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On non-minimal $\mathcal{N} = 4$ supermultiplets in $1D$ and their associated σ-models

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Abstract

We construct the non-minimal linear representations of the $\mathcal{N} = 4$ Extended Supersymmetry in one-dimension. They act on 8 bosonic and 8 fermionic fields. Inequivalent representations are specified by the mass-dimension of the fields and the connectivity of the associated graphs. The oxidation to minimal $\mathcal{N} = 5$ linear representations is given. Two types of $\mathcal{N} = 4$ σ-models based on non-minimal representations are obtained: the resulting off-shell actions are either manifestly invariant or depend on a constrained prepotential. The connectivity properties of the graphs play a decisive role in discriminating inequivalent actions. These results find application in partial breaking of supersymmetric theories.

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1 Introduction

In this work we present a systematical investigation of the inequivalent non-minimal linear supermultiplets carrying a representation of the one-dimensional $\mathcal{N} = 4$-Extended Superalgebra, as well as of their associated $\mathcal{N} = 4$-invariant, $1D$ σ-models.

The $1D$ $\mathcal{N}$-Extended Superalgebra, with $\mathcal{N}$ odd generators Q_I ($I = 1, 2, \ldots, \mathcal{N}$) and a single even generator H satisfying the (anti)-commutation relations

$$\begin{aligned} \{Q_I, Q_J\} &= \delta_{IJ} H, \\ [H, Q_I] &= 0, \end{aligned} \tag{1}$$

is the superalgebra underlying the Supersymmetric Quantum Mechanics [1]. In recent years the structure of its linear representations has been unveiled by a series of works (upon which the present investigation is based) [2]–[13], that we will briefly comment.

The linear representations under considerations (supermultiplets) contain a finite, equal number of bosonic and fermionic fields depending on a single coordinate (the time). The operators Q_I and H act as differential operators. The linear representations are characterized by a series of properties which, for sake of consistency, are reviewed in Appendix **A**. They include the *grading* of the fields (in physical terms, their *mass-dimension*), the *length* and the *field content* of the supermultiplets [2, 4], the *dressing transformations* [14, 15, 2], the association with graphs [3, 5, 6, 9, 10, 11], the *connectivity symbol* [9, 10] characterizing inequivalent representation with a given field content (the notion of inequivalent representations has been discussed in [5, 6]), the *mirror duality* [8], etc.

The *minimal* linear representations (also called *irreducible supermultiplets*) are given by the minimal number n_{min} of bosonic (fermionic) fields for a given value of $\mathcal{N}$. The value n_{min} is given [2] by the formula

$$\begin{aligned} \mathcal{N} &= 8l + m, \\ n_{min} &= 2^{4l} G(m), \end{aligned} \tag{2}$$

where $l = 0, 1, 2, \ldots$ and $m = 1, 2, 3, 4, 5, 6, 7, 8$.

$G(m)$ appearing in (2) is the Radon-Hurwitz function

m	1	2	3	4	5	6	7	8
$G(m)$	1	2	4	4	8	8	8	8

(3)

Non-minimal linear representations have been discussed in [4, 12, 13, 16, 17]. The maximal *finite* number n_{max} of bosonic (fermionic) fields entering a non-minimal representation is given by [12, 13]*

$$n_{max} = 2^{\mathcal{N}-1}. \tag{4}$$

*this result applies to "adinkraic" multiplets, according to the terminology of [12, 13], i.e. to multiplets which can be expressed in graphical form (see the discussion in Appendix **A**). In this paper we restrict our attention to this class of multiplets.

An important subclass of non-minimal representations is given by the reducible but indecomposable supermultiplets (see [4]). For this subclass the associated graph (see Appendix **A**) is connected (there is a path connecting any two given vertices).

For $\mathcal{N} = 4$ we have that $n_{min} = 4$ and $n_{max} = 8$. As a consequence, there are only two subclasses of non-minimal $\mathcal{N} = 4$ representations. Besides the irreducible but indecomposable subclass, we have the subclass of fully reducible representations given by the direct sum of two minimal $\mathcal{N} = 4$ representations (the associated graph is disconnected and given by two separate minimal $\mathcal{N} = 4$ graphs).

In Section **2** we present the inequivalent non-minimal $\mathcal{N} = 4$ supermultiplets, characterized by their admissible field contents and connectivity symbols (in subsection **2.1** we comment about the difference between the overall connectedness of a graph and its associated connectivity symbol). We employ the techniques introduced in [9, 10] and discussed in detail in those works. Therefore, here we limit ourselves to present the main results.

The investigation of the properties of the non-minimal $\mathcal{N} = 4$ supermultiplets continues in Section **3**, where we present the so-called "oxidation diagrams" connecting the non-minimal $\mathcal{N} = 4$ representations with the minimal $\mathcal{N} = 5$ ones.[†] This Section employs the techniques introduced in [10]. The presented results answer the following question: which minimal $\mathcal{N} = 5$ supermultiplets can be obtained by adding an extra supersymmetry operator to a given non-minimal $\mathcal{N} = 4$ supermultiplet in such a way to guarantee an overall $\mathcal{N} = 5$ (1) superalgebra. We recall that, due to (2), the minimal $\mathcal{N} = 5$ representations contain the same number of fields (8 bosons and 8 fermions) as the non-minimal $\mathcal{N} = 4$ representations.

In Section **4** we present the most general unitary group commuting with the $\mathcal{N} = 4$ supersymmetry operators acting on a non-minimal supermultiplet. This Section applies the methods presented in [17] where, for the minimal supermultiplets, the supersymmetric extension of the Schur's lemma was constructed.

The next part of the paper deals with the construction of off-shell, $\mathcal{N} = 4$-invariant supersymmetric σ-models associated with each given non-minimal linear supermultiplet. $1D$ supersymmetric σ-models were first discussed in [19, 20]; minimal $\mathcal{N} = 4$ σ-models were constructed in [21]-[26], while minimal $\mathcal{N} = 8$-invariant σ-models were investigated in [27]. There are also supersymmetric σ-models based on non-linear realizations of the supersymmetry that we are not discussing here (for a partial list of references one can consult [17]).

The fields $x_j(t)$ ($j = 1, \ldots, k$) of lower-dimension in a supermultiplet can be assumed [4] to be bosonic and have 0-mass dimension. They are physically interpreted [7, 28, 17] as the target-coordinates of the associated σ-model. An $\mathcal{N} = 4$-invariant off-shell action $\mathcal{S}$, with the correct dimension of a kinetic term, is obtained [4, 7, 28, 17] through

$$\mathcal{S} \quad = \quad \int dt\mathcal{L} = \frac{1}{m}\int dt Q_1 Q_2 Q_3 Q_4 F(\vec{x}) \qquad (5)$$

(the dimensional parameter m will be normalized to 1 in the following), where the supersymmetry operators Q_i's act as graded derivatives and F is the prepotential. By

[†] We postpone to the Conclusions the discussion about the meaning of the term *oxidation* and of the physical importance of the so-called *oxidation program*, see [18, 7, 17].

construction, the action $\mathcal{S}$ is manifestly $\mathcal{N}=4$-invariant no matter which is the choice of F (unconstrained prepotential).

In the case of a fully reducible supermultiplet given by the direct sum of two $\mathcal{N}=4$ irreducible supermultiplets (whose 0-mass dimension fields are denoted as $\vec{x}$, $\vec{y}$, respectively) we have that interacting terms involving the fields belonging to the irreducible supermultiplets arise provided that

$$F(\vec{x},\vec{y}) \quad \neq \quad A(\vec{x})+B(\vec{y}). \tag{6}$$

As a result, non-trivial interacting Lagrangians can be produced even from fully reducible representations (which are trivial, from the representation theory point of view), therefore justifying the attention we have to pay to them.

The (5) manifest $\mathcal{N}=4$ construction has been discussed in [28]. In several cases, however, this construction does not produce the most general $\mathcal{N}=4$ invariant action. The resulting Lagrangian can be of first order and furthermore, in the presence of fermionic sources[‡], not all fields belonging to the given supermultiplet enter the Lagrangian.

On the other hand, even in those cases, the existence of a second-order Lagrangian involving all fields of the supermultiplet is known [4, 29]. To systematically construct them a novel approach is here presented (it will be referred as "Construction II", while (5) will be referred as "Construction I"). Construction II is outlined as follows. For a reducible length-3 $\mathcal{N}=4$ representation of field content $(k,8,8-k)$, we consider at first (see the definitions in Appendix **A**) its associated root supermultiplet of length-2 (in a different context, the importance of invariant actions induced by the root supermultiplets has also been discussed in [30, 31, 32]). The root supermultiplet contains 8 bosonic fields x_i (the target coordinates) and 8 fermionic fields ψ_i. A Lagrangian $\mathcal{L}$ for the root supermultiplet is at first constructed by setting, as in (5),

$$\mathcal{L} \quad = \quad Q_1Q_2Q_3Q_4W(\vec{x}), \tag{7}$$

where the prepotential $W(\vec{x})$ is now function of the 8 bosonic fields.

An equivalent, up to a total derivative, Lagrangian $\overline{\mathcal{L}}$ functionally depends on the fields and their first-order time-derivatives alone. Therefore

$$\overline{\mathcal{L}} \quad \equiv \quad \overline{\mathcal{L}}(x_i,\dot{x}_i,\psi_i,\dot{\psi}_i). \tag{8}$$

The next step consists in constraining $\overline{\mathcal{L}}$ such that, for $j=k+1,\ldots,8$, we have

$$\frac{\partial\overline{\mathcal{L}}}{\partial x_j} \quad = \quad 0, \tag{9}$$

eliminating its dependence on x_j's. This condition allows us to regard, according to the dressing procedure, the $\dot{x}_j$'s no longer as derivative fields, but as the auxiliary fields g_j of mass-dimension 1 entering the $(k,8,8-k)$ supermultiplet. We can therefore set

$$\begin{array}{rcl} g_j &=& \dot{x}_j \\ \overline{\mathcal{L}} &\equiv& \overline{\mathcal{L}}(x_l,\dot{x}_l,\psi_i,\dot{\psi}_i,g_j), \end{array} \tag{10}$$

[‡]A fermionic source [5, 6, 9] is a fermionic $\frac{1}{2}$-mass dimension field which, in the graphical presentation of the supermultiplet, does not possess edges connecting it to the 0-mass dimension fields (the bosonic target coordinates). For $\mathcal{N}=4$, the connectivity symbol of a graph with r fermionic sources is expressed as $r_4+\ldots$, see Appendix **A**.

($l = 1, \ldots, k$, while $i = 1, \ldots, 8$ and $j = k+1, \ldots, 8$).

Setting (10) is not something innocuous. One has in fact to guarantee that the resulting action, after the "renaming" of the fields, is still $\mathcal{N} = 4$-invariant. Together with (9), this requirement produces a constraint on the prepotential $W(x_l)$.

