsymbol{\alpha}$ monopole. Importantly, because the bound-state of [22,23] occurs only in a gauge theory with $N = 4$ supersymmetry, the degeneracy occurs only in this case. Hence the resulting spectrum, in this limit, is in complete agreement with the results presented in this paper.

We should emphasize that the above refers only to the limit of the spectrum in [22,23], not of the underlying moduli space. In particular the metric on the moduli space given in [22,23] is singular in the limit of non-maximal symmetry breaking. However this divergence is due to the corresponding zero modes becoming non-normalizable, which is due in turn to the onset of long-range non-abelian fields in this limit. It seems likely that this singularity could be removed by adding a third monopole, well separated from the other two, which exactly cancels the long-range non-abelian field.

## Appendix A

In this appendix we consider in more detail the zero-modes corresponding to the global non-abelian colour transformations.

Consider first of all the Laplace equation (5.5) which must be satisfied by the gauge transformations in the background of a monopole. At distances much greater than the size of the core, it is convenient to work in unitary gauge using the two-patch formalism of Wu and Yang [24]. In this gauge, in the monopole background the angular momentum operator is

$$L_i = -i\epsilon_{ijk}x_jD_k - i\frac{x_i}{r}[\mathrm{ad}\, t^3(\boldsymbol{\alpha})]. \tag{A.1}$$

The Laplace equation (5.5) can then be expressed as

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{r^2}L_i^2\right]\Omega = 0. \tag{A.2}$$

The solution can be expanded in terms of the eigenvectors of $[\mathrm{ad}\, t^3(\boldsymbol{\alpha})]$:

$$\Omega(x_i) = \sum_{\boldsymbol{\beta}} \Omega_{\boldsymbol{\beta}}(x_i)E_{\boldsymbol{\beta}} + \Omega_0(x_i)t^3(\boldsymbol{\alpha}). \tag{A.3}$$

The component associated to $E_{\boldsymbol{\beta}}$ can then be expanded in terms of monopole harmonics [24]

$$\Omega_{\boldsymbol{\beta}}(r,\theta,\phi) = f(r)Y_{qlm}(\theta,\phi), \tag{A.4}$$

where $q = -\boldsymbol{\alpha}\cdot\boldsymbol{\beta}/\boldsymbol{\alpha}^2 = \pm 1/2$ is the eigenvalue of $[\mathrm{ad}\, it^3(\boldsymbol{\alpha})]$ on the generator $E_{\boldsymbol{\beta}}$. The quantum numbers $l$ amd $m$ are the total angular momentum and its $x_3$ component, so that $l = 0, 1/2, 1, \ldots$ and $-l \le m \le l$. The allowed values of $l$ are $|q|,\ |q|+1, \ldots$. The radial equation is

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) - \frac{1}{r^2}l(l+1)\right]f(r) = 0, \tag{A.5}$$

and so $f(r) = r^\alpha$ where $\alpha(\alpha+1) = l(l+1)$.

For each pair of roots $\pm\boldsymbol{\beta}$ with $\boldsymbol{\beta}\cdot\boldsymbol{\alpha} = -2\boldsymbol{\alpha}^2/2$, the solutions which are relevant for the global non-abelian colour modes are $r^{1/2}Y_{\pm\frac{1}{2}\frac{1}{2}m}(\theta,\phi)E_{\pm\boldsymbol{\beta}}$, where $m = \pm\frac{1}{2}$. Imposing the reality condition $\Omega^\dagger = -\Omega$, and using the reality properties of the monopole harmonics $Y^*_{qlm}(\theta,\phi) = (-)^{q+m} Y_{-ql-m}(\theta,\phi)$ we find the solutions

