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RENORMALONS at LOW x

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ABSTRACT

The role of infrared and ultraviolet renormalons are discussed in context of leading log(1/x) approximation of perturbative QCD. We generalize the BFKL equation for the case of running coupling QCD constant and show that the uncertainties related to the contribution of infrared renormalons turn out to be smaller than the shadowing correction to the total cross section. The contribution of infrared and ultraviolet renormalons to the BFKL equation are studied, and the solution of the BFKL equation with running coupling constant is discussed.

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1 Introduction.

In this paper we study two problems:

1. How to include the running coupling QCD constant in the BFKL equation [1];

2. The contributions to the solution of the BFKL equation which originate from infrared and ultraviolet renormalons.

To demonstrate why these two problems are important for the low x physics, we review the main points of our strategy in perturbative QCD approach. We start with perturbative QCD (pQCD) in the kinematic region where the parton density and coupling QCD constant (α_s) are small. Each physical observable (i.e. gluon structure function) can be written in pQCD as following series:

$$xG(x,Q^2) \;=\; \Sigma_{n=0}\, C_n(\alpha_s)^n \cdot (L^n + a_{n-1}L^{n-1}...a_0)\;,. \tag{1}$$

This perturbative series has two big problems:

1. The natural small parameter α_s is compensated by large log (L). The value of L depends on the process and kinematic region. For example in deeply inelastic scattering (DIS):

$$L \;=\; \log\frac{Q^2}{Q_0^2} \quad at \;\; Q^2 \gg Q_0^2 \;\; but \quad x \sim 1$$

$$L \;=\; \log(1/x) \quad at \quad Q^2 \sim Q_0^2 \quad and \;\; x \to 0$$

$$L \;=\; \log Q^2 \cdot \log(1/x) \; at \;\; Q^2 \gg Q_0^2 \quad and \;\; x \to 0$$

$$L \;=\; \log(1-x) \;\; at \;\; Q^2 \sim Q_0^2 \quad and \quad x \to 1$$

Of course it is not the full list of scales. The only that we would like to emphasize that L depends on the kinematic region. Thus to calculate $xG(x,Q^2)$ one cannot calculate only the Born Approximation, but has to calculate a huge number of Feynman diagrams.

2. $C_N \;\to\; n! \;\; at \;\; n \gg 1$ [2]. Means that we are dealing with an asymptotic series and we do not know the general rules of what to do with such series. There is only one rule, namely to find the analytic function which has the same perturbative series. Sometimes but very rarely we can find such analytic function. For this case, this is the exact solution of our problem. However, the general approach has been developed based on Leading Log Approximation (LLA). The idea is simple. Let us find the analytic function that sums the series:

$$xG(x,Q^2)_{LLA} \;=\; \Sigma_{n=0} C_n\,(\alpha_s \cdot L)^n\;. \tag{2}$$

Usually we can write the equation for function $xG(x,Q^2)_{LLA}$. The most famous one, the GLAP evolution equation [3], sums eq.(2) if $L=\log(Q^2/Q_0^2)$. However, it turns out that for the region of small x the most important is so called the BFKL [1] equation. It gives the answer for eq.(2) in the case when L=log$(1/x)$.

Using the solution of the LLA equation we build the ratio:

$$R(x,Q^2) \;=\; \frac{xG(x,Q^2)}{xG(x,Q^2)_{LLA}} \;=\; \Sigma_{n=0} r^n \;=\Sigma_{n=0} c_n \cdot (L^{n-1} + a_{n-1}L^{n-2} + ...a_0)\;. \tag{3}$$

This ratio is also asymptotic series, however here we calculate this series term by term. Our hope is that the value of the next term will be smaller then the previous one ($\frac{r_n}{r_{n-1}} \ll 1$) for sufficiently large n. However, we know that at some value of $n = N$, $\frac{r_N}{r_{N-1}} \sim 1$. The only thing that we can say about such situation in general, is that our calculation has intrinsic theoretical accuracy, and the result of calculation should be presented in the form

$$R(x, Q^2) \;=\; \Sigma_{n=0}^{n=N-1} r_n \pm r_N \; . \tag{4}$$

The size of N depends mostly on how well we chose the LLA, and how well we established the value of scale L in the process of interest.

In the region of small x the natural value of the scale L is $L = \ln(1/x)$ and the series of eq.(2) can be summed by the BFKL equation. We have reached substantial understanding of the main properties of this equation during the last two decades (see ref.[4]). The last major advance in such understanding was made by A.Mueller [5] who introduced the correct degrees of freedom - colour dipoles, in terms of which the BFKL equation got the partonic-like probabilistic interpretation (see also related papers [6]). However one problem with the BFKL equation is still open,namely we do not know how to include the running QCD coupling constant in it.

There are two aspects of the problem. The first one is of the fundamental importance. Indeed, only the BFKL equation with running coupling constant can turn into the GLAP equation at large value of virtualities Q^2. We discuss this issue in the next section. The second aspect is more practical one. The first attempts to take into account the running coupling constant (α_S) in the BFKL equation [7] show that the behaviour of the solution at low x which is power-like ($xG(x, Q^2) \propto (\frac{1}{x})^{\omega_0}$) becomes much slower (the value of ω_0 is significantly lower) than for the original version of the BFKL equation with fixed α_S. In section 2 we will generalize the BFKL equation for the case of running α_S.

The question arises: can we guarantee the accuracy of the BFKL equation with running α_S knowing that the calculation even the correction of the order of α_S to the BFKL equation is a very difficult task, which to date has not been performed even by the great experts in the field [8]. To answer this question, we have to discuss the second problem in the perturbative series, namely the $n!$ rise of coefficients C_n (see eq. 1).

By now we have known three sources of the $n!$ behaviour of C_n: infrared (IR) and ultraviolet (UV) renormalons which are intimately related to the running α_S and instanton contribution. In this paper we concentrate our efforts on the first two, because the instantons at high energy give negligible contribution (see refs. [9] for relevant discussion). The $n!$ behaviour of C_n from IR and UV renormalons originate from running α_S and real parameter which governs the inclusion of α_S is $\alpha_S^n\, n!$.

In section 3, we study the first correction to the BFKL equation due to running α_S in detail, and show three principle results. The first one is that we can absorb all uncertainties related to the contribution of the IR renormalons, which have basically nonperturbative origin, to the shadowing correction (SC) to the total cross section. It should be stressed that those SC can be expressed through the correlation length between two gluons, which cannot be calculated in the framework of perturbative QCD. The second result is the fact that we can guarantee the accuracy of the BFKL equation with running α_S summing all terms of the order $\alpha_S^n n!$ in the series of eq. (1). The third result is the nonperturbative contribution of the order of $\sqrt{\frac{\Lambda^2}{Q^2}}$ from ultraviolet renormalons which does not appear in the Wilson Operator Expansion.

In section 4 we are going to discuss the solution to the BFKL equation with running α_S. We start with the numerical estimates of the first correction to the value of ω_0 due to running α_S. We formulate our numerical accuracy in the attempts to solve the BFKL equation with running

α_S and find the Green function for it. We show that running α_S leads to a smaller value of ω_0 ($xG(x,Q^2) \to (\frac{1}{x})^{\omega_0}$, than the original BFKL equation with fixed α_S.

2 The BFKL equation with running QCD coupling constant.

In this section we discuss the BFKL equation [1] with running α_S, but start with the original version of this equation with fixed α_S, both for the sake of completeness of the presentation and to clarify the main steps in the derivation of the BFKL equation. It should be mentioned that we will follow the original derivation of the BFKL equation in the momentum representation since we found it more economic to include the running α_S than the Mueller approach [5]. We firmly believe that it could also be duplicated in dipole picture, but leave this job for further publication.

2.1 The BFKL equation in the lowest order of α_S.

The BFKL equation was derived in so called Leading Log (1/x) Approximation, in which we would like to keep the contribution of the order of $(\alpha_S log(1/x))^n$ and neglect all other contributions, even of the order of $\alpha_S\ log(Q^2/Q_0^2)$. So the set of parameters in LL(1/x)A is obvious:

$$\alpha_S\ log\frac{1}{x}\ \sim\ 1\ ;$$
$$\alpha_S\ log\frac{Q^2}{Q_0^2}\ <\ 1\ ; \qquad (5)$$
$$\alpha_S\ \ll\ 1\ ;$$

Let us consider the simplest process: the quark - quark scattering at high energy at zero momentum transfer . All problems of infrared divergency in such a process is irrelevant since they are canceled in the scattering of two colourless hadrons (see, for example, ref. [5] for details).

In the Born Approximation the only diagram of Fig. 2.1 contributes to the imaginary part of the scattering amplitude (A). It is easy to understand that the result of calculation of this diagram gives:

$$2\,\mathrm{Im}\{A^{BA}(s,t=0)\}\ =\ \int\frac{d^2k_t}{(2\pi)^2}|\ M(2\to 2;\alpha_S\mid s,t=-k_t^2)\ |^2\ =\ s\frac{C_2^2\alpha_S^2}{N^2-1}\int\frac{d^2k_t}{k_t^4}\ , \qquad (6)$$

where $M(2\to 2;\alpha_S\,|\,s,t=-k_t^2)$ denotes the amplitude in the lowest order of α_S for quark - quark scattering at transfer momentum $t=-k_t^2$ through one gluon exchange (see Fig.2.1), N is the number of colours and $C_2\ =\ \frac{N^2-1}{2N}$. One can recognize the Low - Nussinov mechanism [10] of high energy interaction in this simple example.

In the next order we have to consider a larger number of the diagrams, but we can write down the answer in the following general form:

$$2\,\mathrm{Im}\{A^{NBA}(s,t=0)\}\ = \qquad (7)$$

$$\int(2\pi)^4\delta^{(4)}(\mathbf{p_1}+\mathbf{p_2}-\mathbf{p'_1}-\mathbf{p'_2}-\mathbf{q})|\ M(2\to 3;g^3)\ |^2\ \Pi_{i=1}^{i=3}\frac{d^3p'_i}{(2\pi)^3 2E'_i}\ +$$

$$\int (2\pi)^4 \delta^{(4)}(\mathbf{p_1} + \mathbf{p_1} - \mathbf{p'_1} - \mathbf{p'_2}) \cdot 2\ \mathrm{Re}\{M(2 \to 2; \alpha_S^2) \cdot M^*(2 \to 2; \alpha_S)\} \Pi_{i=1}^{i=2} \frac{d^3 p'_i}{(2\pi)^3 2E'_i} ,$$

where g is coupling constant of QCD ($\alpha_S = \frac{g^2}{4\pi}$); $M(2 \to 3; g^3)$ is the amplitude for the production of the extra gluon in the Born Approximation, and is given by the set of Feynman diagram in Fig.2.2 while $M(2 \to 2, \alpha_S^2)$ is the amplitude of the elastic scattering in the next to leading Born Approximation (see Fig. 2.3) at the momentum transfer k_t.

Two terms in eq. (7) have different physical meaning: the first one describes the emission of the additional gluon in the final state of our reaction, while the second term is the virtual correction to the Born Approximation due to the emission of the additional gluon. It corresponds to the same two particle final state, and describes the fact that due to emission the probability to detect this final state becomes smaller (we will see later that the sign of the second term is negative).

In the both terms of eq. (7) we can integrate over $\vec{p_3}'$ as well as over the longitudinal component of $\vec{p}'_1$ (p'_{1L}). Finally, we rewrite the phase space in the following way:

$$\int (2\pi)^4 \delta^{(4)}(\mathbf{p_1} + \mathbf{p_2} - \mathbf{p'_1} - \mathbf{p'_2} - \mathbf{q}\ \Pi_{i=1}^{i=3} \frac{d^3 p'_i}{(2\pi)^3 2E'_i} = \frac{1}{4\pi} \cdot \frac{1}{s} \cdot \int_{x_{min} = \frac{m^2}{s}}^{1} \frac{dx'_3}{x'_3} \int \frac{d^2 p'_{1t} d^2 p'_{3t}}{(2\pi)^4} ;$$

$$\int (2\pi)^4 \delta^{(4)}(\mathbf{p_1} + \mathbf{p_1} - \mathbf{p'_1} - \mathbf{p'_2}) \cdot \Pi_{i=1}^{i=2} \frac{d^3 p'_i}{(2\pi)^3 2E'_i} = \frac{1}{s} \cdot \int \frac{d^2 p'_{1t}}{(2\pi)^2} ; \qquad (8)$$

where x is the fraction of the longitudinal momentum carried by particle. the value of x_{min} depends on the reaction. For example in the deeply inelastic scattering $x_{min} = x_B = \frac{|Q^2|}{s}$. In the case of the quark scattering $x_{min} = \frac{m_t^2}{s}$ where m_t is the transverse mass of produced quark.

From eq. (8) one can see the origin of the $log(1/x_{min})$ contribution: it stems from the phase space integration, if $M(2 \to 3)$ does not go to zero at $x_3 \to 0$. To sum the diagrams of Fig. 2.2 in this limit we can use two tricks. The first one is for each t- channel gluon one can rewrite the numerator of the gluon propagator at high energy in the following way:

$$g_{\mu\nu} = \frac{p_{1\mu} p_{2\nu} + p_{2\mu} p_{1\nu}}{p_1 \cdot p_2} + O(\frac{m^2}{s}) \qquad (9)$$

The second trick is based on the gauge invariance of the QCD. We look on the subset of the diagrams of Fig. 2.2 pictured in Fig. 2.4 as the amplitude of the interaction of gluon k with the quark p_2 (see Fig.2.4). Since all particles in amplitude M_ν except gluon k are on the mass shell, the gauge invariance leads to the relationship:

$$k_\nu M_\nu = 0 . \qquad (10)$$

Using Sudakov variables [11] we can expand vector k as

$$k_\nu = \alpha_k p_{1\nu} + \beta_k p_{2\mu} + k_{t\mu}$$

and rewrite eq. (10) in the form:

$$(\alpha_k p_{1\nu} + \beta_k p_{2\mu} + k_{t\mu}) M_\nu = 0 . \qquad (11)$$

We note that all particle inside M_ν have a large component of their momentum on p_2. It means that we can neglect the projection of vector M_ν on p_1, or in other words

$$M_\nu \ = \ M^{(1)} p_{1\nu} \ + \ M^{(2)} p_{2\nu} \ + \ M_\nu^{(t)}$$

and $M^{(1)} \ll M^{(2)}$. Thus we can conclude from eq. (11) that

$$p_{1\nu} M_\nu \ = \ - \frac{k_{t\nu} M_\nu}{\alpha_k} \, . \tag{12}$$

Using both tricks of eq. (9) and eq. (12) one can easily see that only diagram of Fig. 2.4 (1) contributes in LL(log(1/x)A. Indeed, let us consider for example the diagram of Fig. 2.4 (2), the dominator of the quark propagator $(p_2 + k)^2$ is equal to

$$(p_2 \, + \, k)^2 \ = \ \alpha_k s \ + \ k_t^2 \ = \ - \frac{p_{3t}'^2}{x_3} + k_t^2 \ \sim \ - \frac{p_{3t}'^2}{x_3} \, .$$

Since due to eq. (12) the polarization of gluon k is transverse we cannot compensate the smallness this diagram at $x_3 \rightarrow 0$. Using the same tricks with the upper parts of the diagrams of Fig. 2.2 we arrive at the conclusion that that the set of the diagrams of Fig. 2.2 degenerates into one diagram of Fig. 2.5 with specific vertex for gluon emission:

$$\Gamma_\sigma \ = \ i \, g \, f_{abc} \cdot \frac{2 \, k_{t\mu} \, k'_{t\nu}}{\alpha_k \beta_{k'} s} \gamma_{\mu\sigma\nu} \ , \tag{13}$$

where $\gamma_{\mu\sigma\nu}$ is given by the usual Feynman rules for QCD.

Substituting eq. (13) into the first term of eq. (7) we get the contribution of the emission of one additional gluon to the next to Born Approximation in the form:

$$\mathrm{Im}\{A^{NBA}_{emission}\} \ = \ s \, \frac{\alpha_S^2 \, C_2^2}{N^2 \, - \, 1} \int_0^y dy' \int \frac{d^2 k_t}{\pi \, k^4} \cdot \frac{N \alpha_S}{\pi} \cdot K_{emission}(k_t, k'_t) \cdot \frac{d^2 k'_t}{\pi \, k'^4} \ , \tag{14}$$

where $y' \ = \ log(1/x_3)$, $y \ = \ log(1/x_{min}$ and the kernel $K(k_t, k'_t)$ is equal to

$$K_{emission}(k_t, k'_t) \ = \ \frac{k_t^2 \, k_t'^2}{(\, k_t \ - \ k'_t \,)^2} \tag{15}$$

.