It is worth stressing the fact that while "construction I" is analogous to the construction of D-terms in four-dimensional supersymmetric theories expressed in superspace, "construction II" is inherently one-dimensional. It uses in fact a specific one-dimensional property which has no analog in higher dimension, namely that supersymmetric multiplets can be obtained by dressing their associated root multiplets (see [14, 15, 2]).

In the following we will compare the invariant actions arising from the constructions I and II and discuss the constraints on the prepotential. Depending on which supersymmetric multiplets constructions I and II are applied to, construction II can coincide with construction I, can give a restriction of the construction I result or can produce an altogether different result (like a second-order Lagrangian instead of a first-order Lagrangian).

To derive the off-shell invariant actions we implemented a special package for *Maple 11*. For convenience we had to use different (but equivalent) conventions for the presentations of the non-minimal supermultiplets with respect to the explicit construction given in Appendix **B**. We present in Section **5** and **6** some selected cases which exemplify the general picture. In Section **5** we discuss the construction I. In Section **6** we discuss the construction II. It will be manifest that inequivalent $\mathcal{N} = 4$-invariant actions are obtained for supermultiplets presenting the same field content, but differing in connectivity symbol. In the Conclusions we make comments on the obtained results.

The paper is complemented, for sake of clarity, with the presentation of a few selected (unoriented, color-blind) graphs associated to non-minimal supermultiplets.

2 Non-minimal $\mathcal{N} = 4$ supermultiplets

We present here the complete list of inequivalent $\mathcal{N} = 4$ non-minimal supermultiplets associated to connected graphs (therefore providing reducible but indecomposable representations).

Up to equivalence, there exists a unique length-2 $\mathcal{N} = 4$ root supermultiplet, based on a connected graph, of field content $(8, 8)$ (it is explicitly given in Appendix **B**).

The inequivalent non-minimal supermultiplets of length-3 are given by the table below. One should note that, for field content $(k, 8, 8-k)$ with $k = 2, 3, 4, 5, 6$, inequivalent supermultiplets are discriminated by their respective connectivity symbol (see Appendix **A**) and are named according to the given label. The table further reports the dually related supermultiplet (see Appendix **A**). We have

field content:	*label:*	*connectivity symbol:*	*dual supermultiplet:*
$(1,8,7)$		4_4+4_3	$(7,8,1)$
$(2,8,6)$	A	$2_4+4_3+2_2$	$(6,8,2)_A$
	B	8_3	$(6,8,2)_B$
$(3,8,5)$	A	$1_4+3_3+3_2+1_1$	$(5,8,3)_A$
	B	4_3+4_2	$(5,8,3)_B$
$(4,8,4)$	A	$1_4+6_2+1_0$	self-dual
	B	4_3+4_1	self-dual
	C	$2_3+4_2+2_1$	self-dual
	D	8_2	self-dual
$(5,8,3)$	A	$1_3+3_2+3_1+1_0$	$(3,8,5)_A$
	B	4_2+4_1	$(3,8,5)_B$
$(6,8,2)$	A	$2_2+4_1+2_0$	$(2,8,6)_A$
	B	8_1	$(2,8,6)_B$
$(7,8,1)$		4_1+4_0	$(1,8,7)$

(11)

The explicit construction of a representative supermultiplet in each given inequivalent class is given in the Appendix **B**.

We further complete the list by presenting the non-minimal (reducible but indecomposable) $\mathcal{N}=4$ representations with length $l>3$. They are uniquely characterized by their field content. They are given by

$$(1,4,6,4,1),\quad (1,7,7,1),\quad (1,4,7,4)\leftrightarrow(4,7,4,1),\quad (1,5,7,3)\leftrightarrow(3,7,5,1),$$
$$(1,6,7,2)\leftrightarrow(2,7,6,1),\quad (2,6,6,2),\tag{12}$$

with dually related supermultiplets connected by arrows.

2.1 Connectivity symbol of the $\mathcal{N}=4$ fully reducible representations

It is interesting to compare the connectivity symbol of the non-minimal (reducible, but indecomposable) linear representations of the one-dimensional $\mathcal{N}=4$ Extended Superalgebra, with the connectivity symbol of the fully reducible $\mathcal{N}=4$ representations involving 8 bosonic and 8 fermionic fields. The representations are otained by direct sum of two $(k,4,4-k)$ (for $k=0,1,2,3,4$) $\mathcal{N}=4$ irreducible (minimal) representations.

For length-3 fully reducible representations we have the table[§]

[§]In order to distinguish the fully reducible representations of field content $(8,8)\equiv(4,4,0)\oplus(0,4,4)$, $(7,8,1)$, $(1,8,7)$ from their non-minimal, reducible but indecomposable counterparts, a label is introduced (respectively, either "FR" or "red"). The fully reducible representations will be denoted as $(8,8)_{FR}$, $(7,8,1)_{FR}$, $(1,8,7)_{FR}$; the reducible but indecomposable supermultiplets entering table (11) will be denoted as $(8,8)_{red}$, $(7,8,1)_{red}$, $(1,8,7)_{red}$.

field content:	*label:*	*decomposition:*	*connectivity symbol:*
$(1,8,7)$	FR	$(1,4,3)\oplus(0,4,4)$	4_4+4_3
$(2,8,6)$	a	$(2,4,2)\oplus(0,4,4)$	4_4+4_2
	b	$(1,4,3)\oplus(1,4,3)$	8_3
$(3,8,5)$	a	$(3,4,1)\oplus(0,4,4)$	4_4+4_1
	b	$(2,4,2)\oplus(1,4,3)$	4_3+4_2
$(4,8,4)$	a	$(4,4,0)\oplus(0,4,4)$	4_4+4_0
	b	$(3,4,1)\oplus(1,4,3)$	4_3+4_1
	c	$(2,4,2)\oplus(2,4,2)$	8_2
$(5,8,3)$	a	$(4,4,0)\oplus(1,4,3)$	4_4+4_3
	b	$(3,4,1)\oplus(2,4,2)$	4_2+4_1
$(6,8,2)$	a	$(4,4,0)\oplus(2,4,2)$	4_2+4_0
	b	$(3,4,1)\oplus(3,4,1)$	8_1
$(7,8,1)$	FR	$(4,4,0)\oplus(3,4,1)$	4_1+4_0

(13)

A comment is in order. The reducible but indecomposable representations discussed before and the fully reducible representations here listed are inequivalent. Indeed, their associated graphs are connected in the first case, disconnected in the second case. The so-called *connectivity symbol* introduced in [9] allows to discriminate the inequivalent representations inside each broad class of fully reducible or reducible but indecomposable representations. On the other hand, in several cases the connectivity symbol *alone* cannot discriminate between a fully reducible and its reducible but indecomposable counterpart. For instance, the fully reducible representation $(2,8,6)_b$ possesses the same connectivity symbol as the reducible but indecomposable representation $(2,8,6)_B$. Once more, the extra information allowing to discriminate the two is the overall connectedness of the associated graph (either connected or disconnected).

3 The $(\mathcal{N}=4)\Rightarrow(\mathcal{N}=5)$ oxidation

Non-minimal $\mathcal{N}=4$ representations (both the reducible but indecomposable and the fully reducible ones acting on 8 bosonic and 8 fermionic fields) can be "oxidized" to $\mathcal{N}=5$ minimal representations, adding a 5^{th} supersymmetry transformation compatible with the 4 previous supersymmetry transformations.

The minimal $\mathcal{N}=5$ representations have been classified in [9]. There are inequivalent length-3 $\mathcal{N}=5$ representations of field content $(k,8,8-k)$, for $k=2,3,4,5,6$, which are discriminated by their connectivity symbol.

We present a series of tables specifying which minimal $\mathcal{N}=5$ representation (in a column) results from the oxidation of a non-minimal $\mathcal{N}=4$ representation (in a row). A positive answer is marked by an "X". The representations are expressed in terms of their connectivity symbol. The $\mathcal{N}=4$ reducible but indecomposable representations (associated with connected graphs) appear in the upper rows; the $\mathcal{N}=4$ fully reducible representations (associated with disconnected graphs) appear in the lower rows.

We get the following oxidation tables, for each given field content $(k, 8, 8-k)$ with $k = 2, 3, 4, 5, 6$.

For $(2, 8, 6)$ we have

	$(\mathcal{N}=4) \Rightarrow (\mathcal{N}=5)$:	$2_5+2_4+4_3$	6_4+2_3
Connected:	$2_4+4_3+2_2$	X	X
	8_3		X
Disconnected:	4_4+4_2	X	
	8_3		X

(14)

For $(3, 8, 5)$ we have

	$(\mathcal{N}=4) \Rightarrow (\mathcal{N}=5)$:	$1_5+3_4+4_2$	$2_4+5_3+1_2$
Connected:	$1_4+3_3+3_2+1_1$	X	X
	4_3+4_2		X
Disconnected:	4_4+4_1	X	
	4_3+4_2		X

(15)

For $(4, 8, 4)$ we have

	$(\mathcal{N}=4) \Rightarrow (\mathcal{N}=5)$:	4_4+4_1	$1_4+3_3+3_2+1_1$	4_3+4_2
Connected:	$1_4+6_2+1_0$		X	
	4_3+4_1	X		X
	$2_3+4_2+2_1$		X	X
	8_2			X
Disconnected:	4_4+4_0	X		
	4_3+4_1		X	
	8_2			X

(16)

For $(5, 8, 3)$ we have

	$(\mathcal{N}=4) \Rightarrow (\mathcal{N}=5)$:	$4_3+3_1+1_0$	$1_3+5_2+2_1$
Connected:	$1_5+3_2+3_1+1_0$	X	X
	4_2+4_1		X
Disconnected:	4_3+4_0	X	
	4_2+4_1		X

(17)

For $(6, 8, 2)$ we have

	$(\mathcal{N}=4) \Rightarrow (\mathcal{N}=5)$:	$4_2+2_1+2_0$	2_2+6_1
Connected:	$2_2+4_1+2_0$	X	X
	8_1		X
Disconnected:	4_2+4_0	X	
	8_1		X

(18)

Both the reducible but indecomposable and the fully reducible non-minimal $\mathcal{N} = 4$ representations of field content $(8, 8)$, $(1, 8, 7)$ and $(7, 8, 1)$ are oxidized to the minimal

$\mathcal{N} = 5$ representations which are uniquely specified by the corresponding field content $(8, 8)$, $(1, 8, 7)$ and $(7, 8, 1)$.

Combining these results with the results presented in [10] we get that all non-minimal length-3 representations are oxidized to the maximal number $\mathcal{N}_{max} = 8$ of extended supersymmetries.

