CBPF-NF-001/94

Some Comments on Boltzmann-Gibbs Statistical Mechanics

Constantino TSALLIS

Centro Brasileiro de Pesquisas Físicas

Rua Dr. Xavier Sigaud, 150

22290-180 – Rio de Janeiro – RJ, Brazil

Abstract

A nonexhaustive review is presented of the limits of the impressive and vastly known success of Boltzmann-Gibbs statistics and normal thermodynamics. These limits naturally open the door for the research of generalized formalisms that could enlarge the domain of validity of standard statistical mechanics and thermodynamics. A possible such generalization (recently proposed by the author) is commented along this perspective.

1 Limitations of Boltzmann-Gibbs Statistics

1.1 Introduction

The qualitative and quantitative success of Boltzmann-Gibbs (BG) Statistical Mechanics (and, naturally, of its particular cases, the Fermi-Dirac and Bose-Einstein quantum statistics, with their common high temperature asymptotic limit, the classical Maxwell-Boltzmann statistics) is so *ubiquitous, persistent* and *delicate* that not few physicists and chemists have a kind of strong (not necessarily rationalized) feeling that this brilliant formalism is, in practical terms, *universal, eternal* and *infinitely precise.* A more balanced analysis reveals some of its specific limitations and inadequacies, and consequently the fragility or nonuniversality of some of the basic hypothesis of its foundations. Among these foundation stones, a privileged position is detained by the entropy S, as introduced and used by Boltzmann and Gibbs ($S = -k_B \int dx f(x) \ln f(x)$ for classical distribution laws $f(x)$ defined in phase space), further generalized by von Neumann ($S = -k_B Tr \rho \ln \rho$, ρ being the density operator defined in Hilbert or Fock spaces), with its diagonal form ($S = -k_B \Sigma_{i=1}^{W} p_i \ln p_i$, p_i being the probability of the i-th among W microstates, and its famous equal-probability particular form $S = k_B \ln W$) as finely discussed, in the context of Information Theory, by Shannon.

Let us explicitely state, at this point, a property of S *(extensivity* or *additivity* as frequently referred to) which will play a critical role in what follows. If we have two systems Σ_1 and Σ_2 (with respective probabilities $\{p_i^{(1)}\}$ and $\{p_j^{(2)}\}$; $i = 1, 2, .., W_1$ and $j = 1, 2, ..., W_2$) which are *independent* (in the sense that the probabilities $\{p_{ij}^{(12)}\}$ of the system $\Sigma_1 U \Sigma_2$ satisfy $p_{ij}^{(12)} = p_i^{(1)} p_j^{(2)}, \forall (i, j)$), then

$$S\left(\Sigma_1 U \Sigma_2\right) = S(\Sigma_1) + S(\Sigma_2) \tag{1}$$

Obviously, if we have N independent systems $\{\Sigma_s\}$ ($s = 1, 2, ..., N$), Eq. (1) is generalized into

$$S\left(\underset{s=1}{\overset{N}{U}} \Sigma_s\right) = \Sigma_{s=1}^{N} \; S(\Sigma_s) \tag{2}$$

We can now start quoting, in the words of their authors, the inadequacies of BG statistics, or the precautions that have to be taken in what concerns its domain of applicability, or even some intuitive hints.

1.2 Astrophysics and the Gravitational N-Body Problem

A.M. Salzberg, in his 1965 *"Exact statistical thermodynamics of gravitational interactions in one and two dimensions"* [1], writes *"The exact equilibrium statistical mechanics of one-and two-dimensional gases, in which the particles interact through gravitational forces, is obtained. It is found that these systems are characterized by nonextensive thermodynamics leading to behaviour somewhat reminiscent of the formation of a star from interstellar dust"*, and also *"One interesting complication which arises in these gases is the nonextensive nature of the thermodynamic functions"*.

H.E. Kandrup, in his 1989 *"Mixing and "violent relaxation" for the one-dimensional gravitational Coulomb gas"* [2], writes *"One obvious point is that, whereas no statistical equilibrium exists for a three-dimensional system [thus, e.g., the canonical and micro-canonical distributions are not defined), one can in fact make sense of an equilibrium for one-dimensional gravity"*.

L.G. Taff, in his 1985 *"Celestial Mechanics"* [3], writes *"The nexus of the problem with the application of kinetic theory, of statistical mechanics, or of thermodynamics to self-gravitating systems is contained in Eqs. 12.27 and 12.28. Because the interparticle potential u is equal to $-Gm^2/r$, there is trouble at both ends of the domain of integration. $(\cdots)$ Both of these problems are due to the nonsaturation of gravitational forces (Levy-Leblond 1969). This means that the total energy of any finite collection of self-gravitating mass points does not have a finite, extensive (e.g., proportional to the number of particles) lower bound. Without such a property there can be no rigorous basis for the statistical mechanics of such a system (Fisher and Ruelle 1966). This result is not a consequence of the $1/r^2$ nature of the gravitational force but rather of its unshielded character [cf. Dyson and Lenard (1967) for a discussion of the electrostatic case]. Basically it is that simple. One can ignore the fact that one <u>knows</u> that there is no rigorous basis for one's computer manipulations, one can try to improve the situation, or one can look for another job"*.

W.C. Saslaw, in his 1985 *"Gravitational Physics of Stellar and Galactic Systems"* [4], writes *"This equation of state (30.9) also illustrates another important general aspect of gravitational thermodynamics. When interactions are important the thermodynamic parameters may lose their simple intensive and extensive properties for subregions of a given system. $(\cdots)$ In order for the thermodynamic limit to exist rigorously a system must have an equilibrium ground state. For such a state to have a minimum free energy it should* *be "saturated". This means that an increase of N would leave the free energy per particle unaltered. Saturated free energy is therefore proportional to N. If the free energy increases with a higher power of N, then, in the thermodynamic limit as $N \to \infty$, the binding energy per particle also becomes infinite. Thus the ground state would become more and more condensed as N increases. From a slightly different point of view, N could increase and V decrease indefinitely, making it impossible to achieve a thermodynamic limit with $\rho \to$constant. Gravitational systems, as often mentioned earlier, do not saturate and so do not have an ultimate equilibrium state. A three-dimensional purely gravitational system in virial equilibrium (not a state of ultimate equilibrium) has a binding energy $\propto N^2$. $(\cdots)$ These heuristic results, applied to relativistic fermions, were found in the early studies by Landau and Chandrasekhar of the stability of white dwarfs and neutron stars. They have been made more exact and derived rigorously by examining the thermodynamic limit of quantities with the form $N^{-7/3}F(T,V,N)$ in the ground state $(\cdots)$ The main difficulty is that in three dimensions no one knows how to sum the partition function exactly in a closed form. So we shall have to make do with a more open form which connects the partition function with kinetic theory".*

J. Binney and S. Tremaine, in their 1987 *"Galactic Dynamics"* [5], write *"Ogorodnikov (1965) and Lynden-Bell (1967a) show that this calculation leads to the conclusion that S is extremized if and only if f is the DF* [distribution function] *(4-116) of the isothermal sphere. However, the isothermal sphere is a system with infinite mass and energy. Hence this calculation shows that the maximization of S* [entropy] *subject to fixed M* [mass] *and* E [energy] *leads to a DF that is incompatible with finite M and E. From this contradiction it follows that no DF that is compatible with finite M and E maximizes S: if we constrain only M and E, configurations of arbitrarily large entropy can be constructed by suitable rearrangements of the galaxy's stars".*