To calculate the virtual correction in the next to Born Approximation ($M(\, 2 \, \rightarrow \, 2 \, ; \alpha_S^2 \,)$) we have to estimate the contribution of the set of diagrams of Fig. 2.3. The log (1/x) contribution is hidden in the real part of the amplitude $M(\, 2 \, \rightarrow \, 2 \, ; \alpha_S^2 \,)$, and easiest way to extract this log is to use the dispersion relation:

$$M(\, 2 \, \rightarrow \, 2 \, ; \, \alpha_S^2 \,) \ = \ \frac{1}{\pi} \cdot \{ \int \frac{\mathrm{Im}\{M(\, 2 \, \rightarrow \, 2 \, ; \, \alpha_S^2 \,)\}_s}{s' \, - \, s} \, d \, s' \ + \ \frac{\mathrm{Im}\{M(\, 2 \, \rightarrow \, 2 \, ; \, \alpha_S^2 \,)\}_u}{u' \, - \, u} \, d \, u' \, \} \, . \tag{16}$$

We calculate $\mathrm{Im}\{M(\, 2 \, \rightarrow \, 2 \, ; \, \alpha_S^2 \,)\}_s$ and $\mathrm{Im}\{M(\, 2 \, \rightarrow \, 2 \, ; \, \alpha_S^2 \,)\}_u$ using unitarity (see eq. (6)):

$$\mathrm{Im}\{M(\, 2 \, \rightarrow \, 2 \, ; \, \alpha_S^2 \, | \, t \, = \, -k_t^2 \,)\}_s \ = \tag{17}$$

$$\int \frac{d^2 k'_t}{(2\pi)^2} | \ \mathrm{Re}\{M(2 \rightarrow 2; \, \alpha_S \, | \, s, t = -k_t'^2) \, M(2 \rightarrow 2; \, \alpha_S \, | \, s, t = -(k_t - k'_t)^2 \,)\} \, .$$

The difference between eq. (17) and eq. (6) is that the Born amplitude for one gluon exchange enters these two equations at different values of momentum transferred t. The explicit calculations give:

$$\mathrm{Im}\{M(2\to 2;\alpha_S^2\,|\,t=-k_t^2)\}_s = s\cdot C_s\pi\,\Sigma(k_t^2); \tag{18}$$
$$\mathrm{Im}\{M(2\to 2;\alpha_S^2\,|\,t=-k_t^2)\}_u = s\cdot C_u\pi\,\Sigma(k_t^2);$$
$$\Sigma(k_t^2) = \frac{4\alpha_S}{\pi^2}\int\frac{d^2k'_t}{(k_t-k'_t)^2\,k'^2_t};$$

Using the dispersion relation of eq. (16) one can reconstruct the real part of the amplitude and the answer is

$$\mathrm{Re}\{M(2\to 2;\alpha_S^2\,|\,t=-k_t^2)\} = s\,(C_u-C_s)\cdot\Sigma(k_t^2)\cdot log s \tag{19}$$

The colour coefficients have a very famous relation between them (see Fig. 2.6) which gives, for the difference of the colour coefficients in eq. (19) the same colour structure as for the diagram of Fig. 2.3 (3). Thus $\mathrm{Re}\{M(2\to 2;\alpha_S^2\,|\,t=-k_t^2)\}$ has the same colour structure as one gluon exchange in the Born Approximation. This fact makes it possible to rewrite the second term in eq. (7) as the correction to the gluon trajectory, e.g. instead of gluon with propagator $\frac{1}{k^2}\cdot s$ we can introduce the new propagator

$$\frac{1}{k^2}\cdot s^{\alpha^G(k^2)} \tag{20}$$

$$\alpha^G(k^2) = 1-\frac{\alpha_S N}{\pi^2}\cdot\int\frac{k_t^2d^2k'_t}{(k_t-k'_t)^2\,k'^2_t} = 1-\frac{\alpha_S N}{2\pi^2}\cdot\int\frac{k_t^2d^2k'_t}{[(k_t-k'_t)^2+k'^2_t]k'^2_t}$$

The answer for the second term in eq. (7) can be written in the form

$$\mathrm{Im}\{A^{NBA}_{virtual}\} = s\,\frac{\alpha_S^2C_2^2}{N^2-1}\int\left(\alpha^G(k^2)-1\right)\cdot 2\cdot\frac{d^2k_t}{k_t^4}. \tag{21}$$

We can get the full answer for the amplitude in the next to the Born Approximation (α_S^3 by summing eq. (14) and eq. (21) and it can be written in the form:

$$\mathrm{Im}\{A^{NBA}(s,t=0)\} = s\,\frac{\alpha_S^2\,C_2^2}{N^2-1}\int_0^y dy'\int\frac{d^2k_t}{k_t^2}\cdot\frac{\alpha_S N}{\pi^2}\cdot K(k_t,k'_t)\cdot\frac{d^2k'_t}{k'^2_t} \tag{22}$$

where $y = log(1/x_{min}$ and

$$K(k_t,k'_t)\cdot\frac{1}{k'^2_t} = \frac{1}{(k_t-k'_t)^2}\cdot\frac{1}{k'^2_t} - \frac{k_t^2}{(k_t-k'_t)^2[(k_t-k'_t)^2+k'^2_t]}\cdot\frac{1}{k_t^2}. \tag{23}$$

Using eqs. (22)–(23) we can introduce function $\phi(k_t^2)$ and rewrite the total cross section for quark - quark scattering in the form:

$$\sigma_{qq} = \frac{\alpha_S C_2}{N^2-1}\cdot\int\phi(y,k_t^2)\cdot\frac{dk_t^2}{k_t^2}. \tag{24}$$

For ϕ eq. (22) gives the equation:

$$\phi^{(2)}(y,k_t^2) \;=\; \frac{\alpha_S N}{\pi^2}\cdot\int_0^y dy'\int d^2k_t' K(k_t,k_t')\;\phi^{(1)}(y',k_t'^2)\,. \tag{25}$$

where

$$K(k_t,k_t')\phi(k_t^2) \;=\; \frac{1}{(k_t-k_t')^2}\cdot\phi(k_t'^2)\;-\;\frac{k_t^2}{(k_t-k_t')^2[(k_t-k_t')^2+k_t'^2]}\cdot\phi(k_t^2)\,. \tag{26}$$

and

$$\phi^{(1)} \;=\; \frac{\alpha_S C_2}{k_t^2}\,. \tag{27}$$

2.2 The main property of the BFKL equation.

From the simplest calculation in α_S^3 order one can guess the BFKL equation [1] which looks as follows:

$$\frac{d\phi(\,y\,=\,log(1/x),k_t^2\,)}{dy} \;=\; \frac{\alpha_S N}{\pi^2}\cdot\int K(\,k_t,k_t'\,)\,\phi(\,y,k_t'^2\,)\,d^2k_t'\,, \tag{28}$$

where kernel $K(k_t,k_t')$ is defined by eq. (26). This equation sums the $(\alpha_S log(1/x))^n$ contributions and has "ladder" - like structure (see Fig. 2.7). However, such "ladder" diagrams are only an effective representation of the whole huge set of the Feynman diagrams, as explained in the simplest example of the previous subsection. The first part of the kernel $K(k_t,k_t')$ describes the emission of new gluon, but with the vertex which differs from the vertex in the Feynman diagram, while the second one is related to the reggeization of all t-channel gluons in the "ladder".

The solution of the BFKL equation has been given in ref. [1] and we we would like to recall some main properties of this solution.

2.2.1 Eigenfunctions of the BFKL equation.

The eigenfunction of the kernel $K(k_t,k_t')$ is $\phi_f \;=\; (k_t^2)^{f-1}$. Indeed after sufficiently long algebra we can see that

$$\frac{1}{\pi}\int d^2k_t' K(k_t,k_t')\;\phi_f(k_t'^2) \;=\; \chi(f)\;\phi_f(k_t^2) \tag{29}$$

where

$$\chi(f) \;=\; 2\,\Psi(1)\;-\;\Psi(f)\;-\;\Psi(1-f) \tag{30}$$

and

$$\Psi(f) \;=\; \frac{d\;\ln\Gamma(f)}{d\,f}\,,$$

$\Gamma(f)$ is the Euler gamma function.

2.2.2 The general solution of the BFKL equation.

From eq. (29) we can easily find the general solution of the BFKL equation using double Mellin transform:

$$\phi(y,k_t^2) \;=\; \int\frac{d\omega}{2\pi\,i}e^{\omega\,y}\phi(\omega,k_t^2) \;=\; \int\frac{d\omega df}{(2\pi\,i)^2}e^{\omega\,y}\phi_f(k_t^2)\,C(\omega,f) \tag{31}$$

where the contours of integration over ω and f are situated to the right of all singularities of $\phi(f)$ and $C(\omega, f)$. For $C(\omega, f)$ the equation reads

$$\omega\, C(\omega, f) \;=\; \frac{\alpha_S N}{\pi}\, \chi(f)\, C(\omega, f)\,. \tag{32}$$

Finally, the general solution is

$$\phi(y, k_t^2) \;=\; \frac{1}{2\pi\, i} \int df\, e^{\frac{\alpha_S N}{\pi}\chi(f) y + (f-1) r}\, \tilde{\phi}(f) \tag{33}$$

where $\tilde{\phi}(f)$ should be calculated from the initial condition at $y = y_0$ and $r = ln \frac{k_t^2}{q_0^2}$ (q_0^2 is the value of virtuality from which we are able to apply perturbative QCD) .

2.2.3 Anomalous dimension from the BFKL equation.

We can solve eq. (32) in a different way and find $f = \gamma(\omega)$. $\gamma(\omega)$ is the anomalous dimension in LL(log (1/x)A [1] and for $\gamma(\omega)$ we have the following series [12]

$$\gamma(\omega) \;=\; \frac{\alpha_S N}{\pi} \cdot \frac{1}{\omega} \;+\; \frac{2\alpha_S^4 N^4 \zeta(3)}{\pi^4} \cdot \frac{1}{\omega^4} \;+\; O(\frac{\alpha_S^5}{\omega^5}) \tag{34}$$

The first term in eq. (34) is the anomalous dimension of the GLAP equation [3] in leading order of α_S at $\omega \to 0$, which gives the solution for the structure function at $x \to 0$ and corresponds to so called double log approximation of perturbative QCD (DLA).The DLA sums the contributions of the order $(\alpha_S\, log(1/x)\, log(Q^2/q_0^2))^n$ in the perturbative series of eq. (1).

However, we would like to stress that eq. (34) is valid only at fixed α_S while the anomalous dimension in the GLAP equation can be calculated for running α_S. It means that we have to introduce the running α_S in the BFKL equation to achieve a matching with the GLAP equation in the region where $\omega \ll 1$ and $\frac{\alpha_S}{\omega} < 1$.

The second remark is the fact that we can trust the series of eq. (34) only for the value of $\omega \gg \omega_L$, where

$$\omega_L \;=\; \frac{\alpha_S\, N}{\pi} \cdot \chi(\frac{1}{2}) \;=\; \frac{4N\, ln 2\, \alpha_S}{\pi}\,. \tag{35}$$

In vicinity $\omega \to \omega_L$ we have the following expression for $\gamma(\omega)$:

$$\gamma(\omega) \;=\; \frac{1}{2} \;+\; \sqrt{\frac{\omega \;-\; \omega_L}{\frac{\alpha_S N}{\pi}\, 14\zeta(3)}}\,. \tag{36}$$

Substituting eq. (36) in eq. (31) we have

$$\phi(y, k_t^2) \;=\; \int \frac{d\omega}{2\pi\, i}\, e^{\omega y} \phi(\omega, k_t^2) \;=\; \int \frac{d\omega}{2\pi\, i}\, e^{\omega y \,+\, (\gamma(\omega) - 1) r} \tilde{\phi}(\omega) \;= \tag{37}$$

$$\int \frac{d\omega}{2\pi\, i}\, e^{(\omega - \omega_L) y + (-\frac{1}{2} + \sqrt{\frac{\omega - \omega_L}{\frac{\alpha_S N}{\pi} 14\zeta(3)}}\,) r}\, \tilde{\phi}(\omega)\,.$$

[1] From eq. (31) one can notice that moment variable N defined such that N = ω + 1.

Evaluating the above integral using saddle point approximation we obtain

$$\omega_S = \omega_L + \frac{1}{\frac{\alpha_S N}{\pi} 14\zeta(3)} \cdot \frac{r^2}{4\,y^2} \tag{38}$$

which gives the answer:

$$\phi(y,k_t^2) = \frac{1}{\sqrt{k_t^2\,q_0^2}} \cdot \tilde{\phi}(\omega_S) \cdot \sqrt{\frac{2\,\pi\,(\omega_S - \omega_L)}{y}} \cdot e^{\omega_L\,y - \frac{ln^2 \frac{k_t^2}{q_0^2}}{\frac{\alpha_S N}{\pi} 28\zeta(3)\,y}} \,. \tag{39}$$

We can trust this solution in the kinematic region where $(\,ln\frac{k_t^2}{q_0^2}\,)^2 \leq \frac{\alpha_S N}{\pi} 28\zeta(3)\,y$. The solution of eq. (39) illustrates one very important property of the BFKL equation, namely k_t^2 can be not only large, but with the same probability it can also be very small. It means that if we started with sufficiently big value of virtuality q_0^2 at large value of $y = ln(1/x)$ due to evolution in y the value of k_t^2 could be small ($k_t \sim \Lambda$, where Λ is QCD scale). Therefore, the BFKL equation is basically not perturbative and the worse thing, is that we have not yet learned what kind of assumption about the confinement has been made in the BFKL equation.

Our strategy for the further presentation is to keep $k_t^2 > q_0^2$ and to study what kind of nonperturbative effect we can expect on including the running α_S in the BFKL equation, as well as changing the value of ω_L in the series of eq. (34).

2.2.4 The bootstrap property of the BFKL equation.

We have discussed the BFKL equation for the total cross section, however this equation can also be proved for the amplitude at transfer momentum $q^2 \neq 0$, and not only for colourless state of two gluons in t-channel. The general form of the BFKL equation in ω - representation looks as follows [1] (see eq. (31):

$$(\omega - \omega^G(k_t^2) - \omega^G((q-k_t)^2)\,)\phi(\omega,q,k_t) = \frac{\alpha_S}{2\pi}\lambda_R \int \frac{d^2k'_t}{\pi} K(q,k_t,k'_t)\phi(\omega,q,k'_t)\,, \tag{40}$$

where the kernel $K(q,k_t,k'_t)$ describes only gluon emission and

$$K(q,k_t,k'_t) = \frac{k_t^2}{(k_t-k'_t)^2\,k'^2_t} + \frac{(q-k_t)^2}{(k_t-k'_t)^2\,(q-k'_t)^2} - \frac{q_t^2}{(q_t-k'_t)^2\,k'^2_t}\,, \tag{41}$$

λ_R is colour factor wher $\lambda_1 = 2\lambda_8 = N$ for singlet and (N^2-1) representations of colour SU(N) group and $\omega^G(k_t^2) = \alpha^G(k_t^2) - 1$ (see eq. (20)).

The bootstrap equation means that the solution of the BFKL equation for octet colour state of two gluons (for colour SU(3)), should give the reggeized gluon with the trajectory $\alpha^G(k_t^2)$ (or $\omega^G(k_t^2)$) given by eq. (20). The fact that the gluon becomes a Regge pole have been shown by us in the example of the next to the Born Approximation, and has been used to get the BFKL equation in the singlet state. It means that the solution of the BFKL equation in the octet state should have the form of a Regge pole :

$$\phi(\omega,q,k_t) = \frac{Const}{\omega - \omega^G(q^2)} \tag{42}$$

Assuming eq. (42), one arrives to the following bootstrap equation:

$$\omega^G(q^2) \; - \; \omega^G(k_t^2) \; - \; \omega^G((q-k_t)^2) \; = \; \frac{\alpha_S}{2\pi}\,\lambda_8 \int \frac{d^2 k_t'}{\pi} K(q,k_t,k_t') \; , \qquad (43)$$

It is easy to check that the trajectory of eq. (20) satisfies this equation. It is interesting to mention that we can use eq. (43) to reconstruct the form of the kernel $K(q, k_t, k_t')$, if we know the expression for the trajectory (see ref.[14]).

2.3 The BFKL equation with running α_S in the lowest order.

To take into account the running α_S we shall deal with QCD having large number of massless fermions N_f. In this case we can only insert the chain of fermions bubbles in a gluon line in the Feynman diagrams to calculate the contributions of running α_S [13]. Indeed, each such bubble gives the contribution of the order of $N_f \alpha_S(\mu^2)$, where μ^2 is the renormalization scale and there are no other contributions of the same order. Due to the renormalization property of the QCD we have to replace N_f by $-\frac{3}{2}\,b \; = \; -[\,\frac{11}{2}N \; - \; N_f\,]$ in the final answer, to get the correct contribution of running α_S in our problem.

For example, in the Born Approximation for quark - quark total cross section we have to replace the diagram of Fig. 2.1 by the sum of diagrams of Fig. 2.8 , inserting fermion bubbles in two gluon lines in t-channel. Such a procedure leads to the answer in Born Approximation:

$$2\,\mathrm{Im}\{A^{BA}(s,t=0)\} \; = \; s\frac{C_2^2}{N^2-1} \int \frac{\alpha_S^2(k_t^2)\,d^2 k_t}{k_t^4} \; . \qquad (44)$$

In the next order to Born Approximation we have to:

1. Introduce the fermion loops in the amplitudes for production of one additional gluon (see Fig. 2.2), and to the virtual correction diagrams (see Fig. 2.3) in eq. (7). Which means that one should calculate the sets of diagrams of Fig. 2.9 and Fig. 2.10 respectively.

2. Take into account the additional contribution of produced quark - antiquark pair in the final state (see Fig.2.11) in the unitarity equation (see eq. (7)).

Using the technique developed in ref. [8] we are able to calculate the diagrams of Figs. 2.9 and 2.11, however we have not yet finished these calculations and we intend publishing them elsewhere. Instead of the direct calculations of these sets of diagrams we chose the alternative approach, namely to calculate the contribution due to running α_S only to the virtual correction diagrams of Fig. 2.10, and to use the bootstrap equation (43) to reconstruct the form of the kernal of the BFKL equation with running α_S.

Repeating all technoques that we have used in eqs. (16)–(20) we end up with the function $\Sigma(k_t^2)$ which is equal the diagram of Fig. 2.8 at $t \; = \; -k_t^2$ [2] :

$$\Sigma(k_t^2) \; = \; \frac{4}{\pi^2} \int \frac{\alpha_S(k_t'^2)\,\alpha_S((\,k_t \; - \; k_t'\,)^2)\,d^2 k_t'}{(\,k_t - k_t'\,)^2\,k_t'^2} \; . \qquad (45)$$

Comparing eq. (45) with the Born Approximation (eq. (44) and Fig. 2.8) we get

$$\alpha^G(k_t^2) \, - \, 1 = \omega^G(k_t^2) \; = \; - \, \lambda_8 2\,\pi \int \frac{d^2 k_t'}{\pi} \cdot \frac{\alpha_S(k_t'^2)\,\alpha_S((\,k_t \; - \; k_t'\,)^2)}{\alpha_S(k_t^2)} \cdot \frac{k_t^2}{k_t'^2\,(\,k_t \; - \; k_t'\,)^2} \; . \qquad (46)$$

[2] We absorb one power of α_S in the definition of $\Sigma(k_t^2)$ with respect to eq. (19) to make easier the counting of the power of α_S.

Therefore, we have established the form of the trajectory of reggeized gluon in the next to Born Approximation calculation. In the next subsection we will show if we assume that the reggeization of the gluon is so general property of the BFKL equation that it also holds in the case of running α_S, than the knowledge of $\omega(k_t^2)$ is enough to get the kernel for the BFKL equation with running α_S.