The maximal number $\mathcal{N}_{max}$ of supersymmetries operators (oxidized supersymmetries) acting on non-minimal $\mathcal{N} = 4$ representations of length $l = 4, 5$ is given by the table

field content:	$\mathcal{N}_{max}$:
$(1, 7, 7, 1)$	7
$(2, 6, 6, 2)$	6
$(1, 6, 7, 2) \leftrightarrow (2, 7, 6, 1)$	6
$(1, 5, 7, 3) \leftrightarrow (3, 7, 5, 1)$	5
$(1, 4, 7, 4) \leftrightarrow (4, 7, 4, 1)$	4
$(1, 4, 6, 4, 1)$	4

(19)

4 Non-minimal $\mathcal{N} = 4$ supermultiplets and invariant groups

The Schur's lemma states that the irreducible representations of the Clifford algebras are of three types (real, almost complex or quaternionic), according to the most general matrix commuting with all Clifford generators (see [33]). Since minimal root supermultiplets (see [2]) are uniquely determined by their associated Euclidean Clifford algebra, they inherit the corresponding Schur's property. The determination of the Schur's property of higher-length supermultiplets requires the compatibility with the dressing (this implies that the most general commuting matrix of the root supermultiplet is restricted by the further requirement of commuting with the dressing matrix D discussed in Appendix **A**). The analysis for the minimal supermultiplets has been presented in [17]. We extend here the investigation of [17] to the case of the non-minimal $\mathcal{N} = 4$ supermultiplets.

We are looking for the most general real-valued antisymmetric matrix Σ of the form $\Sigma = \begin{pmatrix} \Sigma^{up} & 0 \\ 0 & \Sigma^{down} \end{pmatrix}$, where Σ^{up} (Σ^{down}) is an 8×8 matrix acting on the bosonic (respectively, fermionic) fields, commuting with the 4 non-minimal supersymmetry operators $\widehat{Q}_I$, i.e.

$$[\widehat{Q}_I, \Sigma] \;=\; 0, \quad for \quad I = 1, 2, 3, 4. \tag{20}$$

For the non-minimal, reducible but indecomposable $(8, 8)$ root supermultiplet, the most general matrix Σ can be written as $\Sigma = \sum_{i=1}^{i=6} \lambda_i \Sigma_i$, where the matrices $\Sigma_i = {\Sigma_i}^{up} \oplus {\Sigma_i}^{down}$ are the generators of the $su(2) \oplus su(2)$ Lie algebra.

Without loss of generality we can work with the conventions of Appendix **B**. We have, explicitly,

$$\begin{array}{lll}\Sigma_1{}^{up} = \tau_2\otimes\tau_2\otimes\tau_A, & \Sigma_2{}^{up} = \mathbf{1}_2\otimes\tau_A\otimes\mathbf{1}_2, & \Sigma_3{}^{up} = \tau_2\otimes\tau_1\otimes\tau_A,\\ \Sigma_4{}^{up} = \tau_A\otimes\tau_1\otimes\mathbf{1}_2, & \Sigma_5{}^{up} = \tau_1\otimes\mathbf{1}_2\otimes\tau_A, & \Sigma_6{}^{up} = \tau_A\otimes\tau_2\otimes\mathbf{1}_2,\end{array} \tag{21}$$

while $\Sigma_i{}^{down} = \Sigma_i{}^{up}$ for $i = 1, 2, 3$ and $\Sigma_i{}^{down} = -\Sigma_i{}^{up}$ for $i = 4, 5, 6$.

The unitary invariant groups, commuting with the 4 supersymmetry operators of the length-3, reducible but indecomposable, non-minimal supermultiplets are given by the table

supermultiplet:	*commuting group:*
$(2,8,6)_A$	$U(1)$
$(2,8,6)_B$	$\mathbf{1}$
$(4,8,4)_A$	$\mathbf{1}$
$(4,8,4)_B$	$SU(2)$
$(4,8,4)_C$	$\mathbf{1}$
$(4,8,4)_D$	$U(1)\otimes U(1)$
$(6,8,2)_A$	$U(1)$
$(6,8,2)_B$	$\mathbf{1}$

(22)

In the remaining cases, for field content $(k, 8, 8-k)$ with k odd, the most general unitary group is just the identity group $\mathbf{1}$.

A similar table can be produced for length $l = 2, 3$ $\mathcal{N} = 4$ non-minimal, fully reducible supermultiplets. We have

supermultiplet:	*commuting group:*
$(8,8)_{FR}$	$SU(2)\otimes SU(2)$
$(1,8,7)_{FR}$	$SU(2)\otimes\mathbf{1}_2$
$(2,8,6)_a$	$SU(2)\otimes U(1)$
$(2,8,6)_b$	$\mathbf{1}_2\otimes\mathbf{1}_2$
$(3,8,5)_a$	$SU(2)\otimes\mathbf{1}_2$
$(3,8,5)_b$	$U(1)\otimes\mathbf{1}_2$
$(4,8,4)_a$	$SU(2)\otimes SU(2)$
$(4,8,4)_b$	$\mathbf{1}_2\otimes\mathbf{1}_2$
$(4,8,4)_c$	$U(1)\otimes U(1)$
$(5,8,3)_a$	$SU(2)\otimes\mathbf{1}_2$
$(5,8,3)_b$	$U(1)\otimes\mathbf{1}_2$
$(6,8,2)_a$	$SU(2)\otimes U(1)$
$(6,8,2)_b$	$\mathbf{1}_2\otimes\mathbf{1}_2$
$(7,8,1)_{FR}$	$SU(2)\otimes\mathbf{1}_2$

(23)

5 Manifestly $\mathcal{N} = 4$ σ-models for non-minimal supermultiplets

For the following length-3, non-minimal, reducible but indecomposable supermultiplets, the Construction I (see (5)) of the $\mathcal{N} = 4$ off-shell invariant actions produces a first-order

Lagrangian:

$$(1,8,7)_{red},(2,8,6)_A,(3,8,5)_A,(4,8,4)_A,(4,8,4)_B. \tag{24}$$

With the only exception of $(4,8,4)_B$, these are the supermultiplets admitting, see the footnote in the Introduction, fermionic sources.

In the remaining cases, namely for

$$\begin{array}{c}(2,8,6)_B,(3,8,5)_B,(4,8,4)_C,(4,8,4)_D,(5,8,3)_A,\\ (5,8,3)_B,(6,8,2)_A,(6,8,2)_B,(7,8,1)_{red},\end{array} \tag{25}$$

the Construction I produces second-order Lagrangians.

For each supermultiplet entering (25), the Constructions I and II (detailed in the Introduction) produce, up to a total derivative, the same Lagrangian. Therefore, the corresponding actions are (as a consequence of Construction I) manifestly $\mathcal{N}=4$-invariant and depend on an unconstrained prepotential. We present, for a few selected cases, the explicit computation of the Lagrangian. We write down the invariant Lagrangian, up to a total derivative, for $(4,8,4)_C$ (with connectivity symbol $2_3+4_2+2_1$) and $(2,8,6)_B$. We compare the latter result with the Lagrangian obtained from Construction I applied to the fully reducible supermultiplet $(2,8,6)_b$ (see (13)), characterized by the same connectivity symbol, 8_3, as $(2,8,6)_B$.

The component fields (the υ's, barred or otherwise, denote the target coordinates, the λ's the fermionic fields and the g's the auxiliary fields) are respectively given by $(\upsilon_1,\upsilon_2,\upsilon_3,\bar{\upsilon}_1;\lambda_0,\lambda_i,\bar{\lambda}_0,\bar{\lambda}_i;g_0,\bar{g}_0,\bar{g}_2,\bar{g}_3)$, with $i=1,2,3$, for $(4,8,4)_C$ and by
$(\upsilon_0,\bar{\upsilon}_0;\lambda_0,\lambda_i,\bar{\lambda}_0,\bar{\lambda}_i;g_i,\bar{g}_i)$, with $i=1,2,3$, for both $(2,8,6)_B$ and $(2,8,6)_b$.
The supersymmetry transformations are here explicitly obtained by dressing the $\mathcal{N}=8$ root supermultiplet expressed in terms of the octonionic structure constants C_{ijk}, see [34, 4]

$$\begin{aligned}\widehat{Q}_i(\upsilon_0,\upsilon_j;\lambda_0,\lambda_j) &= (\lambda_i,-\delta_{ij}\lambda_0-C_{ijk}\lambda_k;-\dot{\upsilon}_i,\delta_{ij}\dot{\upsilon}_0+C_{ijk}\dot{\upsilon}_k),\\ \widehat{Q}_8(\upsilon_0,\upsilon_j;\lambda_0,\lambda_j) &= (\lambda_0,\lambda_j;\dot{\upsilon}_0,\dot{\upsilon}_j),\end{aligned} \tag{26}$$

where $i=1,..,7$ and the fields have been renamed in such a way to respect the $\mathcal{N}=4$ quaternionic subalgebra. For $(2,8,6)_B$ and $(2,8,6)_b$ the associated graphs are given in, respectively, figure 4 and 6.
We have the following results (the Einstein convention over repeated indexes is understood):

For $(4,8,4)_C$ the Lagrangian is

$$\begin{aligned}\mathcal{L} \;=\; &\Phi(\dot{\upsilon}_1^2+\dot{\upsilon}_2^2+\dot{\upsilon}_3^2+\dot{\bar{\upsilon}}_1^2+g_0^2+\bar{g}_0^2+\bar{g}_2^2+\bar{g}_3^2+\dot{\lambda}_0\lambda_0+\dot{\bar{\lambda}}_0\bar{\lambda}_0+\\ &\dot{\lambda}_1\lambda_1+\dot{\lambda}_2\lambda_2+\dot{\lambda}_3\lambda_3+\dot{\bar{\lambda}}_1\bar{\lambda}_1+\dot{\bar{\lambda}}_2\bar{\lambda}_2+\dot{\bar{\lambda}}_3\bar{\lambda}_3)+\\ &\Phi_1[\dot{\upsilon}_3(\bar{\lambda}_0\bar{\lambda}_2+\lambda_2\lambda_0+\bar{\lambda}_1\bar{\lambda}_3+\lambda_1\lambda_3)-\dot{\upsilon}_2(\bar{\lambda}_0\bar{\lambda}_3+\lambda_3\lambda_0-\bar{\lambda}_1\bar{\lambda}_2-\lambda_1\lambda_2)+\\ &+g_0(\bar{\lambda}_2\bar{\lambda}_3-\lambda_2\lambda_3-\bar{\lambda}_0\bar{\lambda}_1+\lambda_1\lambda_0)+\bar{g}_0(\lambda_0\bar{\lambda}_1+\lambda_1\bar{\lambda}_0+\lambda_2\bar{\lambda}_3-\lambda_3\bar{\lambda}_2)+\\ &\bar{g}_2(\lambda_2\bar{\lambda}_1+\lambda_1\bar{\lambda}_2-\lambda_0\bar{\lambda}_3+\lambda_3\bar{\lambda}_0)+\bar{g}_3(\lambda_0\bar{\lambda}_2-\lambda_2\bar{\lambda}_0+\lambda_3\bar{\lambda}_1+\lambda_1\bar{\lambda}_3)]+\\ &\Phi_2[\dot{\upsilon}_3(\bar{\lambda}_0\bar{\lambda}_1+\lambda_1\lambda_0+\bar{\lambda}_2\bar{\lambda}_3+\lambda_2\lambda_3)+\dot{\upsilon}_1(\bar{\lambda}_0\bar{\lambda}_3+\lambda_3\lambda_0-\bar{\lambda}_1\bar{\lambda}_2-\lambda_1\lambda_2)+\end{aligned} \tag{27}$$