S. Tremaine, M. Hénon and D. Lynden-Bell, in their 1986 *"H-functions and mixing in violent relaxation"* [6] provide a hint for solving the above problem. They write *"In collisionless systems Boltzmann's H-function $-\int F \log F\, dxdv$ is only one of a variety of H-functions of the form $-\int C(F)dxdv$, where C is any convex function"*. They deplore *"Lynden-Bell (1967) argued that violent relaxation leads toward a unique statistical equilibrium state which might be identified with the observed stellar distribution in elliptical galaxies, and used the methods of statistical mechanics to determine the distribution of* *stars in the equilibrium state. Unfortunately, the resulting equilibrium state has infinite total mass".*

These and similar arguments make us to understand why R. Balian, in his 1991 *"From Microphysics to Macrophysics"*[7], writes *"Important developments have taken place in mathematical physics. The conditions for the validity of the thermodynamic limit (§ 5.5.2) have been established, showing under what circunstances the entropy is an extensive quantity. This enables us to understand the limitations that exist, for instance in astrophysics, on the stability of matter".* On more general grounds, but within a somehow related philosophy, J.L. Lebowitz, in his very recent *"Boltzmann's entropy and time's arrow"* [8], writes *"Boltzmann's stroke of genius was, first, to make a direct connection between this microscopically defined function $S_B(M)$ and the thermodynamic entropy of Clausius, S_{eq}, which is a macroscopically defined, operationally measurable (also up to additive constants), extensive property of macroscopic systems in equilibrium. For a system in equilibrium having a given energy E (within some tolerance), volume V and particle number N, Boltzmann showed that $S_{eq}(E,V,N) \approx S_B(M_{eq})$ (2), where $M_{eq}(E,V,N)$ is the corresponding macrostate. By the symbol "$\approx$" we mean that for large N, such that the system is really macroscopic, the equality holds up to negligible terms when both sides of equation 2 are divided by N and the additive constant is suitably fixed. We require here that the size of the cells used to defined M_{eq}, that is, the macroscale, be very large compared with the microscale".* In other words, the problem might be far from trivial if the elements of the (macroscopic) system interact across distances comparable to (or larger than) the linear size of that system!

1.3 Black-Holes and Superstrings

P.T. Landsberg, in his 1984 *"Is Equilibrium Always an Entropy Maximum?"*[9], writes (in the Abstract) *"A systematic development is given of the view that in the case of systems with long-range forces and which are therefore nonextensive (in some sense) some thermodynamic results do not hold. Among these is the relation $U - TS + pv = \mu N$ and the Gibbs-Duhem equation. If a search for an equilibrium state is made by maximization of the entropy one may obtain misleading results because superadditivity may be violated. The considerations are worked out for a simple gas model, but they are relevant to black hole thermodynamics. Rather general conclusions can be drawn which transcend special* *systems".* He also writes *"The failure of some thermodynamic results, normally taken to be standard for black hole and other nonextensive systems has recently been discussed. $(\cdots)$. If two identical black holes are merged, the presence of long-range forces in the form of gravity leads to a more complicated situation, and the entropy is not extensive: $(\cdots) S_b(2\mathbf{X}) \neq 2S_b(\mathbf{X})$ (1.7) where an obvious notation has been used. In the merged black hole system one has to use $2M, 2J, 2Q$ and the relation between $S_b(2\mathbf{X})$ and $S_b(\mathbf{X})$ has to be investigated. It can in fact be shown that $S_b(\mathbf{X}_A + \mathbf{X}_B) > S_b(\mathbf{X}_A) + S_b(\mathbf{X}_B)$ (1.8). This means that entropy is "strictly superadditive", and this is consistent with (1.7) if one takes $\mathbf{X}_A = \mathbf{X}_B = \mathbf{X}$. $(\cdots)$ The circumstance that the main variables of thermodynamics in the absence of long-range forces are intensive or extensive, cannot be deduced from the so-called "laws" of thermodynamics. Nonetheless it is a very important characteristic of "normal" thermodynamic systems, and that is why it was recognized as such long ago, before the advent of black holes. $(\cdots)$ In any case, any treatment of "normal" thermodynamics should rule out long-range forces early on in the discussion. Anyone who wants to check carefully which parts of thermodynamics may, or may not, be used when long-range forces play a part will find little in the archival literature".*

Along the same lines, D. Pavón, in his 1987 *"Thermodynamics of Superstrings"* [10], writes *"Likewise, superstring entropy is neither homogeneous, $S_s(kE_s) \neq kS_s(E_s)$, nor concave, but it is superadditive. Superadditivity means that the entropy of a composite system must be greater than the combined entropies of the subsystems making up the total system".*

1.4 Lévy Flights, $1/f^\alpha$ Noise and Fractals

E.W. Montroll and M.F. Shlesinger, in their 1983 *"Maximum Entropy Formalism, Fractals, Scaling Phenomena, and $1/f$ Noise: A Tale of Tails"* [11], write (in the Abstract) *"In this report on examples of distribution functions with long tails we show that the derivation of distributions with inverse power tails from a maximum entropy formalism would be a consequence only of an unconventional auxiliary condition that involves the specification of the average value of a complicated logarithmic function".* They also write *"Hence the wonderful world of clusters and intermittencies and bursts that is associated with Lévy distributions would be hidden from us if we depended on a maximum entropy formalism that employed simple traditional auxiliary conditions".*

E.W. Montroll and B.J. West discuss, in their 1987 *"Fluctuation Phenomena"* [12], the connection with fractals: *"Thus, scaling and power-law distributions are intimately related; both imply a lack of a fundamental scale for the underlying process. We note that in the laws of Lotka, Zipf and Pareto there lurks such a scaling property, if only the process is viewed in the proper manner"*. Further discussions on the connection with $1/f^\alpha$ noise can be found in [13].

1.5 Vortex Physics

We can by no means state that vortex and turbulence physics have been shown to have a connection with nonextensive statistical mechanics (due, for instance, to long-range interactions between the vortex). However, many suggestive hints can be found here and there. Let us illustrate this by quoting fragments of the excellent 1980 review *"Two-dimensional turbulence"* by R.H. Kraichnan and D. Montgomery [14]. They write *"Fluid and plasma turbulence is ubiquitous in nature, at all scales from coffee cup to universe. Two-dimensional turbulence has the special distinction that is nowhere realised in nature or the laboratory but only in computer simulations. Its importance is two-fold: first, that it idealises geophysical phenomena in the atmosphere, oceans and magnetosphere and provides a starting point for modelling these phenomena; second, that it presents a bizarre and instructive statistical mechanics.* $(\cdots)$ *The enstrophy constant leads to equilibrium in which a large fraction of the energy is condensed into the largest spatial scales of motion, a situation closely analogous to the Einstein-Bose condensation in an ideal boson gas. But the present condensation involves negative (higher than infinite) temperatures. At a critical value of negative temperature there is evidence for a supercondensation phenomenon* $\cdots$ $(\cdots)$ *to investigate formally the dependence of cascade dynamics on a* <u>*continuous*</u> *dimensionality parameter* d. $(\cdots)$ *They find unphysical behaviour for* $d < 2...$ $(\cdots)$ *In three dimensional flows the analogy between sub-grid scales and thermal agitation already is imperfect because the turbulent motion in fact has a continuous distribution of scale sizes.* $(\cdots)$ *It is probable that the excitation of sub-grid scales by explicit scales and the subsequent reaction of the sub-grid scales on the explicit scales also involves coherent, phase-locked phenomena that are intrinsically unsuited to any statistical treatment".*

1.6 Granular Matter

The macroscopic stability of powder mixtures and other forms of granular or fibre matter might be associated with *fractal* sets (e.g., the set of grain point contacts of a sandpile [15]). Under these circumstances, there is no reason for having a relevant "thermodynamic" energy *proportional* to a standard power of the mass of the system. Consequently, nonextensive phenomena could be present. In fact, A. Mehta and S.F. Edwards [16] have used *"a new formulation of the statistical mechanics of powders to develop a theory for a mixture of grains of two different sizes"*. Also, they *"discuss the insight afforded by this solution on the "thermodynamic" quantities of interest in the powder mixture"* (see also [17]).