$$\begin{aligned}
\Omega^1(r,\theta,\phi) &= i r^{1/2}\left(Y_{\frac12\frac12-\frac12}(\theta,\phi)E_{\boldsymbol\beta} + Y_{-\frac12\frac12\frac12}(\theta,\phi)E_{-\boldsymbol\beta}\right),\\
\Omega^2(r,\theta,\phi) &= r^{1/2}\left(Y_{\frac12\frac12-\frac12}(\theta,\phi)E_{\boldsymbol\beta} - Y_{-\frac12\frac12\frac12}(\theta,\phi)E_{-\boldsymbol\beta}\right),\\
\Omega^3(r,\theta,\phi) &= r^{1/2}\left(Y_{\frac12\frac12\frac12}(\theta,\phi)E_{\boldsymbol\beta} + Y_{-\frac12\frac12-\frac12}(\theta,\phi)E_{-\boldsymbol\beta}\right),\\
\Omega^4(r,\theta,\phi) &= i r^{1/2}\left(Y_{\frac12\frac12\frac12}(\theta,\phi)E_{\boldsymbol\beta} - Y_{-\frac12\frac12-\frac12}(\theta,\phi)E_{-\boldsymbol\beta}\right).
\end{aligned} \tag{A.6}$$

Let us suppose that in the multi-monopole configuration considered in the text, the monopole is situated along the positive $x_3$ axis relative to the conglomerate. Near the monopole we can ignore to a first approximation the fields of the conglomerate and the global gauge transformations must satisfy the Laplace equation (5.5) in the back-ground of the monopole alone. The question to which we now turn is what linear combinations of the four gauge transformations in (A.6) generate the two global colour rotation modes associated to $E_{\pm\boldsymbol\beta}$ near the monopole? Each of the monopole harmonics $Y_{\pm\frac12\pm\frac12 m}(\theta,\phi)$ vanishes along a particular direction in space and the basis of solutions $\Omega^a(x_i)$ has been chosen to respect the symmetry of the problem. The first two solutions vanish in the negative $x_3$ direction while the second two vanish in the positive $x_3$ direction. The effect of the fields of the conglomerate on the first two solutions $\Omega^1(x_i)$ and $\Omega^2(x_i)$ will be suppressed due to the fact that they vanish in its direction. The other two solutions become unacceptably large in the direction of the conglomerate and would not lead to solutions of the full system. For this reason we identify the first two of these solutions as generating the two global non-abelian modes.

The explicit expressions for the corresponding zero-modes are

$$\begin{aligned}
\delta_1 W_m &= \left(\frac{xt^1(\boldsymbol\beta) - yt^2(\boldsymbol\beta)}{2r\sqrt{r+z}}, \frac{yt^1(\boldsymbol\beta) + xt^2(\boldsymbol\beta)}{2r\sqrt{r+z}}, \frac{\sqrt{r+z}}{2r}t^1(\boldsymbol\beta), -\frac{\sqrt{r+z}}{2r}t^2(\boldsymbol\beta)\right),\\
\delta_2 W_m &= \left(\frac{yt^1(\boldsymbol\beta) + xt^2(\boldsymbol\beta)}{2r\sqrt{r+z}}, \frac{-xt^1(\boldsymbol\beta) + yt^2(\boldsymbol\beta)}{2r\sqrt{r+z}}, \frac{\sqrt{r+z}}{2r}t^2(\boldsymbol\beta), \frac{\sqrt{r+z}}{2r}t^1(\boldsymbol\beta)\right),
\end{aligned} \tag{A.7}$$

which are valid in the northern hemisphere around the monopole. These two solutions are related by one of the three almost complex structures of $\mathbb{R}^4$ namely the one which relates $x_1 \leftrightarrow x_2$ and $x_3 \leftrightarrow x_4$. Explicitly

$$J_{mn} = \begin{pmatrix} 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{A.8}$$

and

$$\mathcal{J}_{ab}\delta_b W_m = -J_{mn}\delta_a W_n, \tag{A.9}$$

with

$$\mathcal{J}_{ab} = \begin{pmatrix} 0 & 1\\ -1 & 0 \end{pmatrix}. \tag{A.10}$$

**References**

[1] A. Sen, Phys. Lett. **B329** (1994) 217-221

[2] C. Vafa and E. Witten, Nucl. Phys. **B431** (1994) 3;
J. A. Harvey, G. Moore and A. Strominger, `hep-th/9501096`;
M. Bershadsky, A. Johansen and C. Vafa, Nucl. Phys. **B448** (1995) 166.