$$
\begin{aligned}
&+g_0(\bar{\lambda}_3\bar{\lambda}_1 - \lambda_3\lambda_1 - \bar{\lambda}_0\bar{\lambda}_2 + \lambda_2\lambda_0) + \bar{g}_0(\lambda_0\bar{\lambda}_2 + \lambda_2\bar{\lambda}_0 + \lambda_2\bar{\lambda}_3 - \lambda_3\bar{\lambda}_2) + \\
&-\bar{g}_2(\lambda_0\bar{\lambda}_0 + \lambda_1\bar{\lambda}_1 - \lambda_2\bar{\lambda}_2 + \lambda_3\bar{\lambda}_3) + \bar{g}_3(\lambda_3\bar{\lambda}_2 + \lambda_2\bar{\lambda}_3 - \lambda_0\bar{\lambda}_1 + \lambda_1\bar{\lambda}_0)] + \\
&\Phi_3[\dot{\upsilon}_2(\bar{\lambda}_0\bar{\lambda}_1 + \lambda_1\lambda_0 - \bar{\lambda}_2\bar{\lambda}_3 - \lambda_2\lambda_3) - \dot{\upsilon}_1(\bar{\lambda}_0\bar{\lambda}_2 + \lambda_2\lambda_0 + \bar{\lambda}_1\bar{\lambda}_3 + \lambda_1\lambda_3) + \\
&+g_0(\bar{\lambda}_1\bar{\lambda}_2 - \lambda_1\lambda_2 - \bar{\lambda}_0\bar{\lambda}_3 + \lambda_3\lambda_0) + \bar{g}_0(\lambda_0\bar{\lambda}_3 + \lambda_3\bar{\lambda}_0 + \lambda_1\bar{\lambda}_2 - \lambda_2\bar{\lambda}_1) + \\
&\bar{g}_2(\lambda_0\bar{\lambda}_1 - \lambda_1\bar{\lambda}_0 + \lambda_2\bar{\lambda}_3 + \lambda_3\bar{\lambda}_2) - \bar{g}_3(\lambda_0\bar{\lambda}_0 + \lambda_1\bar{\lambda}_1 + \lambda_2\bar{\lambda}_2 - \lambda_3\bar{\lambda}_3)] + \\
&\Phi_{\bar{1}}[\dot{\upsilon}_1(\lambda_0\bar{\lambda}_0 - \lambda_1\bar{\lambda}_1 + \lambda_2\bar{\lambda}_2 + \lambda_3\bar{\lambda}_3) - \dot{\upsilon}_2(\lambda_0\bar{\lambda}_3 - \lambda_3\bar{\lambda}_0 + \lambda_2\bar{\lambda}_1 + \lambda_1\bar{\lambda}_2) + \\
&\dot{\upsilon}_3(\lambda_0\bar{\lambda}_2 - \lambda_2\bar{\lambda}_0 - \lambda_3\bar{\lambda}_1 - \lambda_1\bar{\lambda}_3) + g_0(\lambda_2\bar{\lambda}_3 - \lambda_3\bar{\lambda}_2 - \lambda_0\bar{\lambda}_1 - \lambda_1\bar{\lambda}_0) + \\
&\bar{g}_0(-\bar{\lambda}_0\bar{\lambda}_1 + \lambda_1\lambda_0 - \bar{\lambda}_2\bar{\lambda}_3 + \lambda_2\lambda_3) + \bar{g}_2(\bar{\lambda}_0\bar{\lambda}_3 + \lambda_3\lambda_0 + \bar{\lambda}_2\bar{\lambda}_1 + \lambda_2\lambda_1) + \\
&\bar{g}_3(-\bar{\lambda}_0\bar{\lambda}_2 - \lambda_2\lambda_0 + \bar{\lambda}_3\bar{\lambda}_1 + \lambda_3\lambda_1)] + \\
&\Phi_{\bar{1}\bar{1}}(\bar{\lambda}_1\lambda_1\lambda_2\bar{\lambda}_2 + \bar{\lambda}_1\lambda_1\lambda_3\bar{\lambda}_3 + \bar{\lambda}_0\lambda_1\bar{\lambda}_2\lambda_3 + \bar{\lambda}_0\lambda_2\bar{\lambda}_3\lambda_1 + \bar{\lambda}_0\lambda_1\bar{\lambda}_2\lambda_3 + \\
&\bar{\lambda}_0\lambda_3\bar{\lambda}_1\lambda_2 - \bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3\bar{\lambda}_0) + \\
&\Phi_{11}(\bar{\lambda}_0\lambda_0\lambda_1\bar{\lambda}_1 + \bar{\lambda}_0\lambda_2\lambda_3\bar{\lambda}_1 + \bar{\lambda}_1\lambda_1\lambda_2\bar{\lambda}_2 + \bar{\lambda}_1\lambda_1\lambda_3\bar{\lambda}_3 + \lambda_0\lambda_2\bar{\lambda}_3\bar{\lambda}_1 + \lambda_0\lambda_3\bar{\lambda}_1\bar{\lambda}_2 \\
&-\lambda_1\lambda_2\lambda_3\lambda_0) + \\
&\Phi_{22}(\bar{\lambda}_0\lambda_0\lambda_2\bar{\lambda}_2 + \bar{\lambda}_0\lambda_3\lambda_1\bar{\lambda}_2 + \bar{\lambda}_2\lambda_2\lambda_1\bar{\lambda}_1 + \bar{\lambda}_2\lambda_2\lambda_3\bar{\lambda}_3 + \lambda_0\lambda_1\bar{\lambda}_2\bar{\lambda}_3 + \lambda_0\lambda_3\bar{\lambda}_1\bar{\lambda}_2 - \\
&-\lambda_1\lambda_2\lambda_3\lambda_0) + \\
&\Phi_{33}(\bar{\lambda}_0\lambda_0\lambda_3\bar{\lambda}_3 + \bar{\lambda}_0\lambda_1\lambda_2\bar{\lambda}_3 + \bar{\lambda}_3\lambda_3\lambda_1\bar{\lambda}_1 + \bar{\lambda}_3\lambda_3\lambda_2\bar{\lambda}_2 + \lambda_0\lambda_1\bar{\lambda}_2\bar{\lambda}_3 + \lambda_0\lambda_2\bar{\lambda}_3\bar{\lambda}_1 \\
&-\lambda_1\lambda_2\lambda_3\lambda_0) + \\
&\Phi_{\bar{1}1}(\lambda_1\lambda_0\lambda_2\bar{\lambda}_3 - \bar{\lambda}_0\bar{\lambda}_1\bar{\lambda}_2\lambda_3 + \lambda_0\lambda_3\lambda_1\bar{\lambda}_2 + \bar{\lambda}_0\bar{\lambda}_3\bar{\lambda}_1\lambda_2 + \lambda_0\lambda_2\lambda_3\bar{\lambda}_1 - \bar{\lambda}_0\bar{\lambda}_2\bar{\lambda}_3\lambda_1 \\
&-\lambda_1\lambda_2\lambda_3\bar{\lambda}_0 - \bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3\lambda_0) + \\
&\Phi_{\bar{1}2}(\lambda_0\lambda_2\lambda_3\bar{\lambda}_2 + \bar{\lambda}_2\bar{\lambda}_1\lambda_0\bar{\lambda}_0 + \bar{\lambda}_0\bar{\lambda}_2\bar{\lambda}_3\lambda_2 + \lambda_0\lambda_3\lambda_1\bar{\lambda}_1 - \bar{\lambda}_0\bar{\lambda}_3\bar{\lambda}_1\lambda_1 - \lambda_1\lambda_2\lambda_3\bar{\lambda}_3 \\
&-\bar{\lambda}_1\bar{\lambda}_2\lambda_3\bar{\lambda}_3) + \\
&\Phi_{\bar{1}3}(\lambda_0\lambda_2\lambda_3\bar{\lambda}_3 + \bar{\lambda}_3\bar{\lambda}_1\lambda_0\bar{\lambda}_0 + \bar{\lambda}_0\bar{\lambda}_2\bar{\lambda}_3\lambda_3 + \lambda_0\lambda_1\lambda_2\bar{\lambda}_1 - \bar{\lambda}_0\bar{\lambda}_1\bar{\lambda}_2\lambda_1 - \lambda_1\lambda_3\lambda_2\bar{\lambda}_2 \\
&-\bar{\lambda}_1\bar{\lambda}_3\lambda_2\bar{\lambda}_2) + \\
&\Phi_{12}(\bar{\lambda}_0\lambda_0\lambda_2\bar{\lambda}_1 - \lambda_0\bar{\lambda}_0\lambda_1\bar{\lambda}_2 + \bar{\lambda}_0\lambda_2\lambda_3\bar{\lambda}_2 - \lambda_0\bar{\lambda}_2\bar{\lambda}_3\lambda_2 + \bar{\lambda}_0\lambda_3\lambda_1\bar{\lambda}_1 - \lambda_0\bar{\lambda}_3\bar{\lambda}_1\lambda_1 \\
&+\bar{\lambda}_1\lambda_2\lambda_3\bar{\lambda}_3 + \bar{\lambda}_2\lambda_1\lambda_3\bar{\lambda}_3) + \\
&\Phi_{13}(\bar{\lambda}_0\lambda_0\lambda_3\bar{\lambda}_1 - \lambda_0\bar{\lambda}_0\lambda_1\bar{\lambda}_3 + \bar{\lambda}_0\lambda_1\lambda_2\bar{\lambda}_1 - \lambda_0\bar{\lambda}_1\bar{\lambda}_2\lambda_1 + \bar{\lambda}_0\lambda_2\lambda_3\bar{\lambda}_3 - \lambda_0\bar{\lambda}_2\bar{\lambda}_3\lambda_3 \\
&+\bar{\lambda}_1\lambda_3\lambda_2\bar{\lambda}_2 + \bar{\lambda}_3\lambda_1\lambda_2\bar{\lambda}_2) + \\
&\Phi_{23}(\bar{\lambda}_0\lambda_0\lambda_3\bar{\lambda}_2 - \lambda_0\bar{\lambda}_0\lambda_2\bar{\lambda}_3 + \bar{\lambda}_0\lambda_1\lambda_2\bar{\lambda}_2 - \lambda_0\bar{\lambda}_1\bar{\lambda}_2\lambda_2 + \bar{\lambda}_0\lambda_3\lambda_1\bar{\lambda}_3 - \lambda_0\bar{\lambda}_3\bar{\lambda}_1\lambda_3 \\
&+\bar{\lambda}_2\lambda_3\lambda_1\bar{\lambda}_1 + \bar{\lambda}_3\lambda_2\lambda_1\bar{\lambda}_1) + \\
&\Omega(\dot{\upsilon}_1\dot{\bar{\upsilon}}_1 + g_0\bar{g}_0 + \dot{\upsilon}_2\bar{g}_2 + \dot{\upsilon}_3\bar{g}_3 + \lambda_0\dot{\bar{\lambda}}_0 + \lambda_1\dot{\bar{\lambda}}_1 + \lambda_2\dot{\bar{\lambda}}_2 + \lambda_3\dot{\bar{\lambda}}_3) + \\
&\Omega_1(\dot{\upsilon}_1\lambda_1\bar{\lambda}_1 + \dot{\upsilon}_2\lambda_2\bar{\lambda}_1 + \dot{\upsilon}_3\lambda_3\bar{\lambda}_1 + g_0\lambda_0\bar{\lambda}_1 - \bar{g}_0\lambda_2\lambda_3 + \bar{g}_2\lambda_0\lambda_3 - \bar{g}_3\lambda_0\lambda_2) + \\
&\Omega_2(\dot{\upsilon}_1\lambda_1\bar{\lambda}_2 + \dot{\upsilon}_2\lambda_2\bar{\lambda}_2 + \dot{\upsilon}_3\lambda_3\bar{\lambda}_2 - \dot{\bar{\upsilon}}_1\lambda_0\lambda_3 - g_0\bar{\lambda}_2\lambda_0 - \bar{g}_0\lambda_3\lambda_1 + \bar{g}_3\lambda_0\lambda_1) + \\
&\Omega_3(\dot{\upsilon}_1\lambda_1\bar{\lambda}_3 + \dot{\upsilon}_2\lambda_2\bar{\lambda}_3 + \dot{\upsilon}_3\lambda_3\bar{\lambda}_3 + \dot{\bar{\upsilon}}_1\lambda_0\lambda_2 - g_0\bar{\lambda}_3\lambda_0 - \bar{g}_0\lambda_1\lambda_2 - \bar{g}_2\lambda_0\lambda_1) + \\
&\Omega_{\bar{1}}(\dot{\upsilon}_1(\lambda_2\bar{\lambda}_2 + \lambda_3\bar{\lambda}_3) + \dot{\upsilon}_2\bar{\lambda}_0\bar{\lambda}_3 - \dot{\upsilon}_3\bar{\lambda}_0\bar{\lambda}_2 - g_0\bar{\lambda}_2\bar{\lambda}_3 + \bar{g}_0\bar{\lambda}_0\lambda_1 + \bar{g}_2\bar{\lambda}_2\lambda_1 + \bar{g}_3\bar{\lambda}_3\lambda_1) + \\
&\Omega_{\bar{1}1}(\lambda_0\bar{\lambda}_0\lambda_1\bar{\lambda}_1 + \lambda_1\bar{\lambda}_1\lambda_2\bar{\lambda}_2 + \lambda_1\bar{\lambda}_1\lambda_3\bar{\lambda}_3) + \\
&\Omega_{\bar{1}2}(\lambda_0\bar{\lambda}_0\lambda_1\bar{\lambda}_2 + \lambda_1\bar{\lambda}_2\lambda_3\bar{\lambda}_3) + \\
&\Omega_{\bar{1}3}(\lambda_0\bar{\lambda}_0\lambda_1\bar{\lambda}_3 + \lambda_1\bar{\lambda}_3\lambda_2\bar{\lambda}_2) +
\end{aligned}
$$