1.7 Neural Network and Learning Curves

Neural networks (e.g., perceptrons, Hopfield model and others) are artificial devices or computational models which can perform human-like tasks such as recognizing, generalizing and learning. Their "statistical mechanics" is very rich and is being intensively studied nowadays (see [18] and references therein). With extremely rare exceptions (e.g., [19]), the entire "thermodynamical" discussion is done in terms of BG statistics (hence, with a Shannon-like entropy), and, for dynamical purposes, in terms of Langevin and Fokker-Planck equations (the stationnary solution of which is the BG equilibrium distribution). *There is however no imperative reason for such a limitation* (remember that jets fly quicker than *any* biological being!). More than that, there are in fact indications in the opposite sense; for instance, simmulated annealing with a Cauchy machine is much faster than with a Boltzmann machine [20]. This area of research could become a very interesting field of applications of nonextensive physics.

1.8 Economics

Of course, like artificial neural networks, Economics does not belong to the so-called "Physical Sciences". Nevertheless, it can constitute an interesting area for applying nonextensive concepts developed in Physics. This is, in particular, the case of the theory of financial decisions (e.g., stock market and similar operations involving "risk" and "uncertainty"). In fact, the cross-fertilizing interaction between Physics and Economics is not new: an illustrious such example is the 1947 von Neumann and Morgenstern seminal work on game theory [21]. In the words of J. Dow and S.R.C. Werlang in their 1992 *"Uncertainty aversion, risk aversion, and the optimal choice of portfolio"* [22]: *"With a nonadditive probability measure, the "probability" that either of two mutually exclusive events will occur is not necessarily equal to the sum of their two "probabilities". If it is less than the sum, the expected-utility calculations using this probability measure will reflect uncertainty aversion as well as (possibly) risk aversion. The reader may be disturbed by "probabilities" that do not sum to one. It should be stressed that the probabilities, together with the utility function, provide a representation of behavior. They are not objective probabilities"*. Also, in the words of M.H. Simonsen and S.R.C. Werlang in their 1991 *"Subadditive probabilities and portfolio inertia"* [23]: *"Uncertainty means, in fact, incomplete information about the true probabilities. $(\cdots)$ The attractiveness of the concept of subadditive probabilities is that it might provide the best possible description for what is behind the widespread notion of subjective probabilities in the theory of financial decisions"*. See also [24].

The possible connection of these subadditive probabilities with the generalized statistical mechanics discussed in Section II will become obvious later on. Let us just anticipate that the generalized mean value (of an observable $\hat{O}$) is calculated with $\Sigma_{i=1}^{W} p_i^q O_i$ $(q \in \mathbb{R})$, hence $\{p_i^q\}$ play the role of probabilities in standard Theory of Probabilities, *but* $\Sigma_{i=1}^{W} p_i^q$ generically equals unity if and only if $q = 1$ ($q > 1$ yields subadditive "probabilities", e.g., $(p_1 + p_2)^q \geq p_1^q + p_2^q$). On the other hand, the reader should be aware that, in [22–24], Choquet's mean value, rather than the present, is currently used. The most striking difference is that if we consider a real positive constant λ, the Choquet expected value of λ yields λ (as with usual probabilities), whereas the present expected value yields $< \lambda >_q = \lambda \Sigma_i p_i^q$, which is generically *smaller than* λ *if* $q > 1$ (it equals λ *only* if $q = 1$ for all $\{p_i\}$ *or* if $\{p_i\} = \{1, 0, 0, \cdots\}$ for all q). Which expected value better describes mathematically human behavior within a theory of financial decisions might be a controversial matter. In our favor let us mention that we exhibit in Section II.7 how the present mean value enables a well defined prescription for *operationally* dealing with this kind of problem.

2 Looking for a Way Out

2.1 Entropic Forms

Generalized forms for the entropy as a measure of information are commonly discussed within the communities of Information Theory and Statistical Inference. The first attempt was done, as far as we know, by Schützenberger (according to Csiszar [25]) and by Renyi [26] who introduced

$$S_q^{SR} \equiv \frac{\ln \sum_{i=1}^{W} p_i^q}{1-q} \qquad (q \in I\!\!R) \tag{3}$$

This expression: (i) recovers, in the $q \to 1$ limit, that of Shannon, for $q = 2$, that of Pielou [27], and is proportional (through a conveniently chosen coefficient) to those introduced by Varma (see [28]) and by Nath [29]; (ii) is *extensive*, i.e., satisfies Eq. (2), $\forall q$; and (iii) has *not necessarily* a fixed concavity for all $\{p_i\}$ and fixed q. After this generalization, many others followed as possible information functions for statistical inference and related purposes. Let us mention [30]

$$S_q^{HCD} \equiv \frac{1}{1-2^{1-q}} \left[1 - \sum_{i=1}^{W} p_i^q\right] \tag{4}$$

introduced by Havrda and Charvat and by Daroczy (it recovers, in the $q \to 1$ limit, that of Shannon),

$$S^{SM} \equiv \frac{1}{1-2^{1-\beta}} \left\{1 - \left[\sum_{i=1}^{W} p_i^\alpha\right]^{(\beta-1)/(\alpha-1)}\right\} \quad (\alpha > 0, \alpha \neq 1, \beta > 1) \tag{5}$$

introduced by Sharma and Mittal [31],

$$S^A \equiv \frac{R}{R-1} \left\{1 - \left[\sum_{i=1}^{W} p_i^R\right]^{1/R}\right\} \qquad (R > 1) \tag{6}$$

introduced by Arimoto [32]. Van der Lubbe et al [33] have generalized all of them by considering three families, namely

$$S_I^{LBB} \equiv -\delta log_2 \left[\sum_{i=1}^{W} p_i^\rho\right]^\sigma \qquad \text{(logarithmic measure)} \tag{7}$$

$$S_{II}^{LBB} \equiv \delta \left\{1 - \left[\sum_{i=1}^{W} p_i^\rho\right]^\sigma\right\} \qquad \text{(linear measure)} \tag{8}$$

$$\text{and} \quad S_{III}^{LBB} \equiv \delta \left\{ \left[\sum_{i=1}^{W} p_i^\rho \right]^{-\sigma} - 1 \right\} \quad \text{(hyperbolic measure)} \tag{9}$$

($\delta > 0; 0 < \rho < 1$ and $\sigma < 0$, or $\rho > 1$ and $\sigma > 0$). More about these vast classes of information measures (and their particular cases) can be found in [34].

2.2 Generalized Statistical Mechanics: Introduction

Curiously enough, as far as we know, no attempts have been made to explore the richness of the above mentioned functional forms in the sense of looking for possible connections with Physics, and ultimately with Nature.