2.4 The BFKL equation with running α_S from the bootstrap equation.

The bootstrap equation (see eq. (43)) gives the relation between gluon trajectory and the kernel of the BFKL equation. It is complicated functional - integral equation which relates $\omega^G(k_t^2)$ and $K(q,k_t,k_t')$, but in ref. [14] it was found that the solution of this equation can be parametrized in the form:

$$K(q,k_t,k_t') \;=\; \frac{\eta(k_t)}{\eta(k_t')\,\eta(k_t-k_t')} + \frac{\eta(q_t-k_t)}{\eta(q_t-k_t')\,\eta(k_t-k_t')} - \frac{\eta(q)}{\eta(k_t')\,\eta(q_t-k_t')}\,, \tag{47}$$

where the function η is related to the trajectory ω^G by a nonlinear integral equation:

$$\omega^G(q) \;=\; -\frac{\lambda_8}{2\pi}\int\frac{d^2k_t'}{\pi}\,\frac{\eta(q)}{\eta(k_t)\,\eta(k_t-k_t')}\,. \tag{48}$$

One can check that eqs. (47)–(48) satisfy the bootstrap equation (43).

Comparing the equation for the gluon trajectory with running α_S that we have calculated in the previous subsection (see eq. (46)) we get

$$\eta(q) \;=\; \frac{q_t^2}{\alpha_S(q_t^2)}\,. \tag{49}$$

Finally, the BFKL equation with running α_S for the total cross section ($q_t^2 = 0$) has the form:

$$\frac{d\phi(y,k_t^2)}{dy} \;=\; \frac{N}{\pi}\int dk_t'^2\;K(k_t,k_t')\phi(y,k_t'^2)\,, \tag{50}$$

where we can get the expression for kernel K substituting eq. (49) into eq. (47)[3] :

$$K(k_t,k_t')\phi(k_t^2) \;= \tag{51}$$

$$\frac{\alpha_S(k_t'^2)\,\alpha_S(\,(k_t-k_t')^2\,)}{\alpha_S(k_t^2)}\Big\{\frac{1}{(\,k_t\,-\,k_t'\,)^2}\cdot\phi(k_t'^2)\;-\;\frac{k_t^2}{(\,k_t\,-\,k_t'\,)^2\,[\,(\,k_t\,-\,k_t'\,)^2\,+\,k_t'^2\,]}\cdot\phi(k_t^2)\,\Big\}$$

The initial condition for eq. (50) in the case of quark - quark interaction looks as follows:

$$\phi^{(1)} \;=\; \frac{C_2\alpha_S(k_t^2)}{k_t^2} \tag{52}$$

[3] We absorbed the ratio $\frac{k_t^2}{k_t'^2}$ in the definition of ϕ to make a clear correspondence with the BFKL equation at fixed α_S (see eq. (28)).

2.5 The matching with the GLAP evolution equation at $x \to 0$.

In this subsection we are going to demonstrate that eq. (50) naturally gives the double log limit of the GLAP equation in the kinematic region where not only $\alpha_S\, log(1/x) \sim 1$, but also the virtuality k_t^2 is large enough ($\alpha_S\, log(k_t^2/q_0^2) \sim 1$). This limit corresponds the integration over $k_t'^2 \ll k_t^2$ in eq. (50). The BFKL equation (50) has a very simple form in this kinematic region:

$$\frac{d\phi(y,k_t^2)}{dy} = \frac{N}{\pi}\frac{1}{k_t^2}\cdot\int^{k_t^2}\alpha_S(k_t'^2)\,\phi(y,k_t'^2)\,dk_t'^2\,. \tag{53}$$

It should be stressed that the gluon structure function can be written uising the function ϕ in the following way [15]:

$$\alpha_S(Q^2)xG(x,Q^2) = \int^{Q^2}\alpha_S(k_t^2)\cdot\phi(ln(1/x),k_t^2)\,dk_t^2\,. \tag{54}$$

Using this equation we can rewrite eq. (53) in the form of differential equation with respect to $xG(x,Q^2)$:

$$\frac{\partial(xG(x,Q^2))}{\partial ln(1/x)} = \frac{N}{\pi}\cdot\int^{Q^2}\frac{\alpha_S(k_t^2)dk_t^2}{k_t^2}\cdot(xG(x,k_t^2))\,. \tag{55}$$

Taking into account the explicit form of running α_S, namely

$$\alpha_S(k^2) = \frac{2\pi}{b\,ln\frac{k^2}{\Lambda^2}} \tag{56}$$

we can derive the double differential equation for $xG(x,Q^2)$ which looks as follows:

$$\frac{\partial^2(xG(x,Q^2))}{\partial ln(1/x)\,\partial\xi} = \frac{2N}{b}(xG(x,Q^2))\,, \tag{57}$$

where $\xi = lnln(Q^2/\Lambda^2)$ and $b = \frac{3}{2}(\,11N - 2N_f\,)$.

This equation is the GLAP equation in the region of low x. In other words it is the equation which we can get from the anomalous dimension of eq. (34) taking into account only the first term with running α_S.

3 The first correction to the BFKL equation due to running α_S.

3.1 General formula.

As was discussed the BFKL equation describes a generalized "ladder" diagrams (see Fig. 2.7). In this section we are going to calculate the first correction due to running α_S in such a "ladder". The procedure how to take into account this correction is very simple, namely we have to insert the kernel $K(k_t,k_t')$ of eq. (51) in one cell of the "ladder", while in all other cells the kernel remains the kernel of the BFKL equation with fixed α_S (see eq. (23)). Fig. 3.1 illustrates this procedure and gives the graphical picture for the expression that will follow below. We will denote K_r the kernel of the BFKL equation with running α_S (see eq. (51)) and use the notation

K for the kernel of the BFKL equation with fixed α_S so as to avoid any misunderstanding in further presentation. The analytic expression for the diagram of Fig. 3.1 looks as follows:

$$\phi^{[1]}(y,Q^2) \;=\; \int dy' \,\frac{d^2 q_t}{\pi}\,\frac{d^2 q'_t}{\pi}\;\phi(y-y',Q^2,q_t^2)\;K_r(q_t,q'_t)\;\phi(y'-y_0,q'^2_t,q_0^2)\,. \qquad (58)$$

Substituting in eq. (58) the solution of the BFKL equation with fixed α_S in the form:

$$\phi(y,Q^2,q_t^2) \;=\; \int \frac{d\omega}{2\pi\, i}\; e^{\omega y + (\gamma(\omega)-1)\ln\frac{Q^2}{q_t^2}}\;\tilde\phi(\omega)\,, \qquad (59)$$

where $\gamma(\omega)$ is given by eq. (34), one can get the following answer for $\phi^{[1]}$:

$$\phi^{[1]}(y,Q^2) \;=\; \frac{N}{\pi\, Q^2}\int \frac{d\omega_1 d\omega_2}{(2\pi\, i)^2}\,\frac{d^2 q_t}{\pi}\,\frac{d^2 q'_t}{\pi}\;\tilde\phi_1(\omega_1)\,\tilde\phi_2(\omega_2)\,e^{\omega_1(y-y')+\omega_2 y'} \qquad (60)$$

$$\Big\{ e^{\gamma(\omega_1)\ln\frac{Q^2}{q_t^2}}\, e^{\gamma(\omega_2)\ln\frac{q'^2_t}{q_0^2}} \cdot \frac{\alpha_S(q_t^2)\,\alpha_S(\,(q_t - q'_t)^2\,)}{\alpha_S(q'^2_t)} \cdot \frac{1}{q'^2_t\,(\,q_t - q'_t\,)^2} \;-$$

$$e^{\gamma(\omega_1)\ln\frac{Q^2}{q_t^2}}\, e^{\gamma(\omega_2)\ln\frac{q_t^2}{q_0^2}} \cdot \frac{\alpha_S(q'^2_t)\,\alpha_S(\,(q_t - q'_t)^2\,)}{\alpha_S(q_t^2)} \cdot \frac{1}{q'^2_t\,[\,(\,q_t - q'_t\,)^2 + q'^2_t\,]}\;\Big\}\,.$$

It should be stressed that $\tilde\phi_1(\omega_1) \;=\; \frac{d\gamma(\omega_1)}{d\omega_1}$ satisfies the initial condition for $\phi(y-y',Q^2,q_t^2) \;=\; \delta(\ln(Q^2/q_t^2)$ at $y = y'$ for upper part of the diagram of Fig. 3.1, while $\tilde\phi_2(\omega_2)$ can be found from the initial condition related to distributions of gluons in the hadron (or in the quark) at $y' = y_0$.

After integration over y' one gets $\omega_1 \;=\; \omega_2$ and eq. (60) can be reduced to the form:

$$\phi^{[1]}(y,Q^2) \;= \qquad (61)$$

$$\frac{N}{\pi Q^2}\int \frac{d\omega_1}{2\pi\, i}\,\frac{d^2 q'_t}{\pi q'^2_t}\,\tilde\phi_1(\omega_1)\,\tilde\phi_2(\omega_1)\,e^{\omega_1 y + \gamma(\omega_1)\ln\frac{Q^2}{q_0^2}} \cdot \frac{1}{\alpha_S(q'^2_t)}\cdot \tilde K_r(\,\gamma(\omega_1);\alpha_S(q'^2_t)\,)\,,$$

where the kernel $\tilde K_r(\,\gamma(\omega_1);\alpha_S(q'^2_t)\,)$ is equal:

$$\tilde K_r(\,\gamma(\omega_1);\alpha_S(q'^2_t)\,) \;=\; \int \frac{d^2 q_t}{\pi}\cdot\frac{\alpha_S(q_t^2)\,\alpha_S(\,(q_t - q'_t)^2\,)}{(\,q_t - q'_t\,)^2}\cdot\Big\{\, e^{(\gamma(\omega_1)-1)\ln\frac{q'^2_t}{q_t^2}} \;-\; 2\,e^{\ln\frac{q'^2_t}{q_t^2}}\,\Big\} \qquad (62)$$

3.2 Simplification of $\tilde{\mathrm{K}}_{\mathrm{r}}(\,\gamma(\omega_1);\alpha_S(\mathrm{q}'^2_{\mathrm{t}})\,)$.

We now discuss the contribution of infrared and ultraviolet renormalons to the kernel $\tilde K_r(\,\gamma(\omega_1);\alpha_S(q'^2_t)\,)$, but let us first simplify the expression for this kernel.

1. Choosing the renormalization point $\mu^2 \;=\; q'^2_t$ we can rewrite the running α_S in eq. (62) in the form:

$$\alpha_S(q_t^2) \;=\; \frac{\alpha_S(q'^2_t)}{1 \,+\, \frac{b}{2\pi}\,\alpha_S(q'^2_t)\,\ln\frac{q_t^2}{q'^2_t}} \;=\; \frac{\alpha_S(q'^2_t)}{1 \,+\, \tilde\alpha_S(q'^2_t)\,\ln\frac{q_t^2}{q'^2_t}} \qquad (63)$$

$$\alpha_S(\,(q_t - q'_t)^2\,) \;=\; \frac{\alpha_S(q'^2_t)}{1 \,+\, \frac{b}{2\pi}\,\alpha_S(q'^2_t)\,\ln\frac{(q_t-q'_t)^2}{q'^2_t}} \;=\; \frac{\alpha_S(q'^2_t)}{1 \,+\, \tilde\alpha_S(q'^2_t)\,\ln\frac{(q_t-q'_t)^2}{q'^2_t}} \;=\; .$$

Using eq. (63) we can integrate over angle in eq. (62):

$$\int d\phi \frac{\alpha_S((q_t - q_t')^2)}{(q_t - q_t')^2} = \alpha_S(q_t'^2)\Sigma_{i=0}^{\infty}(-1)^i \tilde{\alpha}_S(q_t'^2)^i \frac{d^i}{(d\mu)^i}|_{\mu=0} \int \frac{d\phi (q'^2)^{-\mu}}{(q_t - q_t')^{1-\mu}} = \quad (64)$$

$$\pi\, \alpha_S(q_t'^2)\Sigma_{i=0}^{\infty}(-1)^i \tilde{\alpha}_S(q_t'^2)^i \frac{d^i}{(d\mu)^i}|_{\mu=0} \frac{1}{|q_t^2 - q_t'^2|} \cdot [\frac{|q_t^2 - q_t'^2|}{q_t'^2}]^{2\mu} F(\mu,\mu,1,z) \rightarrow$$

$$\rightarrow \pi \frac{\alpha_S(q_t'^2)}{1 + 2\,\tilde{\alpha}_S(q_t'^2)\, ln\frac{|q_t^2 - q_t'^2|}{q_t'^2}} \cdot \frac{1}{|q_t^2 - q_t'^2|} + O(z\alpha_S^3(q_t'^2)),$$

where $z = \frac{q_t'^2}{q_t^2}$ for $q_t'^2 < q_t^2$ and $z = \frac{q_t^2}{q_t'^2}$ for $q_t^2 < q_t'^2$, $F(\mu,\mu,1,z)$ denotes the Gauss hypergeometric function.

The next trick we suggest to work with the product of α_S in eq. (62):

$$\frac{1}{1 + \tilde{\alpha}_S(q_t'^2)\, ln\frac{q_t^2}{q_t'^2}} \cdot \frac{1}{1 + 2\,\tilde{\alpha}_S(q_t'^2)\, ln\frac{|q_t^2 - q_t'^2|}{q_t'^2}} = \quad (65)$$

$$\frac{1}{1 + \tilde{\alpha}_S(q_t'^2)\, ln\frac{q_t^2}{q_t'^2}} + \frac{1}{1 + 2\,\tilde{\alpha}_S(q_t'^2)\, ln\frac{|q_t^2 - q_t'^2|}{q_t'^2}} - 1 +$$

$$2\, \frac{\tilde{\alpha}_S^2(q_t'^2)\, ln\frac{q_t^2}{q_t'^2}\, ln\frac{|q_t^2 - q_t'^2|}{q_t'^2}}{(1 + \tilde{\alpha}_S(q_t'^2)\, ln\frac{q_t^2}{q_t'^2}) \cdot (1 + 2\,\tilde{\alpha}_S(q_t'^2)\, ln\frac{|q_t^2 - q_t'^2|}{q_t'^2})}.$$

We can neglect the contribution of the last term in eq. (65), since it is proportional to α_S^2 and both logs cannot be large simultaneously. We have checked numerically that this contribution is really very small, but it is even more important for us, that this term do not bring any nonperturbative phenomena in the question of the interest, which have not been included in the first three terms.

Using variable $z = \frac{q_t^2}{q'^2}$ which we have introduced we can rewrite the expression for kernel $\tilde{K}_r(\gamma(\omega_1); \alpha_S(q_t'^2))$ in the form:

$$\tilde{K}_r(\gamma(\omega_1); \alpha_S(q_t'^2)) = \alpha_S^2(q_t'^2) \int \frac{dz}{z\,|1 - z|} \cdot \quad (66)$$

$$\{\frac{1}{1 + \tilde{\alpha}_S(q_t'^2)\, lnz} + \frac{1}{1 + 2\,\tilde{\alpha}_S(q_t'^2)\, ln|1 - z|} - 1\} \cdot \{e^{(1-\gamma(\omega_1))\, lnz} - e^{lnz}\}$$

3.3 Infrared Renormalons.

3.3.1 $\tilde{\mathbf{K}}_r(\gamma(\omega_1); \alpha_S(\mathbf{q}_t'^2))$ at $q_t^2 < q_t'^2$.

The IR renormalons contribution comes from the kinematic region $q_t^2 < q_t'^2$. Let us simplify eq. (66) in this kinematic region introducing the new variable u $z = e^u$:

$$\tilde{K}_r(\gamma(\omega_1); \alpha_S(q_t'^2))|_{q_t^2 < q_t'^2} = \alpha_S^2(q_t'^2) \int_0^{\infty} du \frac{1}{1 - e^{-u}} \cdot \quad (67)$$

$$\{\frac{e^{-(1-\gamma(\omega_1))u} - e^{-u}}{1 - \tilde{\alpha}_S(q_t'^2)\,u} + \frac{2\tilde{\alpha}_S(q_t'^2)\,ln(1 - e^{-u})\cdot[e^{-(1-\gamma(\omega_1))u} - e^{-u}]}{1 - 2\,\alpha_S(q_t'^2)\,ln(1 - e^{-u})}\}\,.$$

Changing the variable of integration in the second term of eq. (67) $1 - e^{-u} \rightarrow e^{-\frac{u}{2}}$ we can rewrite eq. (67) in the form:

$$\tilde{K}_r(\,\gamma(\omega_1);\alpha_S^2(q_t'^2)\,)|_{q_t^2<q_t'^2} = \alpha_S(q_t'^2)\int_0^\infty du\,\frac{1}{1 - \alpha_S(q_t'^2)\,u}\cdot \tag{68}$$

$$\{\frac{e^{-(1-\gamma(\omega_1))u} - e^{-u}}{1 - e^{-u}} + \frac{1}{2}\,\frac{\alpha_S(q_t'^2)\,u\,[(1 - e^{-\frac{u}{2}})^{1-\gamma(\omega_1)} - (1 - e^{-\frac{u}{2}})]}{1 - e^{-\frac{u}{2}}}\}\,.$$

Eq. (68) can be rewritten in more symmetric way for the difference

$$\Delta\tilde{K}_r = \frac{1}{\alpha_S(q_t'^2)}\tilde{K}_r(\,\gamma(\omega_1);\alpha_S(q_t'^2)\,)\,|_{q_t^2<q_t'^2} - \alpha_S(q_t'^2)\,\chi(\gamma(\omega_1)\,)\,|_{q_t^2<q_t'^2} \tag{69}$$

$$= \alpha_S(q_t'^2)\int_0^\infty du\,\frac{\alpha_S(q_t'^2)\,u}{1 - \alpha_S(q_t'^2)\,u}\cdot\{\frac{e^{-(1-\gamma(\omega_1))u} - e^{-u}}{1 - e^{-u}}$$

$$+ \frac{1}{2}\,\frac{[(1 - e^{-\frac{u}{2}})^{1-\gamma(\omega_1)} - (1 - e^{-\frac{u}{2}})]}{1 - e^{-\frac{u}{2}}}\}\,,$$

where $\chi(\gamma(\omega_1)|_{q_t^2<q_t'^2}$ is the contribution to the kernel of the BFKL equation (see eq. (32)) from the kinematic region $q_t^2 < q_t'^2$.