$$\begin{aligned}
&\Omega_{12}(\lambda_2\lambda_0\lambda_3\bar{\lambda}_2 - \lambda_0\lambda_3\lambda_1\bar{\lambda}_1) + \\
&\Omega_{13}(\lambda_1\lambda_0\lambda_2\bar{\lambda}_1 - \lambda_0\lambda_2\lambda_3\bar{\lambda}_3) + \\
&\Omega_{23}(\lambda_1\lambda_0\lambda_2\bar{\lambda}_2 - \lambda_0\lambda_3\lambda_1\bar{\lambda}_3),
\end{aligned}$$

where $F(\upsilon_1, \upsilon_2, \upsilon_3, \bar{\upsilon}_1)$ is the unconstrained prepotential, while

$$\begin{aligned}
\Omega &= \Box F = \partial_{11}F + \partial_{22}F + \partial_{33}F + \partial_{\bar{1}\bar{1}}F, \\
\Phi &= \partial_{1\bar{1}}F
\end{aligned} \tag{28}$$

and the partial derivative of Ω (Φ) w.r.t. υ_i, $\bar{\upsilon}_1$ is expressed as Ω_i, $\Omega_{\bar{1}}$ (Φ_i, $\Phi_{\bar{1}}$), respectively.

Similarly to the results of [28], the constraints $\Box\Phi = 0$ and $\Omega = 0$ arise as a consequence of imposing an extra invariance under an $\mathcal{N} = 5$-Extended Supersymmetry (under such constraints the resulting off-shell action is also automatically $\mathcal{N} = 8$-invariant).

An inequivalent $\mathcal{N} = 4$-invariant action is obtained by applying the Construction I to the supermultiplet $(4, 8, 4)_D$.

For $(2, 8, 6)_B$ the associated Lagrangian is

$$\begin{aligned}
\mathcal{L} = {} & \Phi(\dot{\upsilon}_0^2 + \dot{\bar{\upsilon}}_0^2 + g_i^2 + \bar{g}_i^2 + \dot{\lambda}_0\lambda_0 + \dot{\bar{\lambda}}_0\bar{\lambda}_0 + \dot{\lambda}_i\lambda_i + \dot{\bar{\lambda}}_i\bar{\lambda}_i) + \\
& \Phi_0[\dot{\bar{\upsilon}}_0(\bar{\lambda}_i\lambda_i + \lambda_0\bar{\lambda}_0) + g_i(\bar{\lambda}_0\bar{\lambda}_i - \lambda_i\lambda_0) + \bar{g}_i(\lambda_0\bar{\lambda}_i + \lambda_i\bar{\lambda}_0) \\
& -\frac{\varepsilon_{ijk}}{2}g_i(\bar{\lambda}_j\bar{\lambda}_k - \lambda_j\lambda_k) - \varepsilon_{ijk}\lambda_i\bar{\lambda}_j\bar{g}_k] + \\
& \Phi_{\bar{0}}[\dot{\upsilon}_0(\lambda_i\bar{\lambda}_i + \bar{\lambda}_0\lambda_0) + \bar{g}_i(\bar{\lambda}_0\bar{\lambda}_i - \lambda_i\lambda_0) - g_i(\lambda_0\bar{\lambda}_i + \lambda_i\bar{\lambda}_0) \\
& +\frac{\varepsilon_{ijk}}{2}\bar{g}_i(\bar{\lambda}_j\bar{\lambda}_k - \lambda_j\lambda_k) - \varepsilon_{ijk}\lambda_i\bar{\lambda}_j g_k] + \\
& \Phi_{00}\frac{\varepsilon_{ijk}}{6}[3\bar{\lambda}_0\lambda_i\bar{\lambda}_j\lambda_k + \lambda_0\lambda_i\lambda_j\lambda_k] + \\
& \Phi_{\bar{0}\bar{0}}\frac{\varepsilon_{ijk}}{6}[3\lambda_0\lambda_i\bar{\lambda}_j\bar{\lambda}_k + \bar{\lambda}_0\bar{\lambda}_i\bar{\lambda}_j\bar{\lambda}_k] \\
& -(\Phi_{\bar{0}\bar{0}} + \Phi_{00})\lambda_0\bar{\lambda}_0\lambda_i\bar{\lambda}_i + \\
& \Phi_{0\bar{0}}\frac{\varepsilon_{ijk}}{6}[\lambda_i\lambda_j\lambda_k\bar{\lambda}_0 + \bar{\lambda}_i\bar{\lambda}_j\bar{\lambda}_k\lambda_0 - 3(\lambda_0\lambda_i\lambda_j\bar{\lambda}_k + \bar{\lambda}_0\bar{\lambda}_i\bar{\lambda}_j\lambda_k)] + \\
& \Omega(\dot{\upsilon}_0\dot{\bar{\upsilon}}_0 + \lambda_0\dot{\bar{\lambda}}_0 + \lambda_i\dot{\bar{\lambda}}_i + g_i\bar{g}_i) \\
& -\Omega_0(\dot{\upsilon}_0\bar{\lambda}_0\lambda_0 + g_i\bar{\lambda}_0\lambda_i) + \\
& \Omega_{\bar{0}}(\dot{\bar{\upsilon}}_0\bar{\lambda}_i\lambda_i + \bar{g}_i\bar{\lambda}_i\lambda_0) + \\
& \Omega_{0\bar{0}}\lambda_0\bar{\lambda}_0\lambda_i\bar{\lambda}_i \\
& -\frac{\varepsilon_{ijk}}{6}(\Omega_{00}\bar{\lambda}_0\lambda_i\lambda_j\lambda_k + \Omega_{\bar{0}\bar{0}}\bar{\lambda}_i\bar{\lambda}_j\bar{\lambda}_k\lambda_0),
\end{aligned} \tag{29}$$

where now we have

$$\begin{aligned}
\Omega &= \partial_{00}F + \partial_{\bar{0}\bar{0}}F, \\
\Phi &= \partial_{0\bar{0}}F.
\end{aligned} \tag{30}$$

For $(2, 8, 6)_b$ the associated Lagrangian is

$$\mathcal{L} = \Gamma(\dot{\upsilon}_0^2 + g_i^2 + \dot{\lambda}_0\lambda_0 + \dot{\lambda}_i\lambda_i) - \bar{\Gamma}(\dot{\bar{\upsilon}}_0^2 + \bar{g}_i^2 + \dot{\bar{\lambda}}_0\bar{\lambda}_0 + \dot{\bar{\lambda}}_i\bar{\lambda}_i) + \tag{31}$$

$$
\begin{aligned}
&\bar{\Gamma}_0\dot{\bar{v}}_0(\lambda_i\bar{\lambda}_i+\lambda_0\bar{\lambda}_0)-g_i(\bar{\Gamma}_0\bar{\lambda}_0\bar{\lambda}_i+\Gamma_0\lambda_i\lambda_0)-\bar{\Gamma}_0\bar{g}_i(\lambda_0\bar{\lambda}_i+\lambda_i\bar{\lambda}_0)+\\
&g_i\frac{\varepsilon_{ijk}}{2}(\bar{\Gamma}_0\bar{\lambda}_i\bar{\lambda}_j+\Gamma_0\lambda_j\lambda_k)+\bar{\Gamma}_0\varepsilon_{ijk}\lambda_i\bar{\lambda}_j\bar{g}_k+\\
&\Gamma_{\bar{0}}\dot{v}_0(\lambda_i\bar{\lambda}_i+\lambda_0\bar{\lambda}_0)-\bar{g}_i(\bar{\Gamma}_{\bar{0}}\bar{\lambda}_0\bar{\lambda}_i+\Gamma_{\bar{0}}\lambda_i\lambda_0)-\Gamma_{\bar{0}}g_i(\lambda_0\bar{\lambda}_i+\lambda_i\bar{\lambda}_0)\\
&-\bar{g}_i\frac{\varepsilon_{ijk}}{2}(\bar{\Gamma}_{\bar{0}}\bar{\lambda}_i\bar{\lambda}_j+\Gamma_{\bar{0}}\lambda_j\lambda_k)-\Gamma_{\bar{0}}\varepsilon_{ijk}\lambda_i\bar{\lambda}_j g_k\\
&-\frac{\varepsilon_{ijk}}{6}(3\bar{\Gamma}_{00}\bar{\lambda}_0\lambda_i\bar{\lambda}_j\lambda_k-\Gamma_{00}\lambda_0\lambda_i\lambda_j\lambda_k)+\\
&\frac{\varepsilon_{ijk}}{6}(3\Gamma_{\bar{0}\bar{0}}\lambda_0\bar{\lambda}_i\lambda_j\bar{\lambda}_k-\bar{\Gamma}_{\bar{0}\bar{0}}\bar{\lambda}_0\bar{\lambda}_i\bar{\lambda}_j\bar{\lambda}_k)\\
&-\Gamma_{0\bar{0}}\frac{\varepsilon_{ijk}}{6}(\lambda_i\lambda_j\lambda_k\bar{\lambda}_0+3\lambda_0\lambda_i\lambda_j\bar{\lambda}_k)+\bar{\Gamma}_{0\bar{0}}\frac{\varepsilon_{ijk}}{6}(\bar{\lambda}_i\bar{\lambda}_j\bar{\lambda}_k\lambda_0+3\bar{\lambda}_0\bar{\lambda}_i\bar{\lambda}_j\lambda_k),
\end{aligned}
$$