In 1988 we (independently) proposed [35], inspired by multifractals, the generalized entropy

$$S_q \equiv k \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1} \qquad (q \in \mathbb{R}) \tag{10}$$

as a starting point for generalizing BG statistics (k is a conventional positive constant). It recovers, in the $q \to 1$ limit, Shannon's expression, it is *concave* (convex) for $q > 0 (q < 0)$ and all $\{p_i\}$, but *violates extensivity*. Instead of Eqs. (1) and (2), it satisfies *pseudo-additivity*, i.e.,

$$\frac{S_q(\Sigma_1 U \Sigma_2)}{k} = \frac{S_q(\Sigma_1)}{k} + \frac{S_q(\Sigma_2)}{k} + (1-q)\frac{S_q(\Sigma_1)}{k}\frac{S_q(\Sigma_2)}{k} \tag{11}$$

and, for arbitrary $N \geq 2$,

$$1 + (1-q) S_q \left(\overset{N}{\underset{s=1}{U}} \Sigma_s \right) / k = \prod_{s=1}^{N} [1 + (1-q) S_q(\Sigma_s)/k] \tag{12}$$

Eq. (12) can be rewritten as

$$\frac{\ln \left[1 + (1-q) S_q \left(\overset{N}{\underset{s=1}{U}} \Sigma_s \right) / k \right]}{1-q} = \sum_{s=1}^{N} \frac{\ln[1 + (1-q) S_q(\Sigma_s)/k]}{1-q} \,, \tag{13}$$

but we can easily verify that

$$\frac{\ln[1 + (1-q) S_q/k]}{1-q} = S_q^{SR} \,, \tag{14}$$

hence Eq. (13) reproduces Eq. (2) with $S \equiv S_q^{SR}$![35] Incidentally, let us mention that $q < 1$ implies strict *superadditivity* as wanted in Section I.3 (whereas $q > 1$ implies strict *subadditivity*). Other relevant properties follow:

(i) $$S_q \geq 0, \qquad \forall q, \qquad \forall\{p_i\} \tag{15}$$

(ii) S_q is *expansible* for $q > 0$, i.e.,

$$S_q(p_1, p_2, \cdots, p_W, 0) = S_q(p_1, p_2, \cdots, p_W) \tag{16}$$

(iii) H-theorem: under quite general conditions (less restrictive than detailed balance) [36-38], $dS_q/dt \geq 0, = 0$ and ≤ 0, if $q > 0, = 0$ and < 0, respectively, t being the time.

2.3 Microcanonical Ensemble

Equiprobability (i.e., $p_i = 1/W, \forall i$) extremizes $S_q, \forall q$ (maximal for $q > 0$, minimal for $q < 0$, and constant for $q = 0$). Its value is given by

$$S_q = k\frac{W^{1-q} - 1}{1 - q} \tag{17}$$

which recovers Boltzmann's celebrated formula $S = k_B \ln W$, in the limit $q \to 1$. Eq. (17) implies

$$W = [1 + (1-q)S_q/k]^{\frac{1}{1-q}} \tag{18}$$

This form suggests an important conjecture. Indeed, if we denote by $P(\{p_i\})$ (*likelihood function* as sometimes referred to in statistics) the *probability of having a set* $\{p_i\}$ *different from that which maximizes* S_q, Eq. (18) suggests (in analogy with the $q = 1$ case)

$$P(\{p_i\}) \propto [1 + (1-q)S_q(\{p_i\})/k]^{\frac{1}{1-q}} \tag{19}$$

This relation can be conjectured along a different path (elaborated in a private discussion with M.O. Caceres), which follows that of Einstein [39] for the $q = 1$ case. If we think the macroscopic system as made of practically independent macroscopic subsystems, we have that S_q^{SR} is additive, which suggests (see [39])

$$P(\{p_i\}) \propto e^{S_q^{SR}(\{p_i\})} \tag{20}$$

If we now use here Eq. (14), we immediately recover Eq. (19). This conjecture (adapted for the canonical and grand-canonical ensembles, where supplementary constraints are imposed, besides $\Sigma_{i=1}^{W} p_i = 1$) was first assumed by Chame and Mello [40], who deduced from it a very general form for the fluctuation-dissipation theorem (which successfully reproduced, as particular cases, the fluctuation forms of the specific heat and of the magnetic susceptibility, already proved elsewhere through completely different arguments).

2.4 Canonical Ensemble

The optimization of S_q under the constraints $\Sigma_{i=1}^{W} p_i = 1$ and

$$\sum_{i=1}^{W} p_i^q \varepsilon_i = U_q \tag{21}$$

(where $\{\varepsilon_i\}$ are the eigenvalues of the system Hamiltonian $\mathcal{H}$, and the generalized internal energy U_q a *finite* fixed value) yields the following equilibrium distribution [35,41]:
If $q = 1$ (BG statistics),

$$p_i = e^{-\beta\varepsilon_i}/Z_1 \tag{22}$$

with

$$Z_1 = \sum_{i=1}^{W} e^{-\beta\varepsilon_i} \tag{23}$$

$\beta \equiv 1/kT$ being the Lagrange parameter associated with restriction (21).
If $q < 1$ ("superadditive" statistics),

$$p_i = \begin{cases} [1-\beta(1-q)\varepsilon_i]^{\frac{1}{1-q}}/Z_q, & \text{if } [1-\beta(1-q)\varepsilon_i] > 0 \\ 0 & , \text{ otherwise} \end{cases} \tag{24}$$

with

$$Z_q = \sum_{i=1}^{W}{}' [1-\beta(1-q)\varepsilon_i]^{\frac{1}{1-q}} \tag{25}$$

where Σ' runs *only* over the levels ε_i satisfying $1-\beta(1-q)\varepsilon_i > 0$.
If $q > 1$ ("subadditive" statistics),

$$p_i = \begin{cases} [1-\beta(1-q)\varepsilon_i]^{\frac{1}{1-q}}/Z_q & \text{if } 1-\beta(1-q)\varepsilon^* > 0 \\ \delta_{i,*}/g^* & , \text{ otherwise} \end{cases} \tag{26}$$

with

$$Z_q = \sum_{i=1}^{W} [1-\beta(1-q)\varepsilon_i]^{\frac{1}{1-q}} \tag{27}$$

where $\varepsilon^* \equiv \inf\{\varepsilon_i\}(\varepsilon^* \equiv \sup\{\varepsilon_i\})$ if $\beta > 0$ $(\beta < 0)$ and g^* is the associated degeneracy; $\delta_{i,*}$ equals unity if $\varepsilon_i = \varepsilon^*$, and vanishes otherwise.

In order to clarify some confusion existing in the available literature, let us detail (after enlightening discussions with S.A. Cannas, P. Pury and G. Raggio) the equilibrium distribution (Eqs. (22-27)). To fix the ideas, let us assume that the spectrum $\{\varepsilon_i\}$ present L levels (characterized by $\ell = 1, 2, \cdots, L$;

$\Sigma_{\ell=1}^{L} g_\ell = W$ where $\{g_\ell\}$ are the associated degeneracies), and that its labelling satisfies

$$\begin{aligned}
&\underbrace{\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_{g_1}}_{g_1 \text{ states}} < \underbrace{\varepsilon_{g_1+1} = \varepsilon_{g_1+2} = \cdots = \varepsilon_{g_1+g_2}}_{g_2 \text{ states}} < \cdots < \\
&< \underbrace{\varepsilon_{g_1+g_2+\cdots g_{L-2}+1} = \cdots = \varepsilon_{W-g_L-1} = \varepsilon_{W-g_L}}_{g_{L-1} \text{ states}} < \\
&< \underbrace{\varepsilon_{W-g_L+1} = \varepsilon_{W-g_L+2} = \cdots = \varepsilon_{W-1} = \varepsilon_W}_{g_L \text{ states}}
\end{aligned} \tag{28}$$

where *both* ε_1 and ε_W are assumed finite. In all cases ($q \gtreqless 1$), we have that $\lim_{T\to\pm\infty} p_i = \frac{1}{W}, \forall i$.