3.3.2 IR renormalons: uncertainty of the perturbative series and relation to the shadowing correction.

In this subsection we are going to show that $\Delta\tilde{K}_r$ contains terms of the order of $\alpha_S^n\,n!$ which give the natural limit for perturbative calculation since these contributions originate from integration over small value of q_t. To see such terms we can expand eq. (69) with respect to u and e^{-u} or $e^{-\frac{u}{2}}$. Indeed, $\Delta\tilde{K}_r$ can be written as the following series:

$$\Delta\tilde{K}_r|_{q_t^2<q_t'^2} = \alpha_S(q_t'^2)\Sigma_{n=1}\Sigma_{l=0}\,\tilde{\alpha}_S^n(q_t'^2)\cdot \tag{70}$$

$$[\int_0^\infty du\,u^n\,\{(\,e^{-(1+l-\gamma(\omega_1))\,u} + \frac{1}{2}\,\frac{\Gamma(\gamma(\omega_1) + 1)}{\Gamma(\gamma(\omega_1) + l)\,l!}\,e^{-\frac{(l+1)u}{2}}\,) - (\gamma = 0)\}\,] =$$

$$\frac{2\pi}{b}\cdot\Sigma_{n=1}\,\Sigma_{l=0}[\,\{\,(\,\frac{\tilde{\alpha}_S(q_t'^2)}{1 + l - \gamma(\omega_1)}\,)^{n+1}\cdot\Gamma(n+1) +$$

$$\frac{1}{2}\,\frac{\Gamma(\gamma(\omega_1) + 1)}{\Gamma(\gamma(\omega_1) + l)\,l!}(\,\frac{2\,\tilde{\alpha}_S(q_t'^2)}{l + 1}\,)^{n+1}\cdot\Gamma(n+1)\,\} + \{\gamma = 0\}\,] =$$

$$\alpha_S\,\Sigma_{n=1}\,\Sigma_{l=0}\,\tilde{\alpha}_S^n\,R_{n,l}$$

In eq. (70) the integral representation for Euler gamma function has been used, namely

$$\Gamma(n+1) = \int_0^\infty t^n\,e^{-t}dt$$

Eq. (70) is a typical asymptotic series which is no longer reliable, starting from critical order $n = N$ which can be found from the condition:

$$\alpha_S^N(q_t'^2) R_{N,l} \;=\; \alpha_S^{N+1} R_{N+1,l}\;, \tag{71}$$

as has been discussed in the introduction. The answer for the $\Delta\tilde{K}_r$ can be written in the form:

$$\Delta\tilde{K}_r \;=\; \alpha_S\, \Sigma_{n=1}^{N}\, \Sigma_{l=0} \alpha_S^n R_{n,l} \;\pm\; \alpha_S^{n+2} R_{N+1,l}\;. \tag{72}$$

One can check that the largest uncertainties originates from $l = 0$ in the series of eq. (70) which leads to the value of N equal:

$$N \;=\; \frac{1 - \gamma(\omega_1)}{\bar{\alpha}_S(q_t'^2)} \;=\; (\,1 \;-\; \gamma(\omega_1)\,)\cdot ln \frac{q_t'^2}{\Lambda^2}\;.$$

The corresponding uncertainty in the perturbative expansion is given by

$$\delta(\,\Delta\tilde{K}_r\,)\,|_{q_t^2 < q_t'^2} \;=\; \alpha_S(q_t'^2)\,\sqrt{\pi(1-\gamma(\omega_1))\bar{\alpha}_S(q_t'^2)}\;\{\,\frac{\Lambda^2}{q_t'^2}\,\}^{-1+\gamma(\omega_1)} \tag{73}$$

Substituting eq. (73) in eq. (61) one can calculate the uncertainty in $\phi^{[1]}$:

$$\delta\phi^{[1]} \;=\; \frac{N}{\pi Q^2} \int \frac{d\omega_1}{2\pi\, i}\, \frac{d^2 q_t'}{\pi q_t'^2} \bar{\phi}_1(\omega_1)\,\bar{\phi}_2(\omega_1)\, e^{\,\omega_1 y + \gamma(\omega_1)\, ln \frac{Q^2}{q_0^2}}\;. \tag{74}$$

$$\alpha_S(q_t'^2)\,\sqrt{\pi(1-\gamma(\omega_1))\bar{\alpha}_S(q_t'^2)}\;\{\,\frac{\Lambda^2}{q_t'^2}\,\}^{-1+\gamma(\omega_1)}\;.$$

The fact that $\delta\phi^{[1]}$ is proportional to the QCD scale Λ indicates the nonperturbative origin of the uncertainties. The physical meaning of this phenomena is well known [18]. Indeed, the typical value of the momentum ($(q_{eff})_n$) essential in the integral for $\Gamma(n)$ in eq. (69), is equal to:

$$(q_{eff}^2)_n \;\approx\; q'^2\,\cdot\, e^{-\frac{n}{1-\gamma(\omega_1)}} \tag{75}$$

and at large value of n this momentum can be very small, so small that we cannot use the perturbative QCD in our calculations ($q_{eff} \approx \Lambda$).

Of course we cannot trust the value of eq. (74), we consider this equation as the indication that we should examine the nonperturbative contribution with the same Q^2 dependence as is given by eq. (74). In the case of the e^+e^- - annihilation the uncertainties from IR renormalons can be absorbed in the nonperturbative value of the gluon condensate (see refs. [18]). What nonperturbative phenomena can be responsible for the uncertainties in our case is the subject that we now discuss.

It is very instructive to compare $\delta\phi^{[1]}$ with the first diagram (see Fig.3.2) for the shadowing correction [16] [17]:

$$\phi_{SC}^{[1]} \;=\; \frac{\bar{\gamma}}{Q^2\pi R^2}\cdot\int_0^y dy' \int \frac{dq'^2}{q_t'^4}\;\phi(y-y',Q^2,q_t'^2)\;\frac{\alpha_S(q_t'^2)N}{\pi}\;\phi^2(y',q_t'^2,q_0^2) \;=\; \frac{\bar{\gamma}}{Q^2\,\pi\,R^2}\cdot \tag{76}$$

$$\int \frac{d\omega_1\, d\,\omega'}{(2\,\pi)^2}\,\bar{\phi}_1(\omega_1)\,\bar{\phi}_2(\omega_1-\omega')\,\bar{\phi}(\omega')\,e^{\omega_1\, y}\;\frac{\alpha_S(q_t'^2)N\,dq'^2}{\pi\; q_t'^4}\;e^{\gamma(\omega_1)\,ln\frac{Q^2}{q_t'^2} \,+\, [\gamma(\omega_1-\omega') + \gamma(\omega')]\,ln\frac{q_t'^2}{q_0^2}} \;=$$

$$\frac{\bar{\gamma}}{Q^2\,\pi\,R^2}\cdot\int\frac{d\omega_1}{2\,\pi}\,\tilde{\phi}_1(\omega_1)\,\tilde{\phi}_2^2(\frac{\omega_1}{2})\;e^{\omega_1\,y}\;\frac{\alpha_S(q_t'^2)N\,dq'^2}{\pi\;q_t'^2}\;\frac{1}{\sqrt{2\pi\gamma''(\frac{\omega_1}{2})}}\;e^{\gamma(\omega_1)\,ln\frac{Q^2}{q_t'^2}\;-\;[\,2\,\gamma(\frac{\omega_1}{2}\;-\;1]\,ln\frac{q_t'^2}{q_0^2}}\;,$$

where we substituted ϕ from eq. (31) and integrated over ω' using steepest decent method. $\bar{\gamma}$ is the value of triple "ladder" vertex and R^2 is the two gluon correlation length inside the hadron (all other notations are clear from Fig. 3.2). One can find more detailed calculations of the SC diagram in refs.[16][17].

Comparing eq. (76) with eq. (74) one can see that the shadowing correction to the structure function gives the contribution which looks very similar to the nonperturbative uncertainty from IR renormalons. Indeed

$$\phi_{SC}^{[1]}\;\propto\;\frac{1}{Q^2}\;e^{\gamma(\omega_1)\;ln\frac{Q^2}{q_t'^2}\;+\;[\,2\,\gamma(\frac{\omega_1}{2})\,-\,1\,]\;ln\frac{q_t'^2}{q_0^2}}$$

while

$$\delta\phi^{[1]}\;\propto\;\frac{1}{Q^2}\;e^{\gamma(\omega_1)\;ln\frac{Q^2}{q_t'^2}\;+\;[\;\gamma(\omega_1)\,-\,1\,]\;ln\frac{q_t'^2}{q_0^2}}\;.$$

However, one can conclude from the above expression that the SC correction is always bigger than the uncertainties due to the IR renormalons contributions, since $2\gamma(\frac{\omega_1}{2})\;>\;\gamma(\omega_1)$.

We therefore conclude. If we take into account such nonperturbative phenomena as the shadowing correction we can forget about uncertainties due to the IR renormalons contribution in the kernel of the BFKL equation with running coupling QCD constant.

However, we should be very careful with the above statement, because there is an uncertainty from the second term in eq. (69) which gives the value of N

$$N\;=\;\frac{1}{2\bar{\alpha}_S(q_t'^2)}\;=\;\frac{1}{2}\cdot ln\frac{q_t'^2}{\Lambda^2}\;.$$

This N generates the uncertainty in $\phi[1]$

$$\delta\phi^{[1]}\;\propto\;\frac{1}{Q^2}\;e^{\gamma(\omega_1)\,ln\frac{Q^2}{q_t'^2}\;+\;\frac{1}{2}\,ln\frac{q_t'^2}{q_0^2}}\;.$$

This uncertainty is bigger than the shadowing correction for $2\gamma(\frac{\omega_1}{2})\;<\;\frac{1}{2}$. It means that we cannot trust our calculation of the shadowing correction at $\omega\;>\;\omega_c$, where ω_c can be found from equation $2\,\gamma(\frac{\omega_c}{2})\;=\;\frac{1}{2}$. In other words, for such value of ω the nonperturbative corretions to the BFKL equation with running α_S can be bigger than the SC contribution. The rough estimate for the value of ω_c from the first term of eq. (34) gives $\omega_c\;=\;\frac{4N\,\alpha_S(q_t'^2)}{\pi}$.

It should be stressed that we have found the contribution to the gluon structure function which behaves as $\sqrt{\frac{\Lambda^2}{Q^2}}$ and which does not appear in the Wilson Operator Product Expansion. It originates from the small value of the momentum of emitted gluon ($q_t\,-\,q_t'\;\rightarrow\;0$ in Fig. 3.1). We suspect that this contribution is closely related to our fundamental hypothesis on the completeness of the wave functions of the produced hadrons in the final state of the deeply inelastic processes. This hypothesis allows us to reduce the calculation of the deep inelastic structure function to parton (quark and gluon) degrees of freedom. However, the slow hadrons in the current jet can interact with the slow hadrons in the target jet and such an interaction can violate the assumed completeness of the wave functions of produced partons. We would like to draw your attention to the fact that the ratio of the mass of the current jet of hadrons to the value of Q^2 is equal to $\frac{\mu}{Q}$, where μ is the mass of the lightest hadron. However, we have not found the theoretical description of this nonperturbative contribution and consider it as a good subject for the further investigation.

3.3.3 IR renormalons: singularities in the Borel plane.

Let us go back to the asymptotic series of eq. (69) and try to sum this series. To sum asymptotic series, means to find the analytic function which has the same expansion as the asymptotic series. The general procedure of how to guess the analytic function is to use the Borel representation for the divergent, perturbation expansion for

$$\Delta\tilde{K}_r\,|_{q_t^2 < q_t'^2}(\alpha_S(q_t'^2)) \;=\; \alpha_S\,\Sigma_{n=1}\,\Sigma_{l=0}\,\tilde{\alpha}_S^n\,R_{n,l}\,.$$

Instead of this expansion we define the function:

$$\Delta K_r^B\,|_{q_t^2 < q_t'^2} \;=\; \Sigma_{n=1}\,\Sigma_{l=0}\,R_{n+1,l}\,\frac{b^n}{n!}\,. \qquad (77)$$

It is widely believed that this series has a finite radius of convergence in the b - plane (see refs.[18] for detail discussions how it works in the case of e^+e^- - annihilation). ΔK_r^B is the Borel function corresponding to $\Delta\tilde{K}_r\,|_{q_t^2 < q_t'^2}(\alpha_S(q_t'^2))$ and we get

$$\Delta\tilde{K}_r\,|_{q_t^2 < q_t'^2}(\alpha_S(q_t'^2)) \;-\; \Delta\tilde{K}_r\,|_{q_t^2 < q_t'^2}(0) \;=\; \int_0^\infty db\,\Delta K_r^B\,|_{q_t^2 < q_t'^2}\,e^{-\frac{b}{\tilde{\alpha}_S}}\,. \qquad (78)$$

Eq. (78) is certainly correct order by order in perturbation theory, but in general there are two known reasons why this cannot be true for such asymptotic series.

1.The b - integral in eq. (78) does not converge at $b \to \infty$ [19]. However, this singular behaviour comes from the mass spectrum in the exact expression for the physical observable and is irrelevant for the function $\Delta\tilde{K}_r\,|_{q_t^2 < q_t'^2}(\alpha_S(q_t'^2))$ for which we have only a perturbative expression.

2. There are singularities of ΔK_r^B on the positive real b - axis making the integral in eq. (78) ambiguous [19] [20] . Indeed, substituting $R_{n,l}$ from eq. (69) in eq. (77) one can see that ΔK_r^B has the simple poles in b

$$\frac{1}{b \;-\; b_{0l}}$$

where

$$b_{0l} \;=\; a^{(i)} \qquad (79)$$

$$a^{(1)} \;=\; 1 \;+\; l \;-\; \gamma(\omega_1)\,;$$

$$a^{(2)} \;=\; 1 \;+\; l\,;$$

$$a^{(3)} \;=\; \frac{1 \;+\; l}{2}\,;$$

Therefore the singularities in the Borel plane are as shown in Fig. 3.3 (we want to mention that we introduced the Borel transform with respect to $\tilde{\alpha}_S$ but not with respect to α_S).

These singularities lead to ambiguities in performing of the Borel integral. However we have discussed these ambiguities in the previous section and drove to the conclusion that the uncertainties related to them are smaller than the contributions of the shadowing corrections. It means that we can absorbed all uncertainties related to errors that we can make performing the Borel integral in the vicinities of the the poles on positive b - axis in the nonperturbative gluon correlation (R^2) length in the expression for the SC correction (see eq. (76)).

We choose the following prescription for the definition of the integral

$$\int_0^\infty \frac{db\, e^{-\frac{b}{\bar{\alpha}_S}}}{b\,-\,b_{0l}} \;=\; lim|_{\epsilon\to 0}\, \{ \int_0^{b_{0l}-\epsilon} \frac{db\, e^{-\frac{b}{\bar{\alpha}_S}}}{b\,-\,b_{0l}} \;+\; \int_{b_{0l}+\epsilon}^{\infty}, \frac{db\, e^{-\frac{b}{\bar{\alpha}_S}}}{b\,-\,b_{0l}} \} \;= \tag{80}$$

$$= \,-\, e^{-\frac{b_{0l}}{\bar{\alpha}_S}}\; Ei\left(\frac{b_{0l}}{\bar{\alpha}_S}\right) .$$

The $Ei(x)$ is very good analytic function and it solves our task: to find the analytic function which has the same expansion as our perturbative series.

3.3.4 $\Delta\tilde{\mathbf{K}}_r$ for $\mathbf{q}_t^2 < \mathbf{q}_t'^2$.

Finally the answer for $\Delta\tilde{K}_r$ in the region $q_t^2 < q_t'^2$ is:

$$\Delta\omega_{IR} \;=\; \Delta\tilde{K}_r\,|_{q_t^2 < q_t'^2} \;=\; \frac{\alpha_S}{\bar{\alpha}_S}\cdot \Sigma_{n=0}^{\infty}\cdot \tag{81}$$

$$\{ \left(e^{-\frac{1+n}{\bar{\alpha}_S}}\, Ei\left(\frac{1+n}{\bar{\alpha}_S}\right) \,-\, \frac{\bar{\alpha}_S}{n+1} \,-\, e^{-\frac{1+n-\gamma(\omega_1)}{\bar{\alpha}_S}}\, Ei\left(\frac{1+n-\gamma(\omega_1)}{\bar{\alpha}_S}\right) \,+\, \frac{\bar{\alpha}_S}{n+1-\gamma(\omega_1)} \right) +$$

$$\frac{1}{2}\,\frac{\Gamma(\gamma(\omega_1)+1)}{\Gamma(\gamma(\omega_1)+1)\; n!}\left(\frac{2\bar{\alpha}_S}{n+1} \,-\, e^{-\frac{n+1}{2\bar{\alpha}_S}}\, Ei\left(\frac{n+1}{2\,\bar{\alpha}_S}\right)\right)\}$$

The series of eq. (81) is well convergent, since the general term of this expansion at any given value of α_S falls as $1/n^2$. Figs. 3.4 and 3.5 show the numerical estimates for $\Delta\tilde{K}_r$ versus $\gamma(\omega_1)$ at fixed $\bar{\alpha}_S = 0.2$ (Fig. 3.4) and versus $x = \frac{1}{\bar{\alpha}_S}$ at given $\gamma = \frac{1}{2}$. We would like to draw your attention to the fact that for $\bar{\alpha}_S < 0.33$ the correction to the value of the BFKL kernel with fixed α_S ($\alpha_S\chi(\frac{1}{2})$) is smaller than 12% ($\frac{\Delta K_r}{\alpha_S\chi(\frac{1}{2})} < 12\%$).

3.4 Ultraviolet Renormalons

3.4.1 $\tilde{\mathbf{K}}_r(\,\gamma(\omega_1);\alpha_S(\mathbf{q}_t'^2)\,)$ at $\mathbf{q}_t^2 > \mathbf{q}_t'^2$.