with

$$
\begin{aligned}
\Gamma &= \partial_{00}F(v_0,\bar{v}_0),\\
\bar{\Gamma} &= \partial_{\bar{0}\bar{0}}F(v_0,\bar{v}_0).
\end{aligned}
\tag{32}
$$

Therefore it turns out, comparing (29) with (31), that the two inequivalent $\mathcal{N}=4$ supermultiplets $(2,8,6)_B$ and $(2,8,6)_b$ produce inequivalent (for generic values of the functions $\Omega,\Phi,\Gamma,\bar{\Gamma}$) $\mathcal{N}=4$ off-shell invariant actions.

The extra-requirement of an $\mathcal{N}=5$-invariance produces the constraints $\Box\Phi=0$, $\Omega=0$ for the $(2,8,6)_B$ case and $\Gamma=-\bar{\Gamma}$ (implying $\Box\Gamma=0$) for the $(2,8,6)_b$ case. The resulting constrained Lagrangians coincide and the action possesses an overall $\mathcal{N}=8$ invariance.

It is interesting to present an explicit example of $\mathcal{N}=4$-invariant, first-order action derived from Construction I. For the $(4,8,4)_B$ non-minimal supermultiplet with connectivity symbol 4_3+4_1 and component fields $(v_0,v_i;\lambda_0,\lambda_i,\bar{\lambda}_0,\bar{\lambda}_i;\bar{g}_0,\bar{g}_i)$ $(i=1,2,3)$, we have that the associated Lagrangian is

$$
\begin{aligned}
\mathcal{L} \;=\;& \Omega(\dot{v}_0\bar{g}_0+\dot{v}_i\bar{g}_i)+\Omega(\lambda_0\dot{\bar{\lambda}}_0+\lambda_k\dot{\bar{\lambda}}_k)+\\
&\varepsilon_{ijk}\Omega_j\bar{g}_k\lambda_0\lambda_i-\Omega_i\dot{v}_0\bar{\lambda}_i\lambda_0-\Omega_0\dot{v}_i\bar{\lambda}_0\lambda_i+\\
&\frac{\varepsilon_{ijk}}{2}(\Omega_0\bar{g}_k-\Omega_k\bar{g}_0)\lambda_i\lambda_j-\Omega_j\dot{v}_i\bar{\lambda}_j\lambda_i-\Omega_0\dot{v}_0\bar{\lambda}_0\lambda_0\\
&-\frac{1}{2}\varepsilon_{ijk}\Omega_{0k}\lambda_i\lambda_j\lambda_0\bar{\lambda}_0-\frac{\varepsilon_{ijk}}{2}\delta_{pq}\Omega_{pk}\lambda_0\lambda_i\lambda_j\bar{\lambda}_q\\
&-\frac{\varepsilon_{ijk}}{6}(\Omega_{00}\bar{\lambda}_0-\Omega_{0p}\lambda_p)\lambda_i\lambda_j\lambda_k,
\end{aligned}
\tag{33}
$$

where

$$
\Omega=\partial_{00}F+\partial_{ii}F. \tag{34}
$$

In the next Section we show how, for this one and the other supermultiplets entering (24), by applying Construction II, we obtain a second-order, $\mathcal{N}=4$-invariant action expressed in terms of a constrained prepotential.

6 $\mathcal{N}=4$-invariant σ-models with a constrained prepotential

The application of Construction II to the non-minimal supermultiplet $(4,8,4)_B$ induces an $\mathcal{N}=4$-invariant theory, obtained from a second-order Lagrangian, which is expressed in terms of the two independent functions, derived from the prepotential W entering (7), $\Phi(\upsilon_0,\upsilon_i)$ and $\Omega(\upsilon_0,\upsilon_i)$. The $\mathcal{N}=4$-invariance is however recovered if and only if the constraint

$$\Phi_{00}+\Phi_{ii} \;=\; 0 \tag{35}$$

is satisfied. This feature is not present for the $\mathcal{N}=4$ actions derived from the (25) supermultiplets.

The Lagrangian is explicitly given by

$$\begin{aligned}
\mathcal{L} \;=\;& \Phi(\dot{\upsilon}_0^2+\dot{\upsilon}_i^2+\bar{g}_0^2+\bar{g}_i^2+\dot{\lambda}_0\lambda_0+\dot{\bar{\lambda}}_0\bar{\lambda}_0+\dot{\lambda}_i\lambda_i+\dot{\bar{\lambda}}_i\bar{\lambda}_i)+\\
&\varepsilon_{ijk}\Phi_k\dot{\upsilon}_j(\bar{\lambda}_0\bar{\lambda}_i+\lambda_i\lambda_0)+(\Phi_0\dot{\upsilon}_i-\Phi_i\dot{\upsilon}_0)(\bar{\lambda}_0\bar{\lambda}_i-\lambda_i\lambda_0)\\
&-\varepsilon_{ijk}\Phi_j\bar{g}_k(\lambda_0\bar{\lambda}_i-\lambda_i\bar{\lambda}_0)+(\Phi_0\bar{g}_i+\Phi_i\bar{g}_0)(\lambda_0\bar{\lambda}_i+\lambda_i\bar{\lambda}_0)+\\
&\frac{\varepsilon_{ijk}}{2}(\Phi_i\dot{\upsilon}_0-\Phi_0\dot{\upsilon}_i)(\bar{\lambda}_j\bar{\lambda}_k-\lambda_j\lambda_k)+\frac{1}{2}(\Phi_j\dot{\upsilon}_k-\Phi_k\dot{\upsilon}_j)(\bar{\lambda}_j\bar{\lambda}_k+\lambda_j\lambda_k)+\\
&(\Phi_0\bar{g}_0+\Phi_j\bar{g}_j)(\bar{\lambda}_0\lambda_0+\bar{\lambda}_k\lambda_k)-\Phi_i\bar{g}_j(\bar{\lambda}_i\lambda_j-\lambda_i\bar{\lambda}_j)-\Phi_0\bar{g}_0(\bar{\lambda}_0\lambda_0-\lambda_0\bar{\lambda}_0)+\\
&\varepsilon_{ijk}\lambda_i\bar{\lambda}_j(\Phi_k\bar{g}_0-\Phi_0\bar{g}_k)+(\varepsilon_{ijk}\Phi_{0k}-\Phi_{ij})\lambda_0\bar{\lambda}_0\lambda_i\bar{\lambda}_j+\\
&\Phi_{0j}(\bar{\lambda}_0\lambda_j-\lambda_0\bar{\lambda}_j)\lambda_i\bar{\lambda}_i+\frac{1}{2}\varepsilon_{ijk}\delta_{pq}\Phi_{kp}(\bar{\lambda}_0\lambda_i\lambda_j\bar{\lambda}_q-\lambda_0\bar{\lambda}_i\bar{\lambda}_j\lambda_q)+\\
&\Phi_{jk}\bar{\lambda}_j\lambda_k\lambda_i\bar{\lambda}_i-\frac{1}{2}\varepsilon_{ijk}\delta_{pq}\Phi_{0p}\lambda_i\bar{\lambda}_j\lambda_k\bar{\lambda}_q\\
&-\Phi_{00}\lambda_0\bar{\lambda}_0\lambda_i\bar{\lambda}_i+\frac{\varepsilon_{ijk}}{2}[\Phi_{pp}(\lambda_0\lambda_i\bar{\lambda}_j\bar{\lambda}_k)+\Phi_{00}(\bar{\lambda}_0\lambda_i\bar{\lambda}_j\lambda_k)]\\
&-\frac{\varepsilon_{ijk}}{6}[(\Phi_{00}+\Phi_{pp})\lambda_i\lambda_j\lambda_k\lambda_0]+\\
&\Omega(\dot{\upsilon}_0\bar{g}_0+\dot{\upsilon}_i\bar{g}_i)+\Omega(\lambda_0\dot{\bar{\lambda}}_0+\lambda_k\dot{\bar{\lambda}}_k)+\\
&\varepsilon_{ijk}\Omega_j\bar{g}_k\lambda_0\lambda_i-\Omega_i\dot{\upsilon}_0\bar{\lambda}_i\lambda_0-\Omega_0\dot{\upsilon}_i\bar{\lambda}_0\lambda_i+\\
&\frac{\varepsilon_{ijk}}{2}(\Omega_0\bar{g}_k-\Omega_k\bar{g}_0)\lambda_i\lambda_j-\Omega_j\dot{\upsilon}_i\bar{\lambda}_j\lambda_i-\Omega_0\dot{\upsilon}_0\bar{\lambda}_0\lambda_0\\
&-\frac{1}{2}\varepsilon_{ijk}\Omega_{0k}\lambda_i\lambda_j\lambda_0\bar{\lambda}_0-\frac{\varepsilon_{ijk}}{2}\delta_{pq}\Omega_{pk}\lambda_0\lambda_i\lambda_j\bar{\lambda}_q\\
&-\frac{\varepsilon_{ijk}}{6}(\Omega_{00}\bar{\lambda}_0-\Omega_{0p}\lambda_p)\lambda_i\lambda_j\lambda_k
\end{aligned} \tag{36}$$

Requiring an extra, $\mathcal{N}=5$-invariance for the action implies the further constraint $\Omega=0$. The resulting action is automatically $\mathcal{N}=8$-invariant and coincides with the $\mathcal{N}=5$ constrained action derived from $(4,8,4)_C$.