<u>Case $q = 1$</u>:

For $T \to +0, p_i$ approaches $1/g_1$ if $1 \leq i \leq g_1$, and approaches zero otherwise. For $T \to -0, p_i$ approaches $1/g_L$ if $W - g_L + 1 \leq i \leq W$, and approaches zero otherwise. The region $T = 0$ is thermally *forbidden* (physically inaccessible), and the region $T \neq 0$ is thermally *active*.

<u>Case $q < 1$</u>:

If $\varepsilon_1 \geq 0$, then $T < 0$ is thermally *active*, $T \to -0$ implies $p_i \to \varepsilon_i^{\frac{1}{1-q}} / \Sigma_{i=1}^{W} \varepsilon_i^{\frac{1}{1-q}}$, $T \in [0, (1-q)\varepsilon_1/k]$ is thermally *forbidden*, $T \in ((1-q)\varepsilon_1/k, (1-q)\varepsilon_{g_1+1}/k]$ is thermally *frozen* ($p_i = 1/g_1$ if $1 \leq i \leq g_1$, and zero otherwise), and $T > (1-q)\varepsilon_{g_1+1}/k$ is thermally *active*.

If $\varepsilon_W \leq 0$, then $T < (1-q)\varepsilon_{W-g_L}/k$ is *active*, $T \in [(1-q)\varepsilon_{W-g_L}/k, (1-q)\varepsilon_W/k)$ is *frozen* ($p_i = 1/g_L$ if $W - g_L + 1 \leq i \leq W$, and zero otherwise), $T \in [(1-q)\varepsilon_W/k, 0)$ is *forbidden*, $T \to +0$ implies $p_i \to |\varepsilon_i|^{\frac{1}{1-q}} / \Sigma_{i=1}^{W} |\varepsilon_i|^{\frac{1}{1-q}}$, and $T > 0$ is *active*. If $\varepsilon_1 \leq \cdots \leq \varepsilon_B < 0 < \varepsilon_A \leq \cdots \leq \varepsilon_W$ ($\varepsilon_B \equiv$ first level *below* zero, $\varepsilon_A \equiv$ first level *above* zero; the results for the cases $\varepsilon_B = 0 < \varepsilon_A$ and $\varepsilon_B < 0 = \varepsilon_A$ can be obtained as simple limits of the generic case $\varepsilon_B < 0 < \varepsilon_A$), there is no *finite-temperature* forbidden region. If $\varepsilon_1 = \varepsilon_B$ and $\varepsilon_A = \varepsilon_W$ (two level system), then $T < (1-q)\varepsilon_1/k$ is *active*, $T \in [(1-q)\varepsilon_1/k, 0)$ is *frozen* ($p_i = 1/g_W$ for $W - g_2 + 1 \leq i \leq W$, and vanishes otherwise), $T \in (0, (1-q)\varepsilon_W/k]$ is *frozen* ($p_i = 1/g_1$ for $1 \leq i \leq g_1$, and vanishes otherwise), and $T > (1-q)\varepsilon_W/k$ is *active*; if $\varepsilon_B = \varepsilon_1$ and $\varepsilon_A < \varepsilon_W$ (three or more level system), then $T < 0$ is *active*, $T \in (0, (1-q)\varepsilon_A/k]$ is *frozen* ($p_i = 1/g_1$ for $1 \leq i \leq g_1$, and vanishes otherwise), and $T > (1-q)\varepsilon_A/k$ is *active*; if $\varepsilon_1 < \varepsilon_B$ and $\varepsilon_A = \varepsilon_W$ (three or more level system), then

$T < (1-q)\varepsilon_B/k$ is *active*, $T \in [(1-q)\varepsilon_B/k, 0)$ is *frozen* ($p_i = 1/g_W$ for $W - g_L + 1 \leq i \leq W$, and vanishes otherwise), and $T > 0$ is *active*; if $\varepsilon_1 < \varepsilon_B$ and $\varepsilon_A < \varepsilon_W$ (four or more level system), then $T \neq 0$ is *active*, p_i tends, in the $T \to -0$ limit, to $\varepsilon_i^{\frac{1}{1-q}} / \sum_{i:\varepsilon_i > 0} \varepsilon_i^{\frac{1}{1-q}}$ if $\varepsilon_i \geq \varepsilon_A$ and to zero otherwise, and p_i tends, in the $T \to +0$ limit, to $|\varepsilon_i|^{\frac{1}{1-q}} / \sum_{i:\varepsilon_i < 0} |\varepsilon_i|^{\frac{1}{1-q}}$ if $\varepsilon_i \leq \varepsilon_B$ and to zero otherwise.

Case $q > 1$:

There is no finite-temperature forbidden region in any case.

If $\varepsilon_1 \geq 0$, then $T < (1-q)\varepsilon_W/k$ is *active*, $T \in [(1-q)\varepsilon_W/k, 0)$ is *frozen* ($p_i = 1/g_L$ if $W - g_L + 1 \leq i \leq W$, and vanishes otherwise), and $T > 0$ is *active* $\left(\lim_{T\to+0} p_i = \varepsilon_i^{\frac{1}{1-q}} / \Sigma_{i=1}^W \varepsilon_i^{\frac{1}{1-q}}\right)$. If $\varepsilon_W \leq 0$, then $T < 0$ is *active* $\left(\lim_{T\to-0} p_i = |\varepsilon_i|^{\frac{1}{1-q}} / \Sigma_{i=1}^W |\varepsilon_i|^{\frac{1}{1-q}}\right)$, $T \in (0, (1-q)|\varepsilon_1|/k]$ is *frozen* ($p_i = 1/g_1$ if $1 \leq i \leq g_1$, and vanishes otherwise), and $T > (1-q)|\varepsilon_1|/k$ is *active*. If $\varepsilon_1 \leq \cdots \leq \varepsilon_B < 0 < \varepsilon_A \leq \cdots \leq \varepsilon_W$ (as before, the results for the cases $\varepsilon_B = 0 < \varepsilon_A$ and $\varepsilon_B < 0 = \varepsilon_A$ can be obtained as simple limits of the generic case $\varepsilon_B < 0 < \varepsilon_A$), then $T < (1-q)\varepsilon_W/k$ is *active*, $T \in [(1-q)\varepsilon_W/k, 0)$ is *frozen* ($p_i = 1/g_L$ if $W - g_L + 1 \leq i \leq W$, and vanishes otherwise), $T \in (0, (1-q)\varepsilon_1/k]$ is *frozen* ($p_i = 1/g_1$ if $1 \leq i \leq g_1$, and vanishes otherwise), and $T > (1-q)\varepsilon_1/k$ is *active*.

Let us conclude with two remarks: (i) In all circumstances ($q \gtrless 1$), $T < 0$ ($T > 0$) becomes *forbidden* if $\varepsilon_W(\varepsilon_1)$ diverges, because U_q would diverge (which is incompatible with being a *finite* constraint); (ii) If $q \neq 1$, the $\lim_{T\to\pm 0} S_q$ might be different from zero *even in the absence of any degeneracy* (i.e., if $g_\ell = 1, \forall \ell$), in remarkable contrast with the $q = 1$ case; consequently, *the Third Principle of Thermodynamics might be violated if* $q \neq 1$.