In this subsection we are going to write down the expression for kernel K_r in the kinematic region $q_t^2 > q_t'^2$ which is related to the contribution of so called ultraviolet renormalons. Using all techniques of section 3.2 and introducing variable $z = e^{-u}$ we can rewrite eq. (66) for $u > 0$ in the form:

$$\tilde{K}_r(\,\gamma(\omega_1);\alpha_S(q_t'^2)\,)\,|_{q_t^2 > q_t'^2} \;=\; \alpha_S^2(q_t'^2)\,\int_0^\infty du\,\frac{1}{1\,-\,e^{-u}}\cdot \tag{82}$$

$$\{\frac{e^{-\gamma(\omega_1)\,u}\,-\,e^{-u}}{1\,+\,\bar{\alpha}_S(q_t'^2)\,u} \,-\, \frac{2\bar{\alpha}_S(q_t'^2)\,ln(\,e^u\,-\,1\,)\cdot[\,e^{-\gamma(\omega_1)u}\,-\,e^{-u}\,]}{1\,+\,2\,\bar{\alpha}_S(q_t'^2)\,ln(\,e^u\,-\,1\,)}\}\,.$$

In the second term of eq. (82) we change variable of integration $e^u - 1 \to e^{\frac{u}{2}}$ and the difference ΔK_r can be rewritten in the form:

$$\Delta\tilde{K}_r \;=\; \frac{1}{\alpha_S(q_t'^2)}\tilde{K}_r(\,\gamma(\omega_1);\alpha_S(q_t'^2)\,)\,|_{q_t^2>q_t'^2} \;-\; \alpha_S(q_t'^2)\,\chi(\gamma(\omega_1)\,)\,|_{q_t^2<q_t'^2} \tag{83}$$

$$= \alpha_S(q_t'^2)\int_0^\infty du\,\frac{\alpha_S(q_t'^2)\,u}{1+\alpha_S(q_t'^2)\,u}\cdot\{\,\frac{e^{-\gamma(\omega_1)u}\;-\;e^{-u}}{1\;-\;e^{-u}} + \frac{1}{2}\cdot[\,(\,1\;+\;e^{\frac{u}{2}}\,)^{-\gamma(\omega_1)}\;-\;1\,]\,\}\,.$$

In the same way as has been done for the region $q_t^2 < q_t'^2$ we can expand the above expression with respect to e^{-u} or $e^{-\frac{u}{2}}$ and derive the following series:

$$\Delta\tilde{K}_r|_{q_t^2<q_t'^2} \;=\; \alpha_S(q_t'^2)\Sigma_{n=1}\Sigma_{l=0}\,(\,-\bar{\alpha}_S\,)^n(q_t'^2)\,\cdot \tag{84}$$

$$[\int_0^\infty du\,u^n\,\{\,e^{-(l+\gamma(\omega_1))u}\;+\;e^{-(1+l)u}\;+\;\frac{1}{2}\,(-1)^l\frac{\Gamma(\gamma(\omega_1)\,+\,1\,)}{\Gamma(\gamma(\omega_1)\,+\,l\,)\,l!}e^{-\frac{(l+1)u}{2}}\,\}\,]\;=$$

$$\frac{2\pi}{b}\cdot\Sigma_{n=1}\,\Sigma_{l=0}\,[\,\{\,(\,\frac{-\,\bar{\alpha}_S(q_t'^2)}{l\,+\,\gamma(\omega_1)}\,)^{n+1}\,\cdot\,\Gamma(n+1)\,+$$

$$\frac{1}{2}\,(-1)^l\frac{\Gamma(\gamma(\omega_1)\,+\,1\,)}{\Gamma(\gamma(\omega_1)\,+\,l\,)\,l!}(\,\frac{-\,2\,\bar{\alpha}_S(q_t'^2)}{l\,+\,1\,+\,\gamma(\omega_1)}\,)^{n+1}\,\cdot\,\Gamma(n+1)\,\}\;+\;\{\gamma=0\}\,]\;=$$

$$\alpha_S\;\Sigma_{n=1}\,\Sigma_{l=0}\,(\,-\,\bar{\alpha}_S\,)^n\,R_{n,l}\,.$$

The crucial difference between eq. (84) and eq. (70) is the alternating sign in the general term of the series of eq. (84). Such series can give the well defined analytic function in spite of the $n!$ growth of the general term.

The physical meaning of such essential difference is very simple, because the value of the momentum $(q_{eff}^2)_n$ in the integral for $\Gamma(n)$ in eq. (84) turns out to be big, namely

$$(\,q_{eff}^2\,)_n \;\approx\; q_t^2\,\cdot\,e^{\frac{n}{\gamma(\omega_1)}} \tag{85}$$

and even at large value of n, we are still in the region of the use of perturbative QCD.

To use this as a tool for calculation, let us consider once more the singularities in the Borel plane.

3.4.2 UV renormalons: singularities in the Borel plane.

Introducing the Borel image for $\Delta\tilde{K}_r|_{q_t^2>q_t'^2}$ as it has been described in section 3.3.3 we see that a sum over n at fixed l leads to simple poles in b:

$$\frac{1}{b\;+\;b_{0l}}$$

where

$$b_{0l} \;=\; a^{(i)} \tag{86}$$

$$a^{(1)} \;=\; \gamma(\omega_1)\;+\;l\,;$$

$$a^{(2)} = 1 + l ;$$
$$a^{(3)} = \frac{1 + l + \gamma(\omega_1)}{2}$$

Therefore, all singularities are located on negative real axis as shown in Fig.3.3. It means that we can take the Borel integral without any ambiguity. Indeed

$$\int_0^{\infty} \frac{d\,b\; e^{-\frac{b}{\bar{\alpha}_S}}}{b + b_{0l}} = -\,e^{\frac{b_{0l}}{\bar{\alpha}_S}}\; Ei(-\frac{b_{0l}}{\bar{\alpha}_S})\,. \quad (87)$$

Finally the answer for $\Delta\tilde{K}_r$ in the region $q_t^2 > q_t'^2$ is:

$$\Delta\omega_{UV} = \Delta\tilde{K}_r|_{q_t^2 > q_t'^2} \doteq \frac{\alpha_S}{\bar{\alpha}_S}\cdot\Sigma_{n=0}^{\infty}\cdot \quad (88)$$

$$\{\left(e^{\frac{1+n}{\bar{\alpha}_S}}\, Ei\left(-\frac{1+n}{\bar{\alpha}_S}\right) + \frac{\bar{\alpha}_S}{n+1} - e^{\frac{n+\gamma(\omega_1)}{\bar{\alpha}_S}}\, Ei\left(-\frac{n+\gamma(\omega_1)}{\bar{\alpha}_S}\right) + \frac{\bar{\alpha}_S}{n+\gamma(\omega_1)}\right) +$$
$$\frac{1}{2}(-1)^{n+1}\frac{\Gamma(\gamma(\omega_1)+1)}{\Gamma(\gamma(\omega_1)+1)\,n!}\left(-\frac{2\bar{\alpha}_S}{n+1+\gamma(\omega_1)} - e^{\frac{n+1+\gamma(\omega_1)}{2\bar{\alpha}_S}}\, Ei\left(-\frac{n+1+\gamma(\omega_1)}{2\,\bar{\alpha}_S}\right)\right)\}$$

3.5 The first order correction to the intercept of the BFKL Pomeron.

Let us look back at eq. (61) and Fig. 3.1. and try to understand what is the physical meaning of the $\Delta\tilde{K}_r$. We can neglect at the moment the running $\alpha_S(q_t'^2)$ in eq. (61) for the rough estimate and integrate not with respect to ω_1 and ω_2, but over $\gamma(\omega_1)$ and $\gamma(\omega_2)$. Integrating in a such way we can see that integral over q_t' gives us $\delta(\gamma(\omega_1)-\gamma(\omega_2))$ and and integration over y' leads to y in front of the integral. Finally, in vicinity of $\gamma(\omega) = \frac{1}{2}$ we get for the $\Delta\phi^{[1]}$ the answer:

$$\Delta\phi^{[1]} = \Delta\tilde{K}_r(\gamma(\omega_1) = \frac{1}{2}\cdot y\cdot\phi(Q^2,y)\,, \quad (89)$$

where $\phi(Q^2,y)$ is the solution of the BFKL equation with fixed α_S.

One can see from eq. (89) that $\Delta\tilde{K}_r$ is the correction to the value of ω_L in eq. (35).

Fig. 3.6 shows the numerical calculations for $\Delta\omega_\Sigma = \Delta\omega_{IR} + \Delta\omega_{UV} = \Delta\tilde{K}_r(\alpha_S, \gamma = \frac{1}{2})$ as function of $x = \frac{1}{\bar{\alpha}_S}$[4].

It should be stressed that the calculation that is plotted in this figure is completely non-perturbative one even at $x \approx 10$. One can see this by eye because the perturbative behaviour should be $\frac{1}{x^2}$ at large value of x.

The second remark is that the ratio $\Delta\omega_\Sigma/\alpha_S\chi(1/2)$ is smaller than 12 % at $x > 3.5$.

4 The solution to the BFKL equation with running α_S .

The numerical result of the previous section ($\Delta\tilde{K}_r/\alpha_S\chi(1/2 < 12\%$) we would like to use reducing the complete BFKL equation with running α_S (see eq. (50) and eq. (51)) to the

[4]It should be pointed out that the variable x here, is not Bjorken variable for the deeply inelastic scattering and we hope that the use of the same letter will not be misleading

equation which is much simpler, neglecting the contribution of $\Delta\tilde{K}_r$ in the complete kernel of eq. (51). This reduced equation has the form:

$$\frac{d\,\phi(q_t^2,y)}{d\,y} \;=\; \frac{N}{\pi}\int\frac{d^2q_t'}{\pi}\,\alpha_S(q_t'^2)\;K(q_t,q_t')\;\phi(q_t'^2,y) \tag{90}$$

where $K(q_t,q_t')$ is the kernel of the BFKL equation with fixed α_S (see eq. (26)). The general solution of eq. (90) has been found in the GLR paper [16], however we are going to discuss this solution here for the sake of completeness, and to illustrate the connection between this solution and anomalous dimension of eq. (34). Using the explicit expression for the running $\alpha_S \;=\; \frac{2\pi}{b}\cdot\frac{1}{ln(q^2/\Lambda^2)}$ and the double Mellin transform of eq. (31) we can rewrite eq. (90) for the function $\bar{\phi} \;=\; \alpha_S(q_t^2)\,\phi(q_t^2,y)$ in the form:

$$-\,\omega\,\frac{\partial\,C(\omega,f)}{\partial\,f} \;=\; \frac{2\,N}{b}\,\chi(f)\;C(\omega,f)\,. \tag{91}$$

Therefore the general solution can be given in the form:

$$\bar{\phi}(q_t^2,y) \;=\; \int\frac{d\omega\;d\,f}{(2\,\pi\,i)^2}\tilde{\phi}(\omega)exp\{\frac{2\,N}{b\,\omega}\int_f^{\frac{1}{2}}\chi(f')\,d\,f' \;+\;\omega\,y\;+\;(\,f\,-\,1\,)\,r\,\}\;, \tag{92}$$

where $r \;=\; ln(q_t^2/\Lambda^2)$.

One can simplify eq. (92) integrating over f using the saddle point approximation. The saddle point value of $f \;=\; f_S$ can be found from the equation:

$$\alpha_S(q^2)\,\chi(f_S) \;=\; \omega \tag{93}$$

The solution of eq. (93) is the anomalous dimension of eq. (34) ($f_S \;=\; \gamma(\omega)$). Taking the integral over f' in eq. (92) introducing new variable α_S' from the equation

$$\alpha_S'\;\chi(f') \;=\; \omega$$

and integrating by parts we derive the answer:

$$\bar{\phi}(q_t^2,y) \;=\; \frac{1}{q_t^2}\int\frac{d\,\omega}{2\,\pi\,i}\,\tilde{\phi}(\omega)\;e^{\omega\,y+\int_{\alpha_S(q_t^2)}\gamma(\omega,\alpha_S')\,\frac{d\,\alpha_S'}{\alpha_S'^2}}\;\sqrt{\frac{\pi\,\alpha_S(q_t^2)}{\omega}\cdot\frac{d\gamma(\omega,\alpha_S)}{d\,\alpha_S}} \tag{94}$$

The last factor in eq. (94) comes from integration over f in the vicinity of the saddle point. Taking into account that we can calculate $\frac{d\gamma}{d\,\alpha_S}$ using only the first term in eq. (34) we can get the following expression directly for function ϕ:

$$\phi(q_t^2,y) \;=\; \frac{1}{q_t^2}\int\frac{d\,\omega}{2\,\pi\,i}\,\tilde{\phi}(\omega)\;e^{\omega\,y+\int_{\alpha_S(q_t^2)}\gamma(\omega,\alpha_S')\,\frac{d\,\alpha_S'}{\alpha_S'^2}} \tag{95}$$

It is easy to recognize in eq. (95) the usual result of renormalization group approach [21].

Now let us formulate the problem which we desire to solve. We want to find the Green function ($G(y-y_0,r)$) of the BFKL equation with running α_S which satisfies the initial condition:

$$G(y-y_0,r=r_0) \;=\; \delta(\,y\,-\,y_0\,) \tag{96}$$

but in the kinematic region where the anomalous dimension $\gamma(\omega, \alpha_S)$ is close to the limited value $\gamma = \frac{1}{2}$ (see eq. (36)). To find such Green function it is easier to use the form of eq. (92) for the solution.

We would like to stress that we are going to solve the BFKL equation with running α_S in usual way for the GLAP evolution equation, starting with the x distribution at fixed initial virtuality $q^2 = q_0^2, r = r_0 = ln(q_0^2/\lambda^2)$ and calculating the deep inelastic structure function at larger value of q^2 ($q^2 > q_0^2$). It is worthwhile mentioning that the solution of the problem is

$$\phi(q^2, y) = \int d\, y_0 G(y - y_0, r)\, \phi_{in}(y = y_0, r = r_0) \tag{97}$$

However, the Green function for the BFKL equation is quite different from the Green function for the GLAP one in the kinematic region, where the anomalous dimension is close to the value of $\gamma = \frac{1}{2}$ and we need to take into account the sum of all terms in eq. (34).

In vicinity $\gamma \rightarrow \frac{1}{2}$ we can use the following expansion for $\chi(f)$:

$$\chi(f) = \chi(\frac{1}{2}) + 14\,\zeta(3)\,(f - \frac{1}{2})^2 = \chi(\frac{1}{2}) \cdot [1 + \kappa\,(f - \frac{1}{2})^2] \tag{98}$$

It is easy to see that such an expansion leads to eq. (36) for γ. Substituting eq. (98) in the general solution of eq. (92) one can see that integral over f can be taken and gives the Airy function. The final answer for the Green function is

$$G(y - y_0, r, r_0) = e^{\frac{1}{2}(r - r_0)} \cdot \sqrt{\frac{r}{r_0}} \int \frac{d\,\omega}{2\,\pi\,i} \frac{Ai\left((\frac{\omega}{\omega_L r_0 \kappa})^{\frac{1}{3}} [\, r - \frac{\omega_L}{\omega} r_0 \,] \right)}{Ai\left((\frac{\omega}{\omega_L r_0 \kappa})^{\frac{1}{3}} [\, r_0 - \frac{\omega_L}{\omega} r_0 \,] \right)} \,. \tag{99}$$

The contour of integration with respect to ω in eq. (99) is located to the right of singularities of the integrant (see Fig. 4.1). Therefore, we have to know the singularities of the integrand in ω to take the integral over ω and the position in the ω - plane of the possible saddle point in eq. (99). The Airy function is the analytic function of its argument and the origin of singularities in eq. (99) is the zeros of the dominator. Within good numerical accuracy the zeros of the Airy function can be calculated using the following equation:

$$(\frac{\omega_{0k}}{\omega_L r_0 \kappa})^{\frac{1}{3}} [\, r_0 - \frac{\omega_L}{\omega_{0k}} r_0 \,] = \left(\frac{3}{2} \left(\frac{3}{4}\pi + \pi\, k \right) \right)^{\frac{2}{3}} \,. \tag{100}$$

One can see that at very large value of k $\omega_{0k} \propto \frac{1}{k} \rightarrow 0$. The second interesting observation is the fact that the rightmost singularity (pole) turns out to be considerably smaller than the value of ω_L (See Table I which shows the value of six first ω_{0k}).

Table I

ω_{0k} for the BFKL equation with running α_S.

ω_L	ω_{00}	ω_{01}	ω_{02}	ω_{03}	ω_{04}	ω_{05}
0.65	0.3	0.17	0.12	0.093	0.075	0.064
0.5	0.22	0.135	0.102	0.081	0.068	0.055
0.33	0.17	0.115	0.087	0.072	0.053	0.040
0.25	0.14	0.10	0.078	0.064	0.056	0.050
0.20	0.122	0.09	0.071	0.069	0.051	0.046

Now let us discuss the saddle point in the integral of eq. (99). We found such a saddle point in the kinematic region where we can use the asymptotic expansion for the Airy function in the numerator of eq. (99). Indeed, if we assume that

$$(\frac{\omega}{\omega_L\, r_0\, \kappa})^{\frac{1}{3}}\,[\,r\,-\,\frac{\omega_L}{\omega}\,r_0\,]\;\gg\;1$$

we can use the asymptotic expansion for the Airy function which gives

$$Ai\left((\frac{\omega}{\omega_L\, r_0\, \kappa})^{\frac{1}{3}}\,[\,r\,-\,\frac{\omega_L}{\omega}\,r_0\,]\right)\;\rightarrow\;exp\left(\frac{\omega}{\omega_L\, r_0\, \kappa})^{\frac{1}{2}}\,[\,r\,-\,\frac{\omega_L}{\omega}\,r_0\,]^{\frac{3}{2}}\right) \tag{101}$$

$$\rightarrow\;exp\left(-(\frac{\omega}{\omega_L\,\kappa})^{\frac{1}{2}}\,\Delta r\,(\frac{\Delta\omega}{\omega_L})^{\frac{1}{2}}\right)$$

where $\Delta\omega\;=\;\omega\;-\;\omega_L$ and $\Delta r\;=\;r\;-\;r_0$. To get the last line in the above equation we also assumed that

$$\frac{\Delta\omega}{\omega}\;\gg\;\frac{\Delta r}{r_0} \tag{102}$$

Using eq. (101) we can find the position of the saddle point in ω, namely

$$\omega_S\;=\;\omega_L\;+\;\frac{(\,\Delta r\,)^2}{4\,\kappa\,\omega_L\,y^2} \tag{103}$$

Therefore, the final structure of the ω - plane looks as it is shown in Fig. 4.1. Evaluating the integral by steepest decent method we reproduce the solution to the BFKL equation, namely

$$G(y-y_0,r-r_0)\;=\;e^{-\frac{1}{2}(\,r\,-\,r_0\,)}\,.\,\frac{1}{\sqrt{\pi\,\kappa\,\omega_L\,(\,y\,-\,y_0\,)}}\cdot e^{\omega_L y\,-\,\frac{(\,\Delta r\,)^2}{4\,\kappa\,\omega_L\,(\,y\,-\,y_0\,)}} \tag{104}$$

We can trust the above answer only in the kinematic region where

$$(\,\Delta r\,)^2\;\leq\;4\,\kappa\,\omega_L\,(\,y\,-\,y_0\,)\,. \tag{105}$$

However, we need to sutisfy also eq. (102). Substituting eq. (105) in eq. (102) we get the kinematic region where we can consider eq. (104) as a solution to the problem. Namely

$$y\;-\;y_0\;<\;\left(\frac{r_o}{\omega_L\sqrt{4\kappa\omega_L}}\right)^{\frac{2}{3}}\;\propto\;(\alpha_S(r_0))^{-\frac{5}{3}} \tag{106}$$

This value of $y-y_0$ is still in the region of applicability of the BFKL approach because $\alpha_S(q_0^2)(y-y_0)\approx(\alpha_S(q_0^2))^{-\frac{2}{3}}\;>\;1$, but both theoretically and practically it is very restricted region.