All actions obtained via Construction II from the supermultiplets entering (24) share the same features. In the case of the supermultiplet $(1,8,7)_{red}$, described by the fields

$(v_0;\lambda_0,\lambda_i,\bar{\lambda}_0,\bar{\lambda}_i;g_i,\bar{g}_0,\bar{g}_i)$, the associated Lagrangian is

$$\begin{aligned}\mathcal{L} \;=\;& \Phi(\dot{v}_0^2+\bar{g}_0^2+g_i^2+\bar{g}_i^2+\dot{\lambda}_0\lambda_0+\dot{\bar{\lambda}}_0\bar{\lambda}_0+\dot{\lambda}_i\lambda_i+\dot{\bar{\lambda}}_i\bar{\lambda}_i) \\ & \Phi_0[\bar{g}_0(\lambda_0\bar{\lambda}_0-\lambda_i\bar{\lambda}_i)+g_i(\bar{\lambda}_0\bar{\lambda}_i-\lambda_i\lambda_0)+\bar{g}_i(\lambda_0\bar{\lambda}_i+\lambda_i\bar{\lambda}_0)- \\ & -\varepsilon_{ijk}(\bar{g}_i\bar{\lambda}_j\lambda_k+\frac{g_i}{2}(\bar{\lambda}_j\bar{\lambda}_k-\lambda_j\lambda_k))]+ \\ & \Omega(\dot{v}_0\bar{g}_0+\lambda_0\dot{\bar{\lambda}}_0+\lambda_i\dot{\bar{\lambda}}_i+g_i\bar{g}_i)+ \\ & \Omega_0[(\dot{v}_0\lambda_0+g_i\lambda_i)\bar{\lambda}_0+\frac{1}{2}\varepsilon_{ijk}\bar{g}_i\lambda_j\lambda_k]+ \\ & \Omega_{00}\frac{1}{6}\varepsilon_{ijk}\lambda_i\lambda_j\lambda_k\bar{\lambda}_0,\end{aligned} \tag{37}$$

where Φ, Ω are fuctions of v_0. The $\mathcal{N}=4$-invariance requires

$$\Phi_{00}=0. \tag{38}$$

Implementing the $\mathcal{N}=5$-invariance gives the further constraint $\Omega=0$ (again, the resulting action is automatically fully $\mathcal{N}=8$-invariant).

The $\mathcal{N}=8$ model based on the supermultiplet of field content $(1,8,7)$ was first obtained in [4]. Unlike the present, more general construction, the action was derived through an "$\mathcal{N}=8$ covariantization Ansatz" which cannot be applied neither to deduce the $\mathcal{N}=4$-invariant action, nor to obtain the invariant actions for the other supermultiplets entering (24).

7 Conclusions

We have already sketched in the Introduction the structure of our paper and presented its main results. Here we limit ourselves to make some comments and outline the implications of our work.

It is worth stressing the fact that we obtained here for the first time evidence that supermultiplets, sharing the same field content but differing in connectivity symbol, can induce inequivalent supersymmetric-invariant actions (one should compare, e.g., the actions given in formulas (27) and (36)). It was known, from the analysis of [5, 6, 9, 10], that inequivalent representations, discriminated by their respective connectivity symbol, can be found. On the other hand, so far, no dynamical characterization was associated to the connectivity symbol. In [28] the $\mathcal{N}=5$-supersymmetric off-shell invariant actions, induced with respect to inequivalent $\mathcal{N}=5$ supermultiplets of a given field content, were proven to coincide and possess an overall $\mathcal{N}=8$ supersymmetry invariance. The crucial feature here is the fact that inequivalent $\mathcal{N}=4$ off-shell invariant actions are induced by inequivalent *non-minimal* $\mathcal{N}=4$ linear supermultiplets (with the same field content).

We have to spend, as promised, some words on the concept of *oxidation*. It is a pun, employed in superstring/M-theory literature, see [7], to denote the process opposite to dimensional reduction. Extended Supersymmetries in $1D$ can be used to constrain, see [18, 7, 17], possible higher-dimensional supersymmetric theories (for instance, constraining the number of auxiliary fields in supergravity theories).

A much more ambitious task, see [35, 36], would consist in the reconstruction of a higher-dimensional theory from its one-dimensional supersymmetric data.

An N-extended (SuperYang-Mills or supergravity) supersymmetric theory in the ordinary $D = 4$ Minkowskian spacetime produces a $1D$ dimensionally-reduced supersymmetric theory with $\mathcal{N} = 4N$ supercharges. On the other hand, $N = 2$ can be obtained from the dimensional reduction of $D = 6$ (SYM or sugra), $N = 4$ from the dimensional reduction of $D = 10$ (SYM or sugra) and $N = 8$ from the dimensional reduction of the $D = 11$ sugra. As a result, a *necessary condition* for higher-dimensional oxidation consists in producing large $\mathcal{N}$-Extended supersymmetric theories in $1D$. Following [10], the word *oxidation* has been here consistently used in a specific and restricted sense, referring to the operation of enlarging the number of extended supersymmetries (from $\mathcal{N}$ to $\mathcal{N}+1$) acting on a supermultiplet with the same number of component fields. As such, the non-minimal $\mathcal{N} = 4$ linear supermultiplets are progressively oxidized to minimal $\mathcal{N} = 5, 6, 7, 8$ linear supermultiplets possessing 8 bosonic and 8 fermionic component fields. It is clear, from these considerations, that the $\mathcal{N} = 4$ off-shell invariant actions based on non-minimal supermultiplets are not of mere academic interest. Indeed, an $N = 2$, $D = 4$ theory, dimensionally reduced to $1D$, produces a supersymmetric model with $\mathcal{N} = 8$ extended supersymmetries; on the other hand the partial spontaneous breaking of $N = 2$ into $N = 1$ produces an $\mathcal{N} = 4$ invariant $1D$ supersymmetric model whose component fields belong to $\mathcal{N} = 8$ supermultiplets and are therefore non-minimal supermultiplets w.r.t. the $\mathcal{N} = 4$ invariant supersymmetries. The inequivalent $\mathcal{N} = 4$ non-minimal supermultiplets and their inequivalent $\mathcal{N} = 4$-invariant off-shell actions can therefore be regarded as building blocks for constructing supersymmetric models obtained from dimensional reduction of partial spontaneous supersymmetry breaking of $N = 2$, $D = 4$ supersymmetry.

At the end it is worth mentioning a recent paper [37] in which the $\mathcal{N} = 4$-invariance for a non-minimal supermultiplet in presence of a Yang monopole is discussed (see also [16]). In [38] a biharmonic superspace formalism has been discussed. It could be adapted to recast the non-minimal supermultiplets in the language of the harmonic superspace.

Appendix A: definitions and conventions

For completeness we report the definitions, applied to the cases used in the text, of the properties characterizing the linear representations of the one-dimensional $\mathcal{N}$-Extended Superalgebra. In particular the notions of *mass-dimension*, *field content*, *dressing transformation*, *connectivity symbol*, *dual supermultiplet* and so on, as well as the association of linear supersymmetry transformations with graphs, will be reviewed following [2, 4, 3, 9, 10, 28]. The Reader can consult these papers for broader definitions and more detailed discussions.

Mass-dimension:

A grading, the mass-dimension d, can be assigned to any field entering a linear representation (the hamiltonian H, proportional to the time-derivative operator $\partial \equiv \frac{d}{dt}$, has a mass-dimension 1). Bosonic (fermionic) fields have integer (respectively, half-integer) mass-dimension.

Field content:

Each finite linear representation is characterized by its "field content", i.e. the set

of integers $(n_1, n_2, \ldots, n_l)$ specifying the number n_i of fields of mass-dimension d_i ($d_i = d_1 + \frac{i-1}{2}$, with d_1 an arbitrary constant) entering the representation. Physically, the n_l fields of highest dimension are the auxiliary fields which transform as a time-derivative under any supersymmetry generator. The maximal value l (corresponding to the maximal dimensionality d_l) is defined to be the *length* of the representation (a root representation has length $l = 2$). Either $n_1, n_3, \ldots$ correspond to the bosonic fields (therefore $n_2, n_4, \ldots$ specify the fermionic fields) or viceversa.
In both cases the equality $n_1 + n_3 + \ldots = n_2 + n_4 + \ldots = n$ is guaranteed.

Dressing transformation:

Higher-length supermultiplets are obtained by applying a dressing transformation to the length-2 root supermultiplet. The root supermultiplet is specified by the $\mathcal{N}$ supersymmetry operators $\widehat{Q}_i$ ($i = 1, \ldots, \mathcal{N}$), expressed in matrix form as

$$\widehat{Q}_j = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \gamma_j \\ -\gamma_j \cdot H & 0 \end{pmatrix}, \qquad \widehat{Q}_{\mathcal{N}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \mathbf{1}_n \\ \mathbf{1}_n \cdot H & 0 \end{pmatrix}, \tag{39}$$

where the γ_j matrices ($j = 1, \ldots, \mathcal{N} - 1$) satisfy the Euclidean Clifford algebra

$$\{\gamma_i, \gamma_j\} = -2\delta_{ij}\mathbf{1}_n. \tag{40}$$

The length-3 supermultiplets are specified by the $\mathcal{N}$ operators Q_i, given by the dressing transformation

$$Q_i = D\widehat{Q}_i D^{-1}, \tag{41}$$

where D is a diagonal dressing matrix such that

$$D = \begin{pmatrix} \widetilde{D} & 0 \\ 0 & \mathbf{1}_n \end{pmatrix}, \tag{42}$$

with $\widetilde{D}$ an $n \times n$ diagonal matrix whose diagonal entries are either 1 or the derivative operator ∂.

Association with graphs:

The association between linear supersymmetry transformations and $\mathcal{N}$-colored oriented graphs goes as follows. The fields (bosonic and fermionic) entering a representation are expressed as vertices. They can be accommodated into an $X - Y$ plane. The Y coordinate can be chosen to correspond to the mass-dimension d of the fields. Conventionally, the lowest dimensional fields can be associated to vertices lying on the X axis. The higher dimensional fields have positive, integer or half-integer values of Y. A colored edge links two vertices which are connected by a supersymmetry transformation. Each one of the $\mathcal{N}$ Q_i supersymmetry generators is associated to a given color. The edges are oriented. The orientation reflects the sign (positive or negative) of the corresponding supersymmetry transformation connecting the two vertices. Instead of using arrows, alternatively, solid or dashed lines can be associated, respectively, to positive or negative signs. No colored line is drawn for supersymmetry transformations connecting a field with the time-derivative of a lower dimensional field. This is in particular true for the auxiliary fields (the fields of highest dimension in the representation) which are necessarily mapped, under supersymmetry transformations, in the time-derivative of lower-dimensional fields.

Each irreducible supersymmetry transformation can be presented (the identification is not unique) through an oriented $\mathcal{N}$-colored graph with $2n$ vertices. The graph is such that precisely $\mathcal{N}$ edges, one for each color, are linked to any given vertex which represents either a 0-mass dimension or a $\frac{1}{2}$-mass dimension field. An unoriented "color-blind" graph can be associated to the initial graph by disregarding the orientation of the edges and their colors (all edges are painted in black).