It can be proved [41] in general that

$$\frac{1}{T} = \frac{\partial S_q}{\partial U_q} \tag{29}$$

$$U_q = -\frac{\partial}{\partial \beta} \frac{Z_q^{1-q} - 1}{1-q} \tag{30}$$

and

$$F_q \equiv U_q - TS_q = -\frac{1}{\beta} \frac{Z_q^{1-q} - 1}{1-q} \tag{31}$$

2.5 Mean Values

As already appeared in Eq. (21), the relevant mean values within the present generalization are calculated with p_i^q rather than with p_i. In other words, if we have an observable $\hat{0}$, the quantity of interest (associated with the density operator $\hat{\rho}$) is

$$< \hat{O} >_q \equiv Tr\hat{\rho}^{\ q}\ \hat{O} = Tr\hat{\rho}\left(\hat{\rho}^{\ q-1}\hat{O}\right) =< \hat{\rho}^{\ q-1}\hat{O} >_1 \qquad (32)$$

and not $< \hat{O} >_1$ as usually. This is so in order to preserve the Legendre-transformation structure of Thermodynamics [41], or quivalently, in order to satisfy Jaynes' requirements (duality relations) for a formalism to be acceptable within Information Theory [42].

If we introduce the *entropy operator*

$$\hat{S}_q \equiv k\frac{1-\hat{\rho}^{1-q}}{1-q} \qquad (33)$$

we immediately verify, by using Eq. (32), that

$$S_q =< \hat{S}_q >_q \qquad (34)$$

and that

$$< \hat{O} >_q =< \hat{O}_q >_1 \qquad (35)$$

with

$$\hat{O}_q \equiv \frac{\hat{O}}{1-(1-q)\hat{S}_q/k} \qquad (36)$$

So, *the generalized mean value of a standard observable can be thought as the standard mean value of a generalized observable whose definition incorporates the information on the system!* Of course, this statement was already implicit in Eq. (32), but is particularly striking in Eq. (35).

Let us now address an interesting property concerning mean values. For facility, we shall work with diagonalized operators. Our system has W states characterized by the possible couples $\{(i,j)\}$ $(i = 1, 2, \cdots I; j = 1, 2, \cdots, J; IJ = W)$ with probability $\{p_{ij}\}$ $(\Sigma_{i,j} p_{ij} = 1)$. Our observable attains the value O_{ij} in the (i,j) state. Then,

$$\begin{aligned} < \hat{O} >_q &= \sum_{i,j} p_{ij}^q O_{ij} \\ &= \sum_i \left(\sum_{j'} p_{ij'}\right)^q \sum_j \left(\frac{p_{ij}}{\sum_{j'} p_{ij'}}\right)^q O_{ij} \end{aligned} \qquad (37)$$

If we introduce now the *marginal probabilities*

$$p_i \equiv \sum_j p_{ij} \qquad (i = 1, 2, \cdots, I; \sum_i p_i = 1) \tag{38}$$

and the *conditional probabilities*

$$p_j^{(i)} \equiv \frac{p_{ij}}{p_i} \qquad (j = 1, 2, \cdots, J; \sum_j p_j^{(i)} = 1) \tag{39}$$

we can rewrite Eq. (37) as follows:

$$< \hat{O} >_q = \sum_i p_i^q < O >_q^{(i)} \tag{40}$$

with

$$< \hat{O} >_q^{(i)} \equiv \sum_j \left(p_j^{(i)}\right)^q O_{ij} \tag{41}$$

Let us consider now the particular case $O_{ij} = O_i$ (degeneracy of the j index). Eq. (41) becomes

$$\begin{aligned} < \hat{O} >_q^{(i)} &= O_i \sum_j \left(p_j^{(i)}\right)^q \\ &= O_i [1 + (1-q) S_q(p_1^{(i)}, p_2^{(i)}, \cdots, p_J^{(i)})/k] \end{aligned} \tag{42}$$

hence

$$< \hat{O} >_q = \sum_i p_i^q O_i [1 + (1-q) S_q(p_1^{(i)}, p_2^{(i)}, \cdots, p_J^{(i)})/k] \tag{43}$$

which explicitely exhibits how *the information on the j-index affects the weight of the value O_i!* This curious effect just does not exist for $q = 1$. We can consider now the particular case $\hat{O} = \hat{1}$ (hence $O_{ij} = 1, \forall (i,j)$). Then

$$< \hat{O} >_q = 1 + (1-q) S_q(p_{11}, p_{12}, \cdots, p_{ij}, \cdots, p_{IJ})/k \tag{44}$$

where we used the definition (33). Hence, replacing Eq. (44) in Eq. (43) we obtain

$$\begin{aligned} &1 + (1-q) S_q(p_{11}, p_{12}, \cdots, p_{IJ})/k = \\ &\quad = \sum_i p_i^q + (1-q) \sum_i p_i^q S_q(p_1^{(i)}, p_2^{(i)}, \cdots, p_J^{(i)})/k \end{aligned} \tag{45}$$

Using finally that $\Sigma_i p_i^q = 1 + (1-q) S_q(p_1, p_2, \cdots, p_I)/k$ we obtain

$$S_q(\{p_{ij}\}) = S_q(\{p_i\}) + \sum_i p_i^q S_q(\{p_i^{(i)}\}) \tag{46}$$

which exhibits how is gained the *global* information if we are informed, in a *first step*, on the index i, and, in a *second step*, on the index j. The particular case $I = 2$ recovers Eq. (7) of [41], which in turn recovers, for $q = 1$, the celebrated Shannon property.

2.6 Some More Properties and Physical Applications

Plastino and Plastino have shown [42, 43] that both the Ehrenfest theorem and the von Neumann equation are *form-invariant for all values of* q. This has a very important implication (private communication of A.R. Plastino), namely, *it is impossible to determine* q, *for a particular system, through dynamical measurements.* However, it can be determined through statistical measurements (e.g., specific heat, susceptibility, equation of states).

Further standard results that have been generalized for arbitrary q are the quantum statistics [44], the Langevin and Fokker-Planck equations [45], the single-site Callen identity [46], the Bogolyubov inequality [47], a criterion for nonparametric testing [48], the black-body radiation Planck law [49], the simulated annealing [50] among others. One-body and many-body systems that have been studied include the two level system [35, 51], the free particle [52], the Larmor procession [43], $d = 1$ Ising ferromagnet [53, 54], $d = 2$ Ising ferromagnet [46, 55, 56] and, very recently, the localized-spin ideal paramagnet [57] where, for the first time, the existence of a (numerically) well defined thermodynamic limit was exhibited, and where a curious (Bose-Einstein-condensation-like) phase transition was shown.

At the present moment, two physical systems have been shown to present substantial advantages if treated within $q \neq 1$ statistics. The first of them refers to the gravitational effect on the polytropic model for stellar systems, as discussed by Chandrasekhar and others. This model is known to yield an unphysical result, namely *infinite mass*, within BG statistics. This difficulty has been recently overcome by Plastino and Plastino [58] by considering $q \neq 1$: if q sufficiently differs from unity, the mass becomes *finite*.

The second system refers to Lévy flights. This random motion is characterized by eventual *long* jumps which construct, if iterated many times, a fractal (with fractal dimension γ). This jump distribution is long-tailed, and is known to be incompatible with a variational formalism extremizing Shannon entropy with acceptable a priori constraints. This problem has been recently overcome, by Alemany and Zanette [59], by extremizing the generalized entropy S_q; they obtain $q = (3 + \gamma)/(1 + \gamma)$. Also, possible (though yet unfound) applications have been suggested [60] for Condensed Matter (or Plasma or Elementary Particles or other) *droplets* the elements of which interact over lengths comparable to (or larger than) the (linear) size of the droplet.