For larger value of $y=ln(1/x)$ we have to close our contour on the poles of the dominator and we get the solution which behaves as

$$G(y-y_0,r-r_0)\;\propto\;\frac{1}{x^{\omega_{00}}} \tag{107}$$

The solution of the BFKL evolution equation with running α_S is given by eq. (97) which can be rewritten in the ω - representation in the form:

$$\phi(y,q_i^2)\;=\;\int\,\frac{d\,\omega}{2\,\pi\,i}\cdot G(\omega,r-r_0)\cdot\phi_{in}(\omega,r_0)\;, \tag{108}$$

where $\phi_{in}(\omega, r_0)$ is the initial gluon distribution in the ω - representation. We can distinguish two case with different solutions:

1. the initial distribution $\phi_{in} \propto x^{-\lambda}$ with $\lambda > \omega_{00}$. In this case we have to close the contour on the singularities of $\phi_{in}(\omega) = \frac{\phi_0}{\omega - \lambda}$ and the solution in the region of small x looks as follows:

$$\phi(y = ln(1/x), q_t^2) \;=\; \phi_0 \cdot G(\omega = \lambda, r - r_0) \cdot (\,\frac{1}{x}\,)^{\lambda}$$

2. the second case is $\lambda < \omega_{00}$, when we have to close the contour on the singularities of the Green function. The resulting behaviour of the solution is closely related to the value of ω_{00} and looks as follows:

$$\phi(y = ln(1/x), q_t^2) \;=\; \phi_{in}(\omega = \omega_{00}) \cdot \{\, G(\omega, r - r_0) \cdot (\,\omega - \omega_{00}\,)\,\}\,|_{\omega = \omega_{00}} \cdot (\,\frac{1}{x}\,)^{\omega_{00}} \;=$$

$$\phi_{in}(\omega = \omega_{00}) \cdot e^{\frac{1}{2}\,(r - r_0)} \cdot e^{-\frac{2}{3}\left(\frac{\omega_{00}}{\omega_L r_0 \kappa}\right)^{\frac{1}{2}}\left(-\frac{\omega_L}{\omega_{00}} r_0 - r\right)^{\frac{3}{2}}} \cdot \left(\frac{1}{x}\right)^{\omega_{00}} .$$

The above formula solves the problem.

5 Conclusion.

In this paper we attempted to discuss the BFKL equation with the running α_S in a systematic way. Our results look as follows:

1. We found that this equation has the form of eq. (50) with the kernel of eq. (51). The weakness of the argumentation stems from the fact that we assumed the bootstrap equation to reconstruct the form of the kernel. It is not clear how general the bootstrap property of the BFKL equation is. To check this we have to calculate the sets of the Feynman diagrams shown in Figs. 2.9 and 2.11. These calculations are now in progress and will be published elsewhere soon.

2. The uncertainties from the contribution of the infrared renormalons in the BFKL equation with running α_S were estimated and it was shown that they are smaller that the nonperturbative contribution describing the shadowing correction in the deeply inelastic scattering.

3. It was shown that the infrared renormalons give rise to the corrections of the order of $\frac{1}{\sqrt{Q^2}}$ to the gluon structure function in the region of small x. The physical origin of such sort corrections as well as their selfconsistent nonperturbative description is still unclear and has to be clarified in future.

4. The analytic function summing the infrared and ultraviolet renormalons in the BFKL equation with running α_S was suggested, and numerical estimates were given which led to simplification of the answer in the deeply inelastic kinematic region.

5. The reduced evolution equation with the running α_S based on the numerical estimates of the nonperturbative contribution to the kernel of the BFKL equation with running α_S was proposed and solved. The result of the solution shows much slower behaviour of the gluon structure function in the region of small x, than it was predicted in the original version of the BFKL equation with fixed α_S.

6. The legitimate theoretical region $\alpha_S \ln(1/x) \;<\; (\alpha_S(q_0^2))^{\frac{2}{3}}$ for the BFKL equation with fixed α_S was found.

We hope that this paper will be useful in understanding what kind of nonperturbative phenomena has been taking into account in the BFKL equation and for more elegant and comprehensive theoretical description of nonperturbative QCD in the region of low x (high energies).

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Figure Captions

Fig. 2.1 : The Born Approximation of perturbative QCD for quark - quark scattering.

Fig. 2.2 : The next to Born Approximation of perturbative QCD for quark - quark scattering: emission of one extra gluon.

Fig. 2.3 : The next to Born Approximation of perturbative QCD for quark - quark scattering: α_S^2 correction to elastic amplitude.

Fig. 2.4 : The next to Born Approximation of perturbative QCD for quark - quark scattering: gauge invariance trick.

Fig. 2.5 : The next to Born Approximation of perturbative QCD for quark - quark scattering: the resulting answer for emission of one extra gluon.

Fig. 2.6 : The relation between colour coefficients.

Fig. 2.7 : The BFKL equation.

Fig. 2.8 : The Born Approximation of perturbative QCD for quark - quark scattering for a running coupling constant.

Fig. 2.9 : Insertion of fermion bubles in the amplitude of emission of one extra gluon.

Fig. 2.10 : Corrections to the reggeization of the gluon due to running α_S.

Fig. 2.11 : Emission of one quark - antiquark pair.

Fig. 3.1 : The first correction to the BFKL equation due to running α_S.

Fig. 3.2 : The shadowing correction to the deep inelastic scattering.

Fig. 3.3 : The structure of singularities in the Borel plane.

Fig. 3.5 : $\Delta\omega_{IR}$ versus γ at fixed $\alpha_S = 0.25$.

Fig. 3.5 : $\Delta\omega_{IR}$ versus $x = \frac{1}{\alpha_S}$ at fixed $\gamma = \frac{1}{2}$.

Fig. 3.6 : $\Delta\omega_{\Sigma}$ versus $x = \frac{1}{\alpha_S}$ at fixed $\gamma = \frac{1}{2}$.

Fig. 4.1 : The structure of singularities in the ω - plane for the solution of the reduced BFKL equation with running α_S.

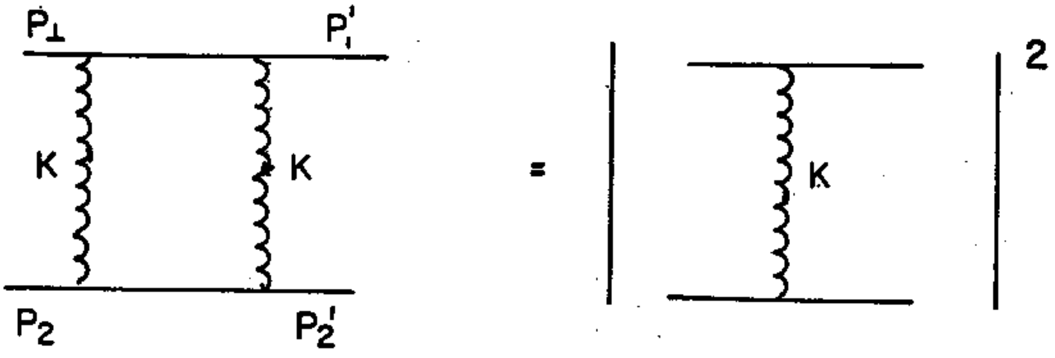

P1
P1'
K
K
P2
P2'
=
K
2

Fig. 2.1

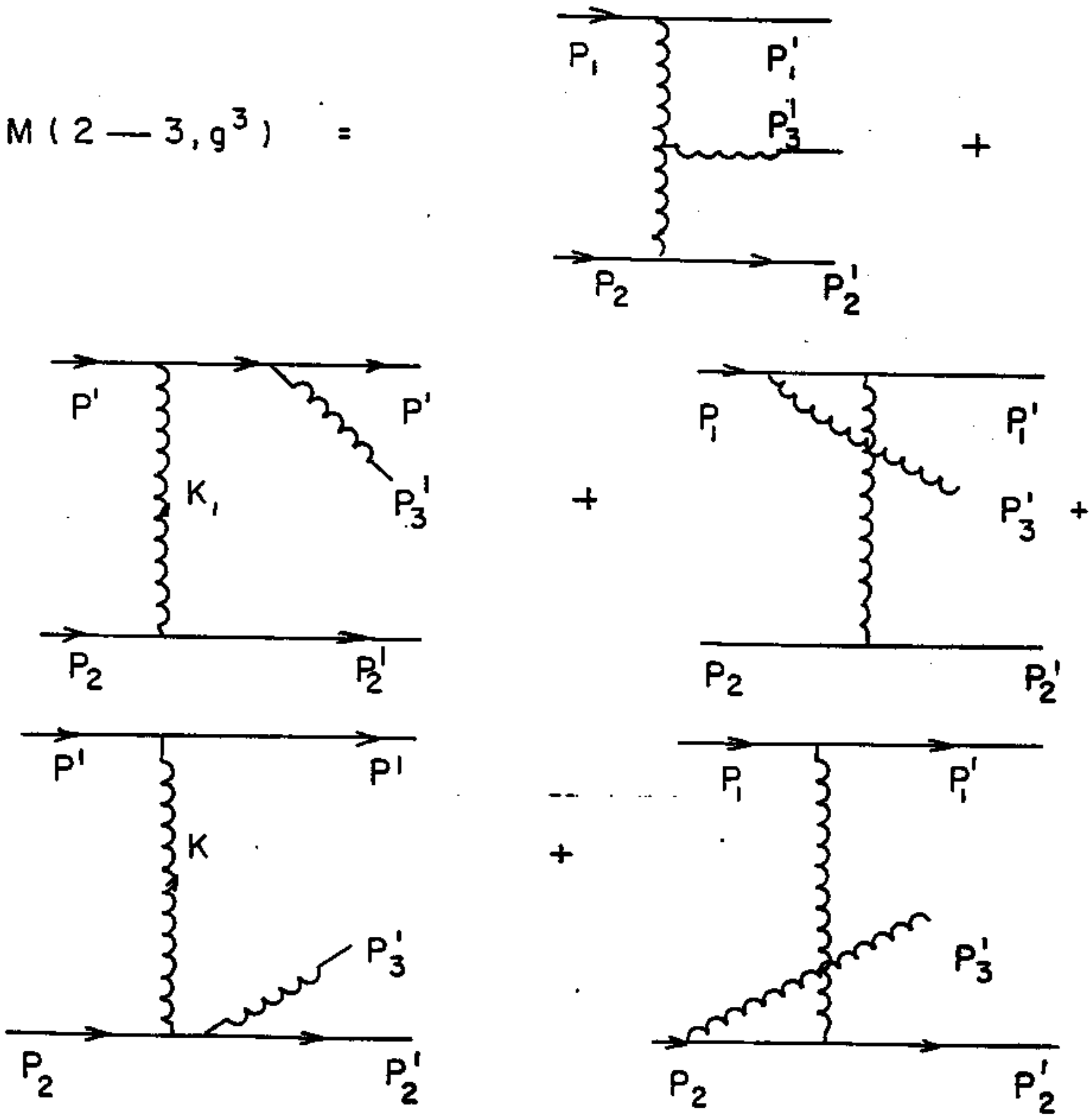
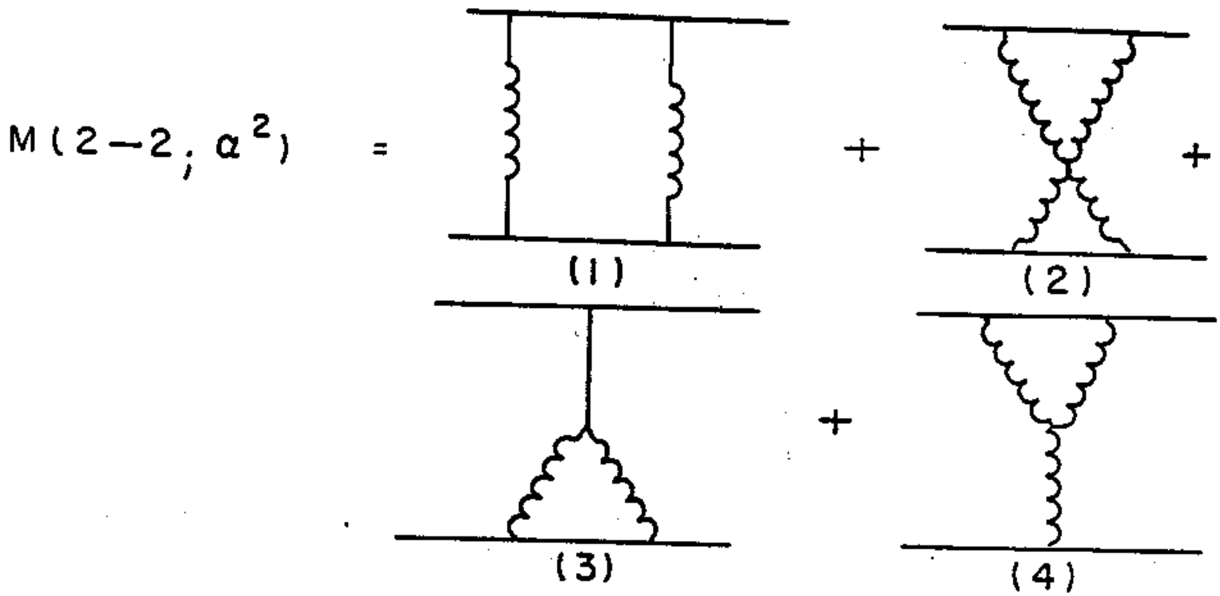

M (2 — 3, g3) =
P1
P1'
P3'
P2
P2'
+
P'
P'
K1
P3'
P2
P2'
+
P1
P1'
P3'
+
P2
P2'
P'
P'
K
P3'
P2
P2'
+
P1
P1'
P3'
P2
P2'

Fig. 2.2



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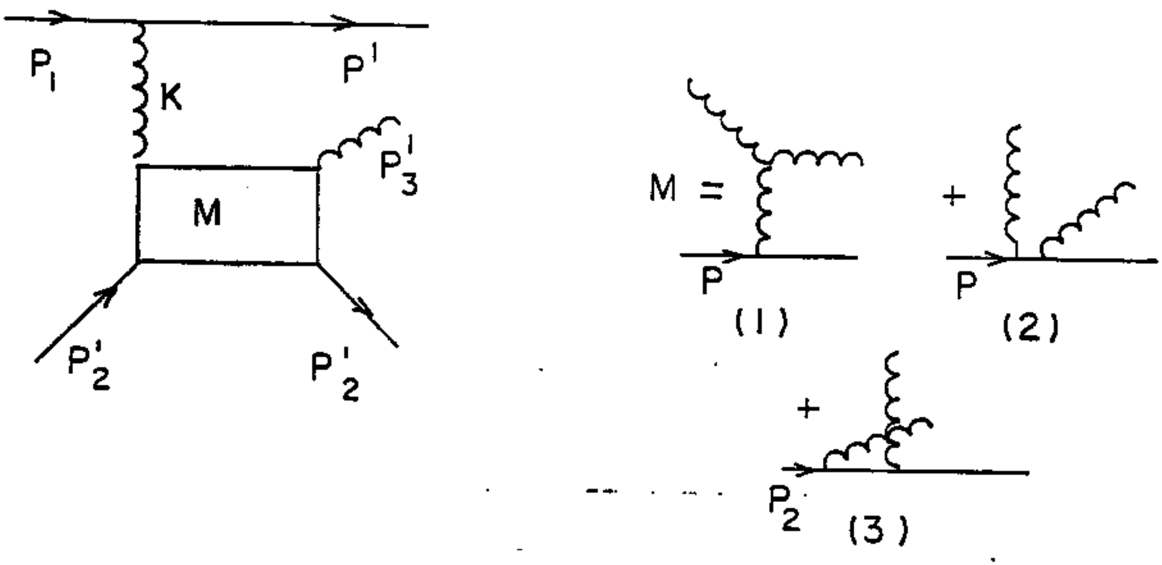


Fig. 2. 4

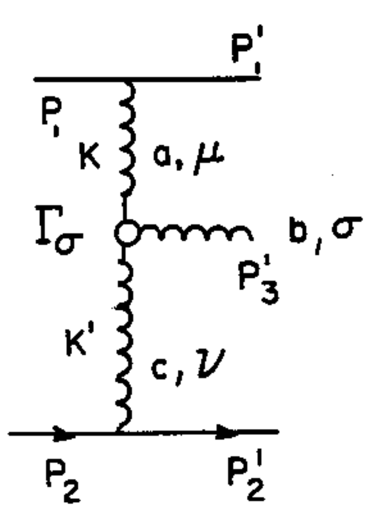


Fig. 2.5

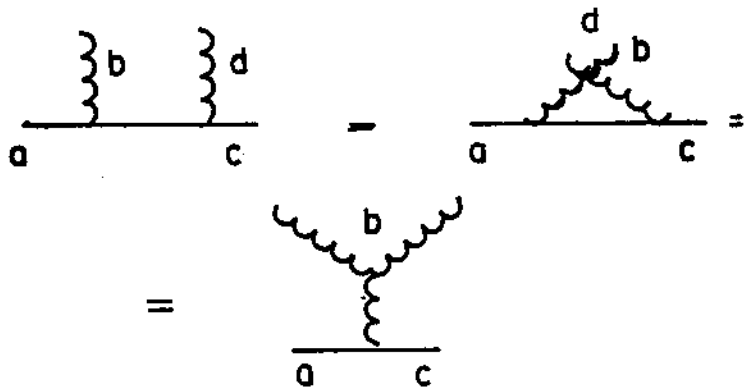


Fig. 2.6

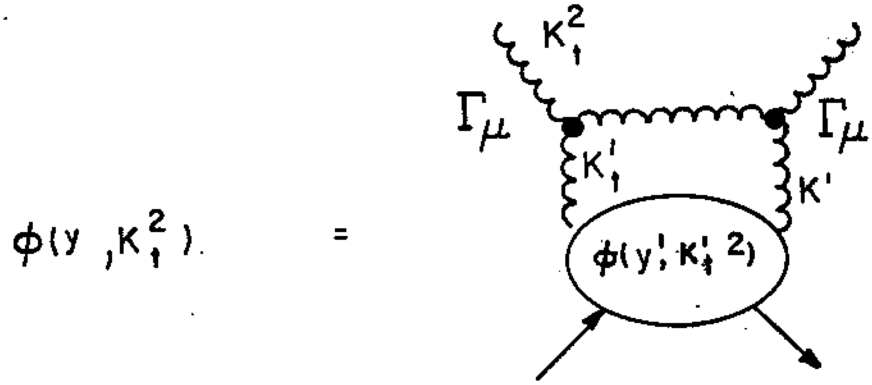
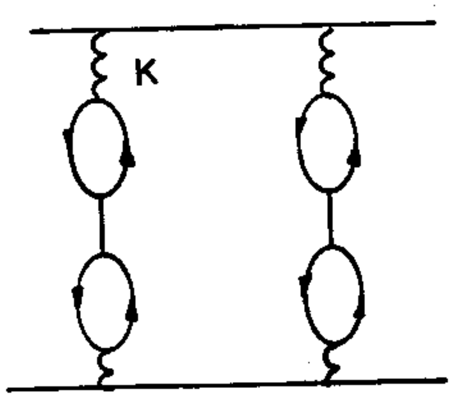