[3] C. Montonen and D. Olive, Phys. Lett. **B72** (1977) 117

[4] H. Osborn, Phys. Lett. **B83** (1979) 321

[5] P. Goddard, J. Nuyts and D. Olive, Nucl. Phys. **B125** (1977) 1

[6] R. Ward, Phys. Lett. **B158** (1985) 424
R. Leese, Nucl. Phys. **B344** (1990) 33

[7] A. Abouelsaood, Phys. Lett. **B125** (1983) 467

[8] A. Abouelsaood, Nucl. Phys. **B226** (1983) 309

[9] E.J. Weinberg, Nucl. Phys. **B167** (1980) 500

[10] P. Nelson and A. Manohar, Phys. Rev. Lett. **50** (1983) 943

[11] P. Nelson, Phys. Rev. Lett. **50** (1983) 939

[12] A.P. Balachandran, G. Marmo, N. Mukunda, J.S. Nilsson, E.C.G. Sudarshan and F. Zaccaria, Phys. Rev. **D29** (1984) 2919

[13] P. Nelson and S. Coleman, Nucl. Phys. **B237** (1984) 1

[14] E.J. Weinberg, Nucl. Phys. **B203** (1982) 445

[15] J.P. Gauntlett, Nucl. Phys. **B411** (1994) 443

[16] N. Manton, Phys. Lett. **B110** (1982) 54

[17] C.J. Callias, Commun. Math. Phy. **62** (1978) 213

[18] J.D. Blum, Phys. Lett. **B333** (1994) 92

[19] L. Alvarez-Gaumé, Commun. Math. Phys. **90** (1983) 161
E. Witten, Nucl. Phys. **B202** (1982) 253

[20] E.J. Weinberg, Phys. Lett. **B119** 151

[21] F.A. Bais, Phys. Rev. **D18** (1978) 1206

[22] J.P. Gauntlett and D. A. Lowe, 'Dyons and S-duality in $N = 4$ Supersymmetric Gauge Theory', `hep-th/9601085`

[23] K. Lee, E.J. Weinberg and P. Yi, 'Electromagnetic Duality and SU(3) Monopoles', `hep-th/9601097`

[24] T.T. Wu and C.N. Yang, Nucl. Phys. **B107** (1976) 365

[25] G. 't Hooft, Nucl. Phys. **B79** (1976) 276
A.M. Polyakov, JETP Lett. **20** (1974) 194

[26] F. Englert and P. Windey, Phys. Rev. **D14** (1976) 2728

[27] M.K. Murray, Commun. Math. Phys. **96** (1984) 539; Commun. Math. Phys. **125** (1989) 661
M.C. Bowman, Phys. Rev. **D32** (1985) 1569

[28] M.K. Prasad and C.M. Sommerfield, Phys. Rev. Lett. **35** (1975) 760

[29] E.J. Weinberg, Phys. Rev. **D20** (1979) 936

[30] B. Julia and A. Zee, Phys. Rev. **D11** (1975) 2227

[31] A. Abouelsaood, Phys. Lett. **B137** (1984) 77

[32] J.A. Harvey and A. Strominger, Commun. Math. Phys. **151** (1993) 221

[33] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press 1978

[34] W. Lerche, C. Vafa and N.P. Warner, Nucl. Phys. **B324** (1989) 427

[35] M.R. Bott, Bull. Soc. Math. France, **84** (1956) 251

[36] J.F. Cornwell, *Group Theory in Physics Vols. 1,2,3*, Academic Press 1984

[37] J.P. Gauntlett and J.A. Harvey, 'S duality and the dyon spectrum in N=2 super Yang-Mills theory', `hep-th/9508156`

[38] A. Klemm, W.Lerche, S. Theisen and S. Yankielowicz, Phys. Lett. **B344** (1995) 169
P.C. Argyers and A. Farragi, Phys. Rev. Lett. **74** (1995) 3931
P.C. Argyers, M.R. Plesser and A.D. Shapere, Phys. Rev. Lett. **75** (1995) 1699

[39] N. Seiberg, Nucl. Phys. **B435** (1995) 129