Finally, an interesting possibility for application in Nonequilibrium Statistical Mechanics appeared very recently (during a private discussion with T. Tomé). Indeed, a variety of standard (and important) transport equations can be deduced from a variational principle applied to the usual entropy if appropriate time-dependent constraints are imposed which involve a *short memory* function (of the type $e^{\varepsilon t'}$ with $\varepsilon > 0$ and $t' \leq 0$) [61]. The whole theory relies on the *convergence* of the integral (in the interval $-\infty < t' \leq 0$) of the memory function. What happens if we are dealing with a *long-memory* phenomenon (of the type $1/(-t')^{1-\alpha}$ for $t' << -1$, with $\alpha \geq 0$)? What are the corresponding transport equations? Can we deduce them from a variational principle applied to an entropy? If so, this entropy *cannot* be the usual one because the above mentioned integral *diverges*! The mathematical problems involved in this type of long-duration phenomena look very similar to those involved in long-range-interaction systems [1–13]. If $q \neq 1$ statistics can provide satisfactory issues [58, 59] for the spatial case, is it not reasonable to expect for something analogous for the time case?

2.7 Approaching the Theory of Financial Decisions

Let us now return to the important financial problem related to human aversion to uncertainty or risk. We shall illustrate the present approach through an example discussed in a very recent survey of the *"Frontiers of Finance"* [62], where we read *"Would you rather have \$85,000, or an 85% chance of \$100,000? Most people would take the money. Would you rather lose \$85,000, or run an 85% risk of losing \$100,000? Most people would take the chance. When Amos Tversky of Stanford University posed people these dilemmas he was interested in their understanding of, and attitude to, probability, time and risk. His work has implications for the study of how a financial market works. It demonstrates that people are "non-linear". They are risk-averse when expecting a gain and risk-seeking when facing a loss".*

Let us interpret this text within the present mathematical language.

<u>Gain expectation:</u>

"Take the money"-choice: $p_1 = 1$, hence $< gain >_q^{(1)} = 1^q \times 85,000 = 85,000$.

"Run a risk"-choice: $p_1 = 0.85$ and $p_2 = 0.15$, hence

$$< gain >_q^{(2)} = (0.85)^q \times 100,000 + (0.15)^q \times 0 = (0.85)^q \times 100,000$$

A person which prefers the *first* choice (as most do) evaluates $< gain >_q^{(1)} > < gain >_q^{(2)}$

hence his (or her) value of q is *above unity.*

Loss expectation:

"Lose the money"-choice: $p_1 = 1$, hence $< gain >_q^{(1)} = 1^q \times (-85,000) = -85,000$.

"Take the chance"-choice: $p_1 = 0.85$ and $p_2 = 0.15$, hence $< gain >_q^{(2)} = -(0.85)^q \times 100,000$.

A person which prefers the *second* choice (as most do) evaluates $< gain >_q^{(2)} > < gain >_q^{(1)}$, hence, his (or her) value of q is *once more above unity.*

So $q > 1$ "explains" *both* gain and loss expectation cases!

Let us now focus the question: How can we measure q? We illustrate this with the gain-expectation dilemma. To fix ideas, suppose the person prefers the "take the money"-choice. Then we pose again the dilemma with a value V slightly *below* 85,000 (say 84,000) for the "take the money"-choice, and the *same* conditions as before for the "run a risk"-choice. If the person still prefers the "take the money" choice, we *decrease even more* V. A *critical* value V_c will be achieved such that the person *just changes* his (or her) mind. In this case, it is $< gain >_q^{(1)} = < gain >_q^{(2)}$, hence $V_c = (0.85)^q \times 100,000$, hence $q = \ln(V_c/100,000)/\ln(0,85)$, which provides the value of q for that person (for that dilemma, at that moment). q monotonically decreases from $(+\infty)$ to $(-\infty)$ while V_c increases from zero to infinity; $q > 1$ if $V_c < 85,000$ (risk-averse attitude), $q = 1$ if $V_c = 85,000$ (ideally rational attitude), $0 \leq q < 1$ if $85,000 < V_c \leq 100,000$ (risk-seeking attitude), and finally $q < 0$ if $V_c > 100,000$ (which would be a completely irrational attitude!).

If we test a *large* number of individuals we will approach the theoretical distribution $R(q)$ of that population $\left(\int_{-\infty}^{\infty} R(q)dq = 1\right)$. *Both* statements *"Most people would take the money"* and *"Most people would take the chance"* in the above text, are interpreted with a *single* mathematical statement namely

$$0 < \int_{-\infty}^{1} R(q)dq < \frac{1}{2} < \int_{1}^{\infty} R(q)dq < 1.$$

3 Conclusion

We tried, in Section I, to convince the reader of the *necessity* of enlarging the horizons of Boltzmann-Gibbs statistics and extensive thermodynamics. In Section II we presented, besides a review of entropic forms that have appeared within the communities of Information Theory and Statistical Inference, a generalized thermostatistical formalism which (i) satisfactorily connects with a consistently generalized thermodynamics, (ii) seems to be mathematically coherent, and (iii) of course embraces, as a particular case, standard thermostatistics. This formalism proved successful for two applications (polytropic model for stellar matter, and Lévy flights), both of them presenting *unshielded long-tailed interactions* of the type which cause the troubles shown in Section I. It is obviously premature to say whether this formalism is the "correct answer" for at least some of the existing difficulties, but it seems that this possibility should not be excluded. Only further studies will fix the question.

I have tremendously benefited from discussions with very many scientists from Brazil, Argentina, France, USA and other countries. Their large number makes it not appropriate to name them all here $\cdots$ but they know who they are!: to all of them my gratitude.

REFERENCES

[1] A.M. Salzberg, J. Math. Phys. **6**, 158 (1965).

[2] H.E. Kandrup, Phys. Rev. A **40**, 7265 (1989).

[3] L.G. Taff, "Celestial Mechanics" (John Wiley and Sons, New York, 1985), pages 437 and 438.

[4] W.C. Saslaw, "Gravitational Physics of Stellar and Galactic Systems" (Cambridge University Press, Cambridge, 1985), pages 217 and 218.

[5] J. Binney and S. Tremaine, "Galactic dynamics" (Princeton University Press, Princeton, 1987), page 268.

[6] S. Tremaine, M. Hénon and D. Lynden-Bell, Mon. Not. R. Astr. Soc. **219**, 285 (1986).

[7] R. Balian, "From Microphysics to Macrophysics" (Springer-Verlag, Berlin, 1991) Vol. I, page 134.

[8] J.L. Lebowitz, Physics Today (September 1993), pages 32-38.

[9] P.T. Landsberg, J. Stat. Phys. **35**, 159 (1984).

[10] D. Pavón, Gen. Rel. and Grav. **19**, 375 (1987).

[11] E.W. Montroll and M.F. Shlesinger, J. Stat. Phys. **32**, 209 (1983).

[12] E.W. Montroll and B.J. West, "Fluctuation Phenomena" (Elsevier, Amsterdam, 1987) page 186.

[13] B.J. West and M.F. Shlesinger, Int. J. Modern Phys. B **3**, 795 (1989).

[14] R.H. Kraichnan and D. Montgomery, Rep. Progress in Physics **43**, 547 (1980).

[15] H.J. Herrmann, private communication (1993).