Fig. 2.7



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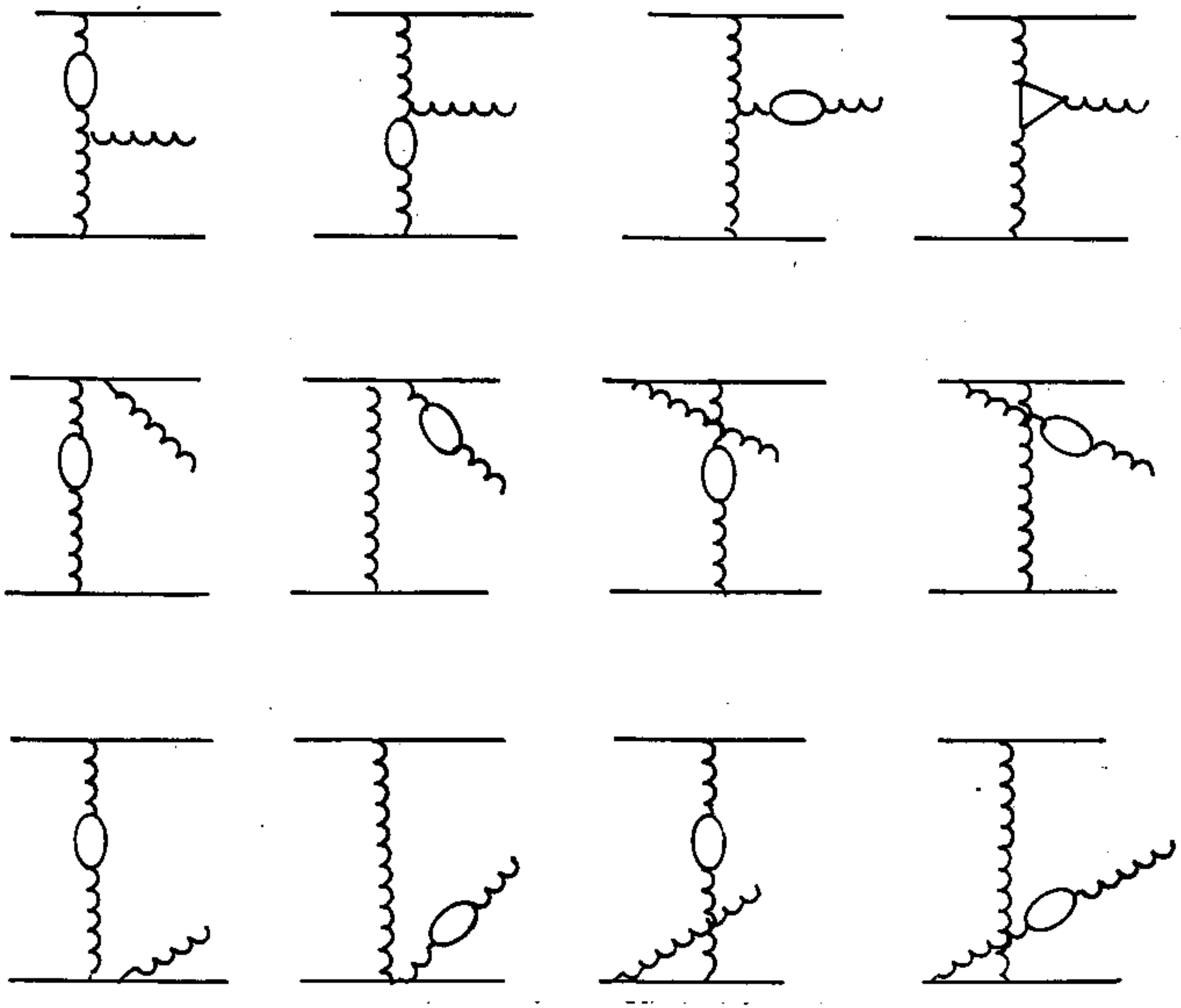

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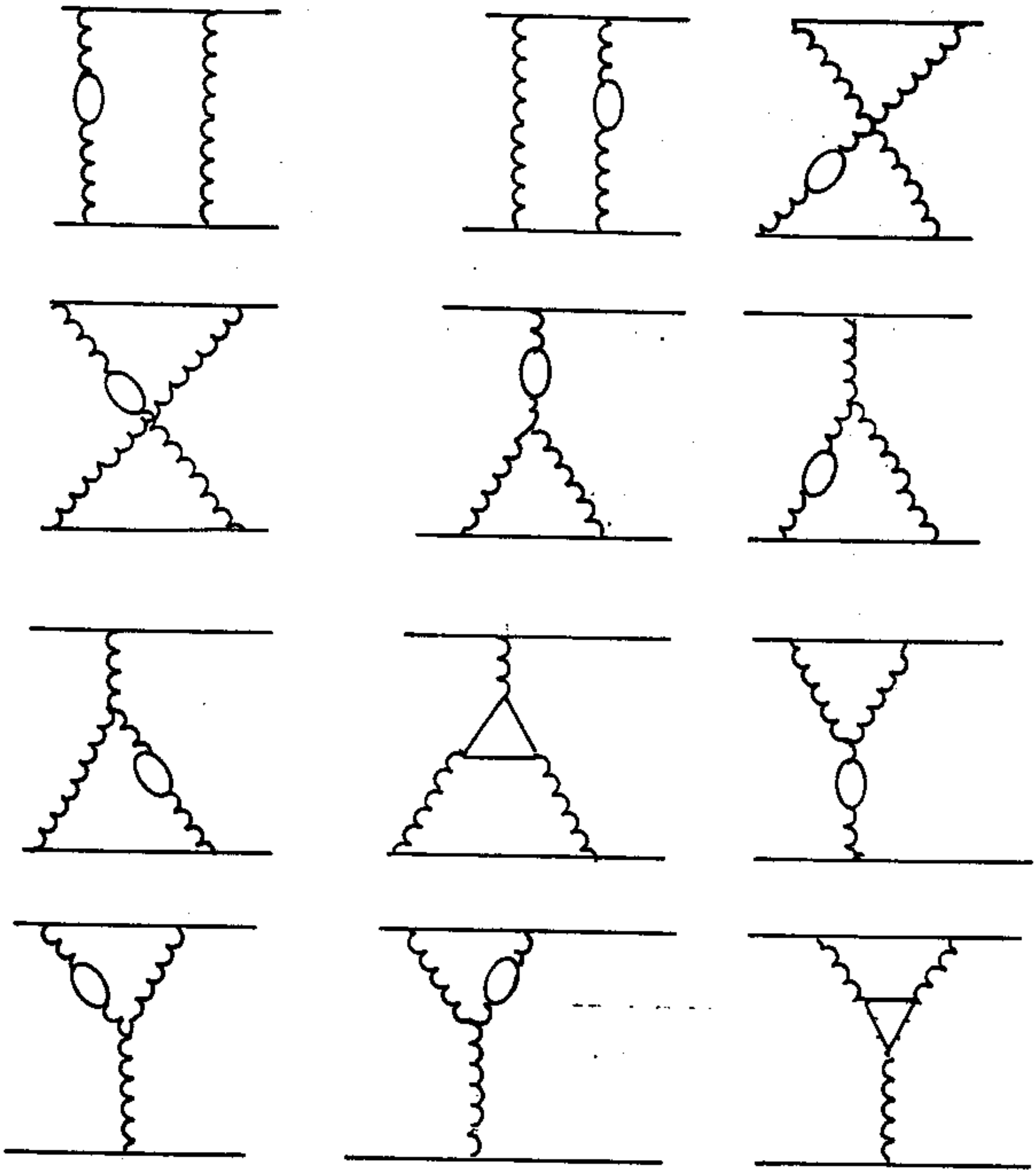
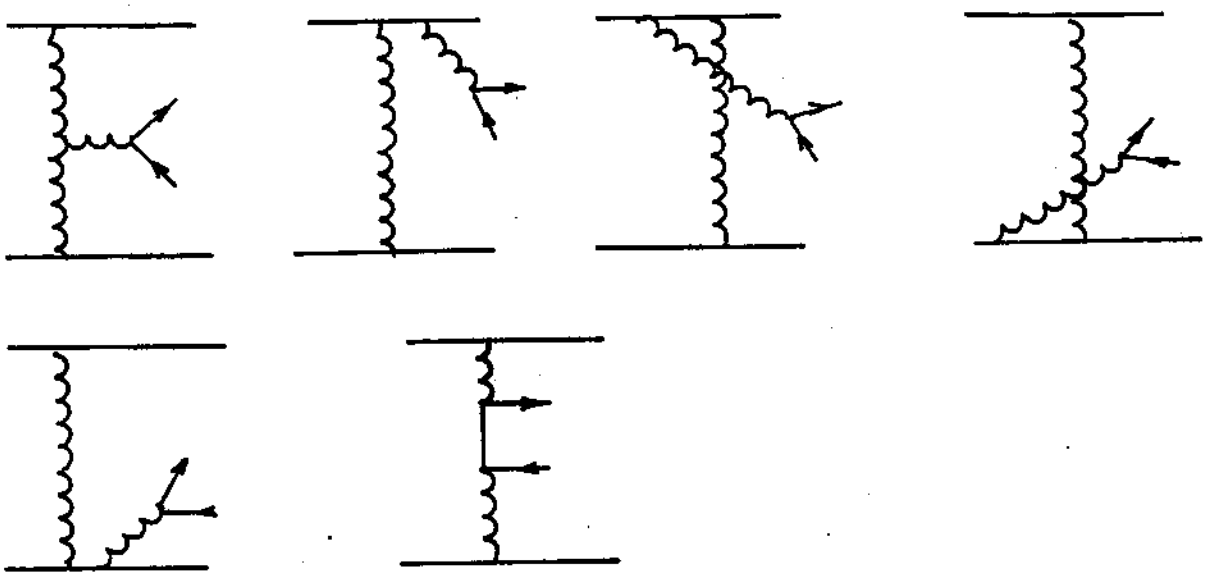


Fig . 2.10



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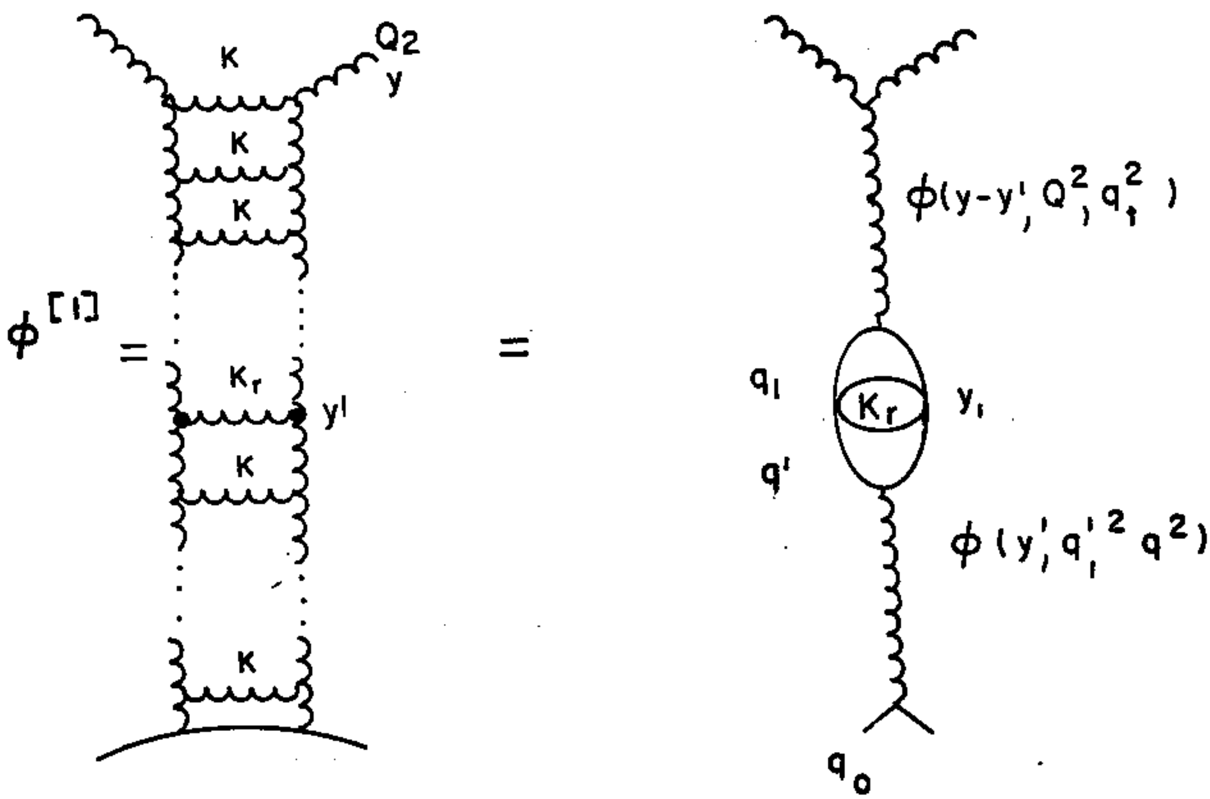