Connectivity symbol:

A characterization of length $l = 3$ color-blind, unoriented graphs can be expressed through the connectivity symbol ψ_g, defined as follows

$$\psi_g \;=\; (m_1)_{s_1} + (m_2)_{s_2} + \ldots + (m_Z)_{s_Z}. \qquad (43)$$

The ψ_g symbol encodes the information on the partition of the n $\frac{1}{2}$-mass dimension fields (vertices) into the sets of m_z vertices ($z = 1, \ldots, Z$) with s_z edges connecting them to the $n-k$ 1-mass dimension auxiliary fields. We have

$$m_1 + m_2 + \ldots + m_Z \;=\; n, \qquad (44)$$

while $s_z \neq s_{z'}$ for $z \neq z'$.

Dual supermultiplet:

A dual supermultiplet is obtained by mirror-reversing, upside-down, the graph associated to the original supermultiplet.

Appendix B: explicit construction of non-minimal supermultiplets

We present here an explicit construction of the non-minimal, reducible but indecomposable $\mathcal{N} = 4$ supermultiplets of length $l = 2, 3$.

We start with the $\mathcal{N} = 8$ $(8, 8)$ root supermultiplet expressed through (39), with the 7 matrices γ_j given by

$$\begin{array}{llll}\gamma_1 = \tau_1 \otimes \mathbf{1}_2 \otimes \tau_A, & \gamma_2 = \tau_2 \otimes \mathbf{1}_2 \otimes \tau_A, & \gamma_3 = \tau_A \otimes \tau_1 \otimes \mathbf{1}_2, & \gamma_4 = \tau_A \otimes \tau_2 \otimes \mathbf{1}_2, \\ \gamma_5 = \mathbf{1}_2 \otimes \tau_A \otimes \tau_1, & \gamma_6 = \mathbf{1}_2 \otimes \tau_A \otimes \tau_2, & \gamma_7 = \tau_A \otimes \tau_A \otimes \tau_A, & \end{array}$$

where τ_1, τ_2, τ_A, $\mathbf{1}_2$ are 2×2 matrices given by (e_{mn} is the 2×2 matrix with entry 1 at the m^{th} row, n^{th} column and 0 otherwise)

$$\tau_1 = e_{12} + e_{21}, \quad \tau_2 = e_{11} - e_{22}, \quad \tau_A = e_{12} - e_{21}, \quad \mathbf{1}_2 = e_{11} + e_{22}.$$

We can select, e.g., the 4 operators producing the non-minimal $\mathcal{N} = 4$ supermultiplet of length $l = 2$ (with connected graph) to be given by $\widehat{Q}_2$, $\widehat{Q}_5$, $\widehat{Q}_6$, $\widehat{Q}_7$. For this choice of root operators, the dressing transformations (41) producing the inequivalent length $l = 3$ non-minimal, reducible but indecomposable $\mathcal{N} = 4$ supermultiplets are obtained by applying the following diagonal dressing matrices $\widetilde{D}$, see (42):

field content:	*label:*	*dressing matrix* $\widetilde{D}$*:*
$(1,8,7)$		$diag(1,\partial,\partial,\partial,\partial,\partial,\partial,\partial)$
$(2,8,6)$	A	$diag(1,1,\partial,\partial,\partial,\partial,\partial,\partial)$
	B	$diag(1,\partial,\partial,\partial,\partial,\partial,\partial,1)$
$(3,8,5)$	A	$diag(1,1,1,\partial,\partial,\partial,\partial,\partial)$
	B	$diag(1,1,\partial,\partial,\partial,\partial,\partial,1)$
$(4,8,4)$	A	$diag(\partial,1,1,1,1,\partial,\partial,\partial)$
	B	$diag(1,1,1,1,\partial,\partial,\partial,\partial)$
	C	$diag(1,1,1,\partial,1,\partial,\partial,\partial)$
	D	$diag(1,1,\partial,\partial,\partial,\partial,1,1)$
$(5,8,3)$	A	$diag(1,1,1,1,1,\partial,\partial,\partial)$
	B	$diag(\partial,1,1,1,1,1,\partial,\partial)$
$(6,8,2)$	A	$diag(1,1,1,1,1,1,\partial,\partial)$
	B	$diag(\partial,1,1,1,1,1,1,\partial)$
$(7,8,1)$		$diag(1,1,1,1,1,1,1,\partial)$

(45)

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Tables with a few selected (unoriented, color-blind) graphs

For sake of clarity we present a few selected (unoriented, color-blind) graphs associated to the non-minimal, both reducible but indecomposable and fully reducible, linear supermultiplets.

We list the graphs associated to the inequivalent $\mathcal{N}=4$ non-minimal linear supermultiplets of field content $(8,8)$ and $(2,8,6)$, respectively. The graphs associated to the length-3 supermultiplets encode, in particular, the information of their connectivity symbol. We have

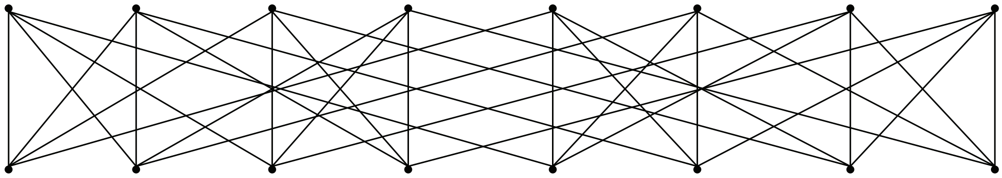

Figure 1: $(8,8)_{red}$, reducible but indecomposable.

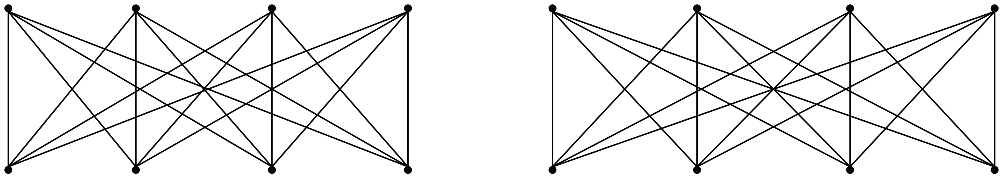

Figure 2: $(8,8)_{FR}$, fully reducible.

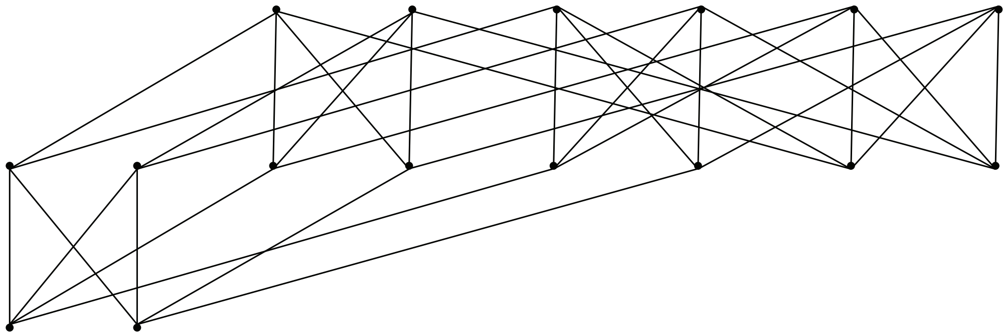

Figure 3: $(2,8,6)_A$, reducible but indecomposable (connectivity symbol $2_4+4_3+2_2$).

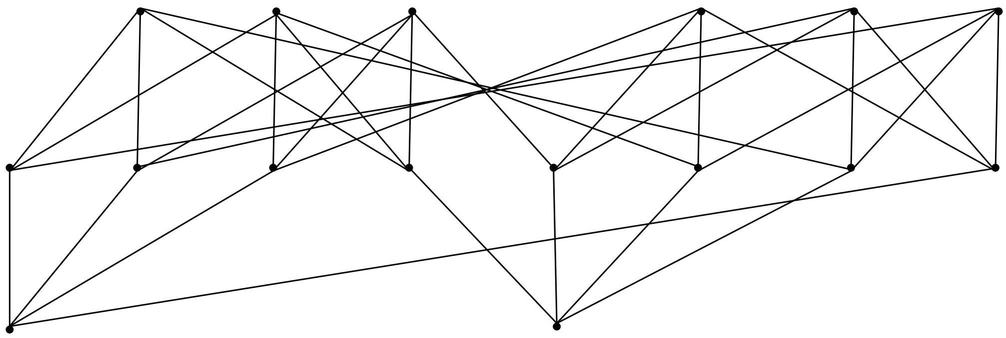

Figure 4: $(2,8,6)_B$, reducible but indecomposable (connectivity symbol 8_3).

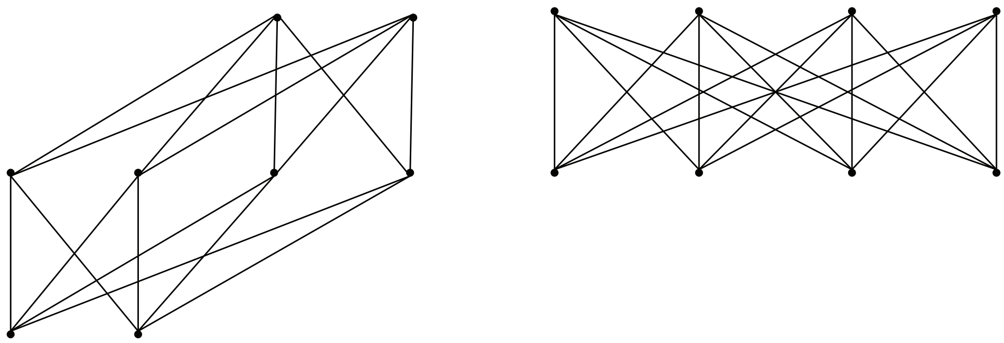

Figure 5: $(2,8,6)_a$, fully reducible (connectivity symbol $4_4 + 4_2$).

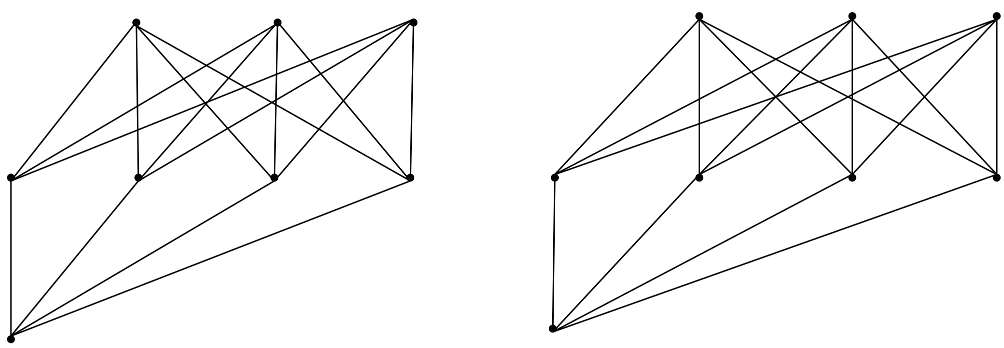

Figure 6: $(2,8,6)_b$, fully reducible (connectivity symbol 8_3).

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