[16] A. Mehta and S.F. Edwards, Physica A **157**, 1091 (1989).

[17] S.F. Edwards and R.B.S. Oakeshott, Physica A **157**, 1080 (1989); S.F. Edwards, J. Stat. Phys. **62**, 889 (1991).

[18] H.S. Seung, H. Sompolinsky and N. Tishby, Phys. Rev. A **45**, 6056 (1992); T.L.H. Watkin, A. Rau and M. Biehl, Rev. Mod. Phys. **65**, 499 (1993).

[19] A. Crisanti, M. Falcioni and A. Vulpiani, J. Phys. A **26**, 3441 (1993).

[20] H. Szu, Proceedings Conference Snowdown, Colorado 1986, pages 420-425.

[21] J. von Neumann and O. Morgenstern, "Theory of Games and Economic Behavior" (Princeton University Press, Princeton, 1947).

[22] J. Dow and S.R.C. Werlang, Econometrica **60**, 197 (1992).

[23] M.H. Simonsen and S.R.C. Werlang, R. Econometria **11**, 1 (1991).

[24] J. Dow and S.R.C. Werlang, European Economic Review **36**, 631 (1992).

[25] I. Csiszar, "Information measures: A critical survey", in "Transactions of the Seventh Prague Conference on Information Theory, Statistical Decision Functions, Random Processes" and the European Meeting of Statisticians, 1974 (Reidel, Dordrecht, 1978) page 73.

[26] A. Renyi, "On measures of entropy and information", in Proc. Fourth Berkeley Symposium on Mathematical Statistics and Probability, $N^{\underline{o}}$ 1, 547 (1960), and "Probability Theory" (North-Holland, Amsterdam, 1970).

[27] E.C. Pielou, "Mathematical Ecology" (Wiley, New York, 1975; revised edition of "An Introduction to Mathematical Ecology, 1969).

[28] P.N. Rathie, "Generalized entropies in coding theory", Metrika **18**, 216 (1972); B.D. Sharma, "On amount of information of type-β and other measures", Metrika **19**, 1 (1972).

[29] P. Nath, "Entropy, inaccuracy and information", Metrika **13**, 136 (1968).

[30] J. Havrda and F. Charvat, Kybernetica **3**, 30 (1967); Z. Daroczy, Information and Control **16**, 36 (1970).

[31] B.D. Sharma and D.P. Mittal,"New non-additive measures of entropy for discrete probability distributions", J. Math. Sci. **10**, 28 (1975).

[32] S. Arimoto, "Information theoretical considerations on estimation problems", Inform. and Control **19**, 181 (1971).

[33] J.C.A. van der Lubbe, Y. Boxma and D.E. Boekee, Inform. Sci. **32**, 187 (1984); D.E. Boekee and J.C.A. van der Lubbe, Inf. and Control **45**, 136 (1980).

[34] J. Aczel, B. Forte and C.T. Ng, Advances in Applied Probability **6**, 131 (1974); A. Wehrl, Rev. Mod. Phys. **50**, 221 (1987); G. Jumarie, Cybernetics and Systems **19**, 169 and 311 (1988); K.E. Eriksson, K. Lindgren and B.A. Mansson, "Structure, content, complexity, organization" (World Scientific, Singapore, 1988); R.M. Losee Jr., "The Science of Information-Measurements and Applications" (Academic Press, San Diego, 1990).

[35] C. Tsallis, J. Stat. Phys. **52**, 479 (1988); in fact, the idea of this generalization first appeared to me in 1985 during the 1^{st} Workshop of Statistical Mechanics (Mexico, 2-13 September 1985), but it was implemented only in 1987, after the encouraging and stimulating comments (which I now gratefully acknowledge) of H.J. Herrmann and E.M.F. Curado during the Workshop on Cellular Automata (Maceió, 24-28 August 1987).

[36] A.M. Mariz, Phys. Lett. A **165**, 409 (1992).

[37] J.D. Ramshaw, Phys. Lett. A **175**, 169 (1993).

[38] J.D. Ramshaw, Phys. Lett. A **175**, 171 (1993).

[39] A. Einstein, Ann. Phys. **33**, 1275 (1910) and "Investigations on the Theory of Brownian Movement" (Methuen and Co, London, 1926); H.B. Callen, "Thermodynamics" (John Wiley, New York, 1960) Chapter 15, page 269; L. Landau and L. Lifchitz, "Physique Statistique", in "Physique Théorique" (MIR, Moscow, 1967) Vol. 5, Chapter 12, page 414; L.E. Reichl, "A modern course in Statistical Physics" (University of Texas Press, Austin, 1980) page 309.

[40] A. Chame and E. Mello, J. Phys. A **27**, 3663 (1994).

[41] E.M.F. Curado and C. Tsallis, J. Phys. A **24**, L69 (1991); Corrigenda: J. Phys. A **24**, 3187 (1991) and **25**, 1019 (1992).

[42] A.R. Plastino and A. Plastino, Phys. Lett. A **177**, 177 (1993).

[43] A.R. Plastino and A. Plastino, Physica A **202**, 438 (1994).

[44] F. Buyukkiliç and D. Demirhan, Phys. Lett. A **181**, 24 (1993).

[45] D.A. Stariolo, Phys. Lett. A **185**, 262 (1994).

[46] E.F. Sarmento, "Generalization of Singe-Site Callen's Identity within Tsallis Statistics, preprint (1994).

[47] A. Plastino and C. Tsallis, J. Phys. A **26**, L893 (1993).

[48] C. Tsallis, "Generalized Entropy-Based Criterion for Consistent Nonparametric Testing", preprint (1994).

[49] C. Tsallis and F.C. Sá Barreto, "Generalization of Planck Radiation Law: Possible Testing of the Nature of Space-Time", preprint (1994).

[50] C. Tsallis and D.A. Stariolo, "Generalized Simulated Annealing", preprint (1994).

[51] N. Ito and C. Tsallis, N. Cim. D **11**, 907 (1989).

[52] E.P. da Silva, C. Tsallis and E.M.F. Curado, Physica A **199**, 137 (1993); Erratum: Physica A **203**, 160 (1994).

[53] R.F.S. Andrade, Physica A **175**, 285 (1991).

[54] R.F.S. Andrade, Physica A **203**, 486 (1994).

[55] S.A. Cannas and C. Tsallis, "Self-Dual Planar Lattice Ising Ferromagnet within Generalized Statistics", preprint (1994).

[56] L.R. da Silva, "Duality-Based Approximations for the Critical Point of the Square-Lattice Ising Ferromagnet within Tsallis Statistics", preprint (1994).

[57] F. Nobre and C. Tsallis, "Localized-Spins Ideal Paramagnet within Nonextensive Statistical Mechanics", preprint (1994); see also F. Nobre and C. Tsallis, "Infinite-range Ising Ferromagnet: Thermodynamic Limit within Generalized Statistical Mechanics", Physica A (1994), in press.

[58] A.R. Plastino and A. Plastino, Phys. Lett. A **174**, 384 (1993).

[59] D.A. Alemany and D.H. Zanette, Phys. Rev. E **49**, 936 (1994).

[60] C. Tsallis, in "New Trends in Magnetism, Magnetic Materials and Their Applications", eds. J.L. Morán-López and J.M. Sánchez (Plenum Press, New York, 1994), 451.

[61] D.N. Zubarev, "Nonequilibrium Statistical Thermodynamics" (Plenum, New York, 1974) Chapter IV, 439-448.

[62] "Frontiers of finance", The Economist (October 9^{th} 1993) pages 6-20.