Fig 3.1

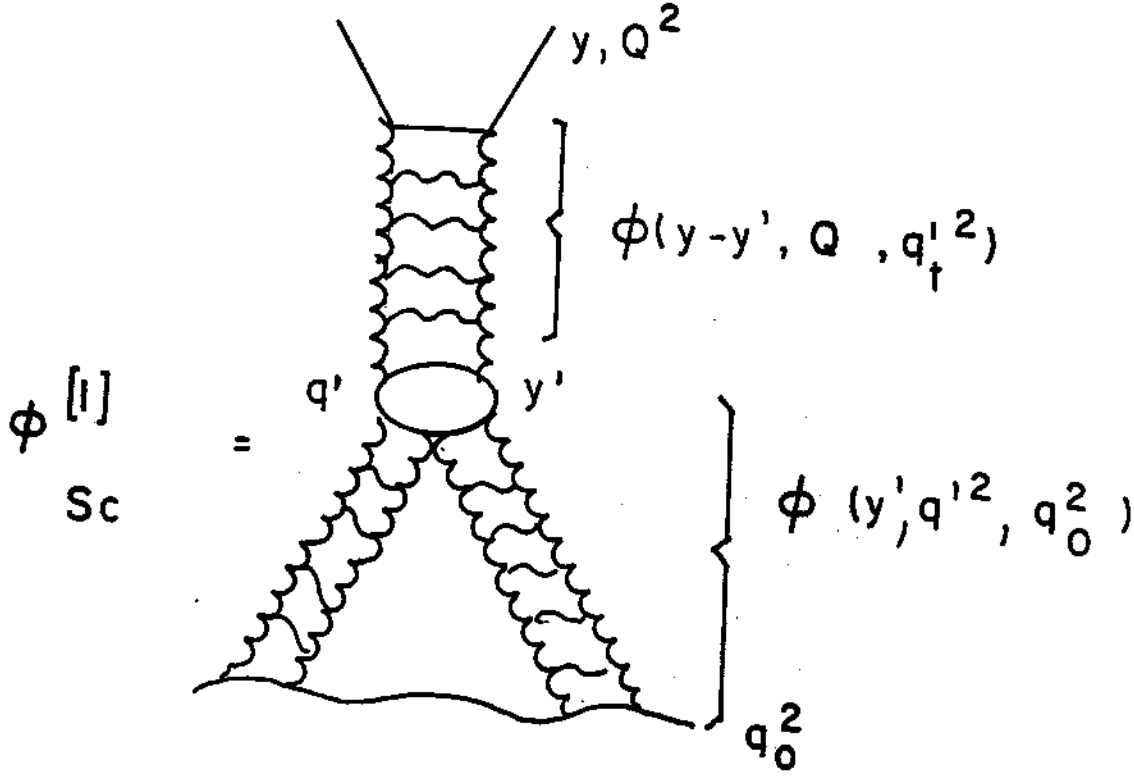


Fig. 3.2

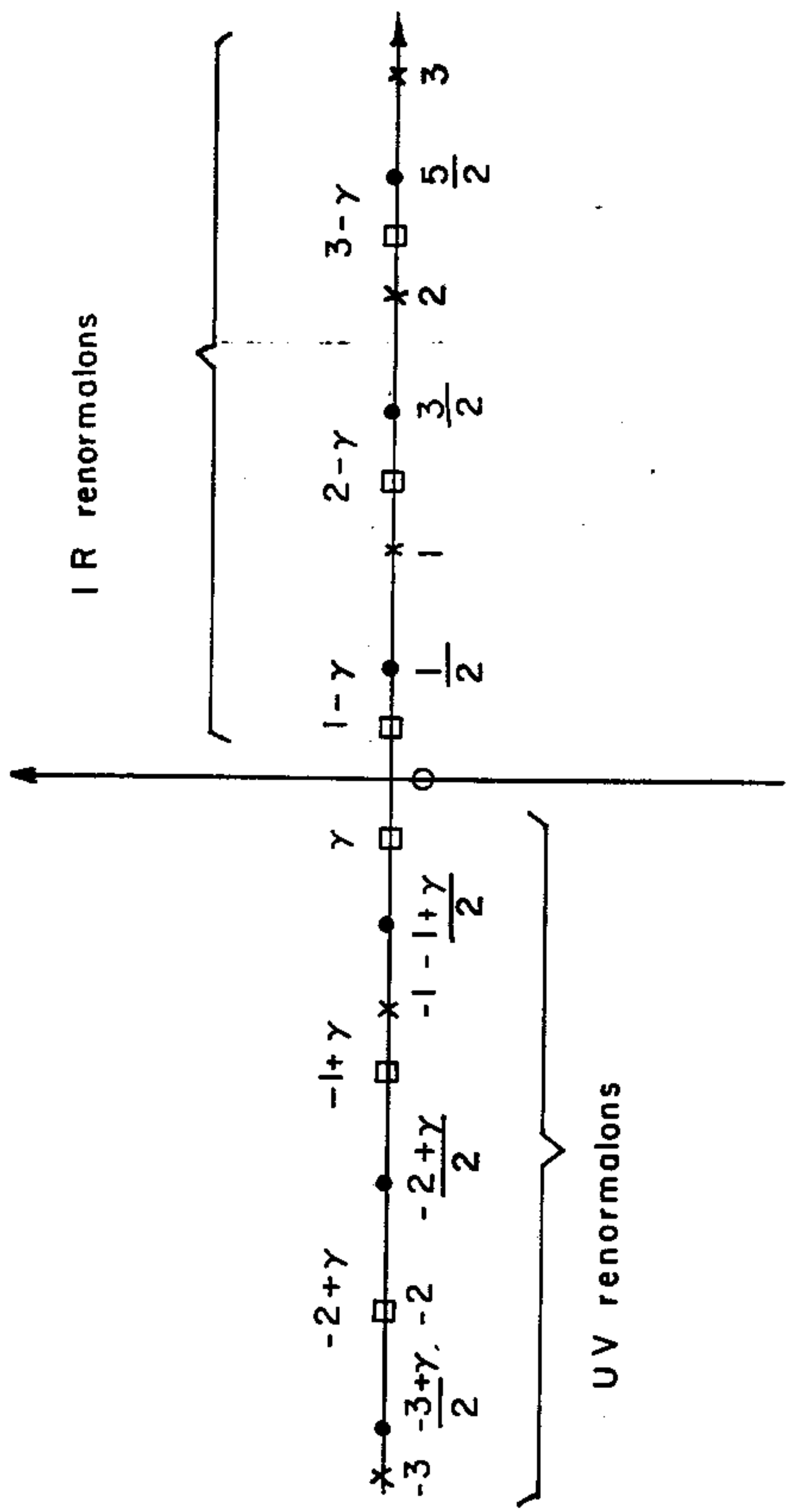
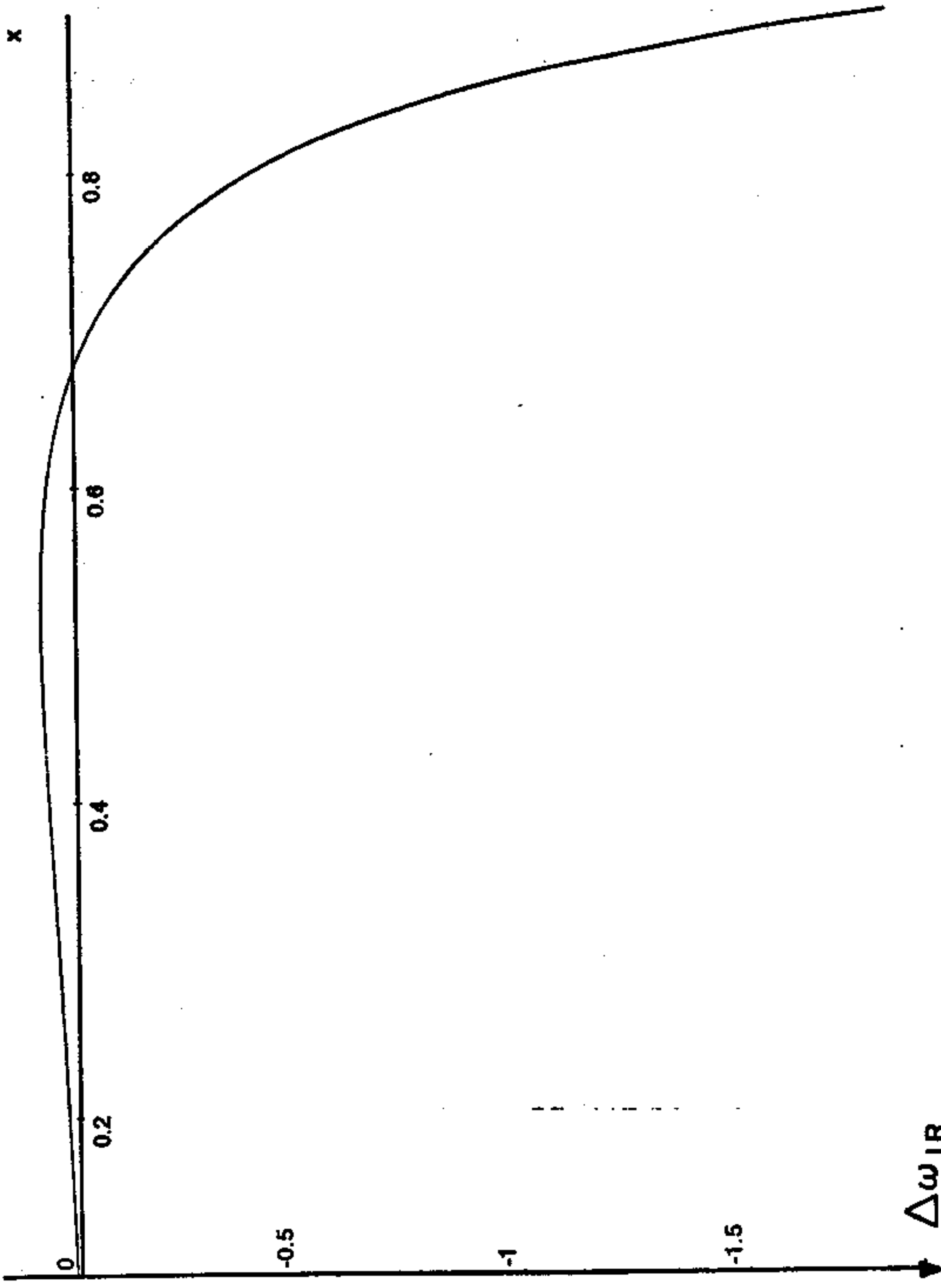

IR renormalons
UV renormalons
3-γ
2-γ
1-γ
γ
-1+γ
-2+γ
3
5/2
2
3/2
1
1/2
0
-1 - (1+γ)/2
-(2+γ)/2
-2
-(3+γ)/2
-3

Fig. 3.3



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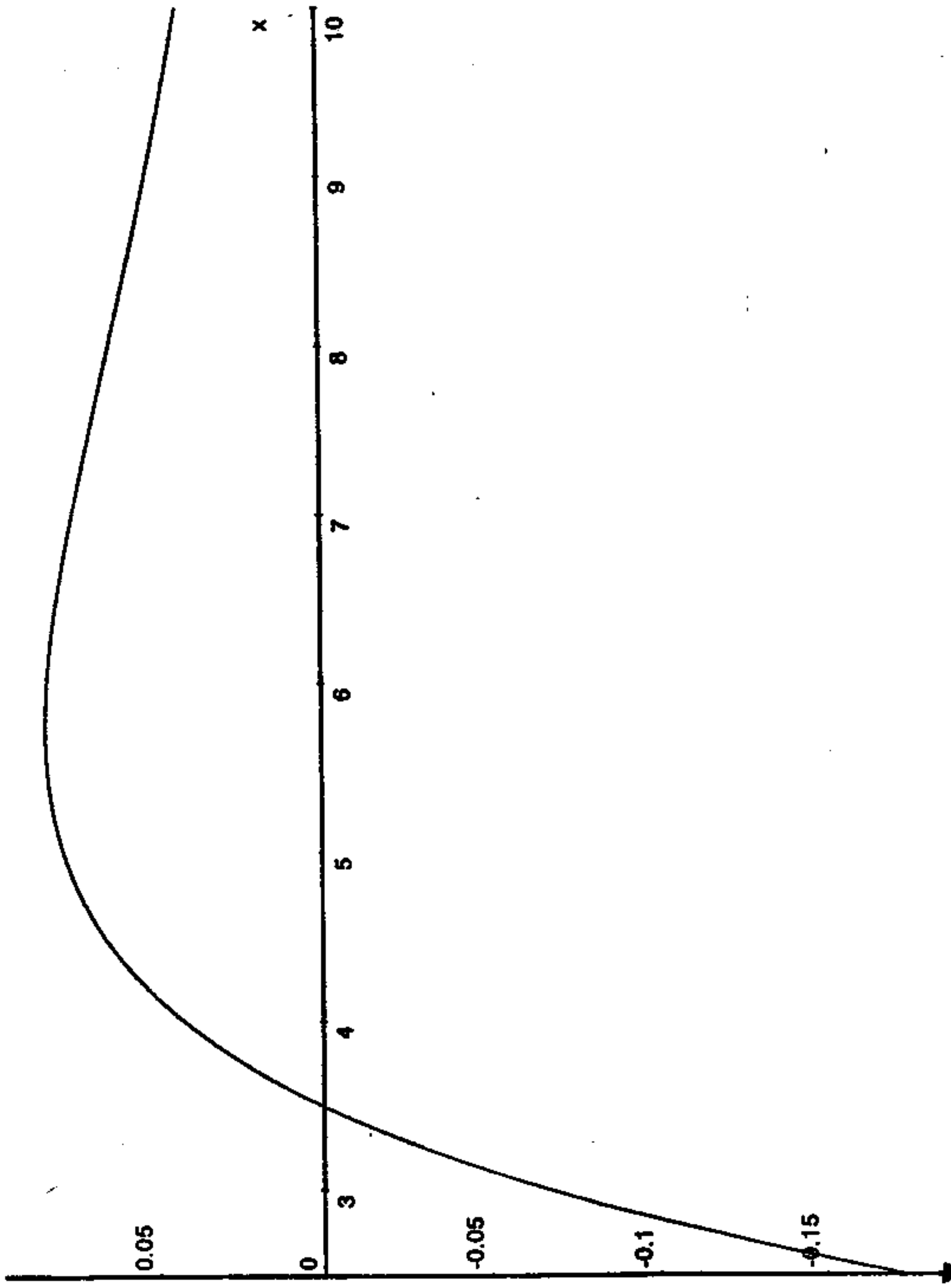

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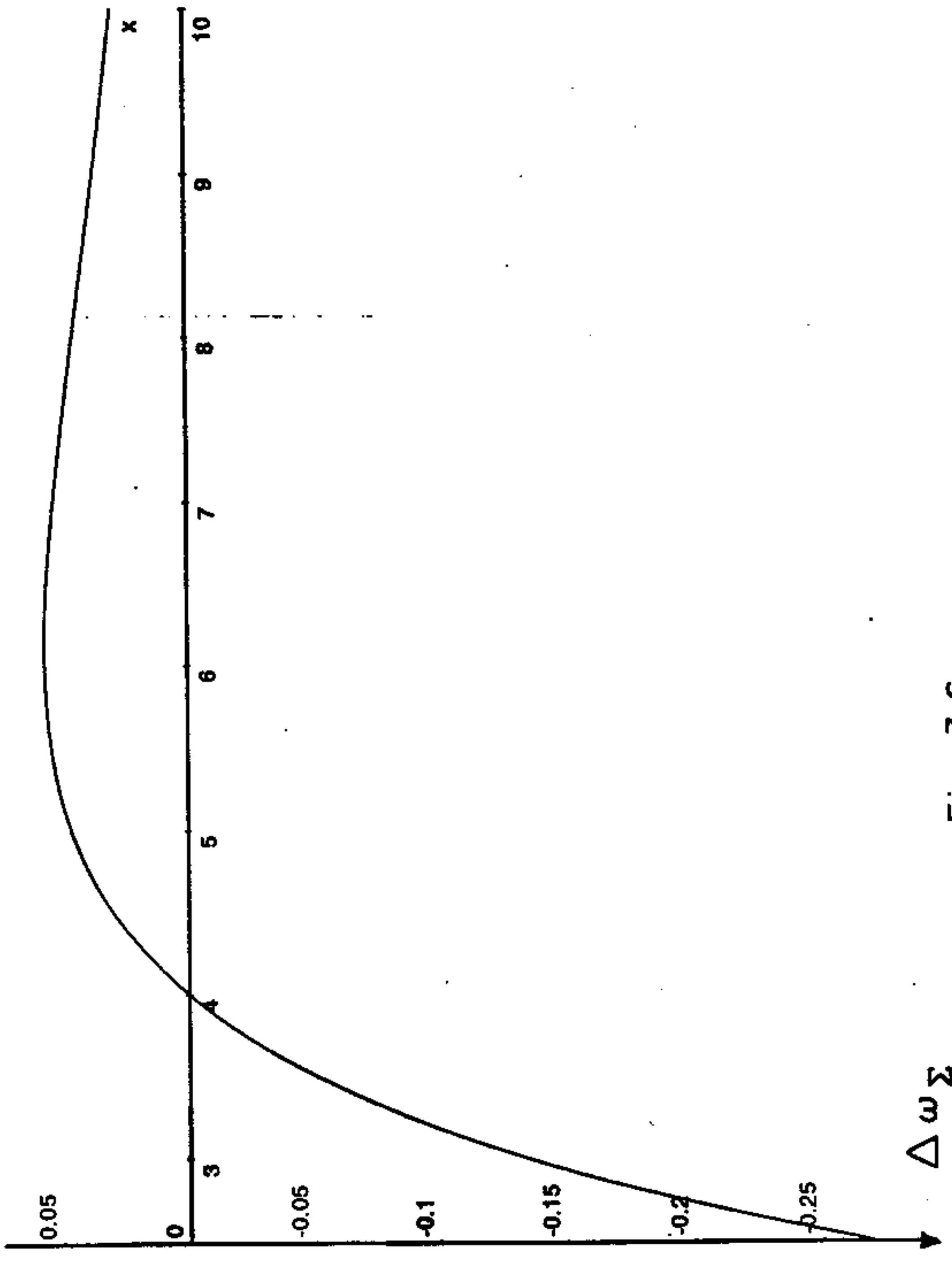

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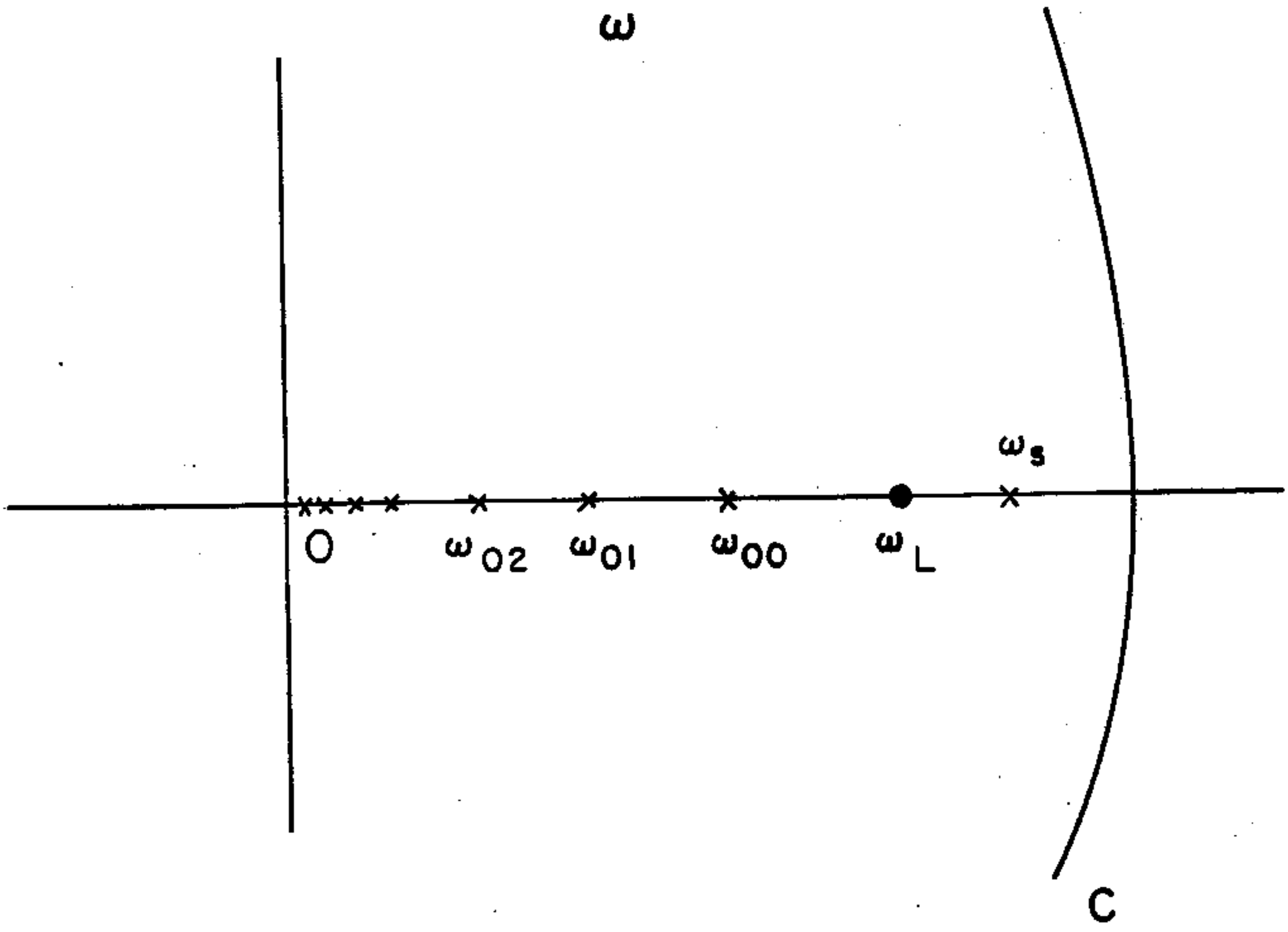

Fig. 4.1