IFTUWr-03/2002/01
CBPF-NF-001/02

Generalized Space-time Supersymmetries, Division Algebras and Octonionic M-theory

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Abstract

We describe the set of generalized Poincaré and conformal superalgebras in $D = 4, 5$ and 7 dimensions as two sequences of superalgebraic structures, taking values in the division algebras $\mathbf{R}, \mathbf{C}$ and $\mathbf{H}$. The generalized conformal superalgebras are described for $D = 4$ by $OSp(1; 8|\mathbf{R})$, for $D = 5$ by $SU(4, 4; 1)$ and for $D = 7$ by $U_\alpha U(8; 1|\mathbf{H})$. The relation with other schemes, in particular the framework of conformal spin (super)algebras and Jordan (super)algebras is discussed. By extending the division-algebra-valued superalgebras to octonions we get in $D = 11$ an octonionic generalized Poincaré superalgebra, which we call *octonionic M-algebra*, describing the octonionic M-theory. It contains 32 real supercharges but, due to the octonionic structure, only 52 real bosonic generators remain independent in place of the 528 bosonic charges of standard M-algebra. In octonionic M-theory there is a sort of equivalence between the octonionic M2 (supermembrane) and the octonionic M5 (super-5-brane) sectors. We also define the octonionic generalized conformal M-superalgebra, with 239 bosonic generators.

Key-words: M-theory, supersymmetry, division algebras.

*Supported by KBN grant 5P03B05620

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1 Introduction

We shall call generalized space-time supersymmetries the ones which go beyond the standard HLS scheme [1]. In four dimensions using the framework of local field theory and the arguments from S-matrix theory it was shown [1, 2] that the bosonic sector B of Poincaré or conformal superalgebra has the following tensor product structure[1]:

$$B \quad = \quad B_{geom} \oplus B_{int}, \tag{1}$$

where B_{geom} describes space-time Poincaré or conformal algebras and the remaining generators spanning B_{int} are Lorentz scalars. It is easy to show that one can introduce the standard Poincaré superalgebra, satisfying the relation (1), in any dimension (see e.g. [3]), but one arrives at a difficulty in trying to impose in any dimension standard conformal superalgebras. It appears [4, 5, 6] that one can introduce only at $D = 3, 4$ and 6 an infinite series of standard conformal superalgebras, which can be denoted in a unified way as $U_\alpha U(4; n|\mathbf{F})$[2] [6, 7] ($\mathbf{F} = \mathbf{R}$ for $D = 3$, $\mathbf{F} = \mathbf{C}$ for $D = 4$ and $\mathbf{F} = \mathbf{H}$ for $D = 6$). More explicitly[3]

$$\begin{aligned} D = 3: &\quad U_\alpha U(4; n|\mathbf{R}) &\equiv&\quad OSp(n; 4|\mathbf{R}), \\ D = 4: &\quad U_\alpha U(4; n|\mathbf{C}) &\equiv&\quad SU(2, 2; n), \\ D = 6: &\quad U_\alpha U(4; n|\mathbf{H}) &\equiv&\quad O^*Sp(8; 2n). \end{aligned}$$

It appears that if we wish to use the notion of conformal superalgebra in any dimension we should extend the standard Poincaré superalgebra (see e.g. [10, 11, 12, 13, 14]). The best known case is in $D = 11$, where the generalized Poincaré superalgebra going beyond the HLS axioms is called the M-algebra and looks as follows ($r, s = 1, 2, \ldots, 32$; $\mu, \nu = 0, 1, \ldots, 10$):

$$\{Q_r, Q_s\} = Z_{rs} = (C\Gamma_\mu)_{rs} P^\mu + (C\Gamma_{[\mu\nu]})_{rs} Z^{[\mu\nu]} + (C\Gamma_{[\mu_1\ldots\mu_5]})_{rs} Z^{[\mu_1,\ldots,\mu_5]}, \tag{2}$$

where $C = \Gamma_0$ is the $D = 11$ real Majorana charge conjugation matrix. The generalized $D = 11$ conformal superalgebra is obtained by adding a second copy of the superalgebra (2), with the extension of the conformal accelerations sector to the 32×32 symmetric matrices $\tilde{Z}_{rs}$:

$$\{S_r, S_s\} = \tilde{Z}_{rs} = (C\Gamma_\mu)_{rs} K^\mu + (C\Gamma_{[\mu\nu]})_{rs} \tilde{Z}^{[\mu\nu]} + (C\Gamma_{[\mu_1\ldots\mu_5]}) \tilde{Z}^{[\mu_1,\ldots,\mu_5]}. \tag{3}$$

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[1] Such a definition can be used also for standard super-de Sitter algebras (for both signs of the radii).

[2] $U_\alpha(n|\mathbf{F})$ describes the antiunitary $\mathbf{F}$-valued matrix group of transformations preserving the $\mathbf{F}$-valued antiHermitian bilinear form ${q_i}^\dagger A_{ij} q_j = inv$ ($A_{ij} = -{A_{ji}}^\dagger$) where $q_i \in \mathbf{F}$ and $q_i \mapsto {q_i}^\dagger$ is the main conjugation in $\mathbf{F}$. For quaternions ($\mathbf{F} = \mathbf{H}$) one can show that $U_\alpha(n|\mathbf{H}) = O(2n; \mathbf{C}) \cap U(n, n) = O^*(2n)$. The supergroup $U_\alpha U(n; m|\mathbf{F})$ describes the $\mathbf{F}$-valued matrix supergroup of graded transformations preserving the $\mathbf{F}$-valued bilinear form ${q_i}^\dagger A_{ij} q_j + {\theta_k}^\dagger \theta_k = inv$, where θ_k ($k = 1, \ldots, m$) are $\mathbf{F}$-valued Grassmann variables. For $\mathbf{F} = \mathbf{H}$ one gets $U_\alpha U(n; m|\mathbf{H}) = SU(n, n; m) \cap OSp(2m, 2n|\mathbf{C})$, which is usually denoted as $OSp^*(2n; 2m)$.

[3] We add here for completeness that for $D = 5$ there is a unique "exotic" standard conformal superalgebra F_4, with bosonic sector $\overline{O(5,2)} \times SU(2)$ (see e.g. [8, 9]).

Both sets of generators Z_{rs}, $\tilde{Z}_{rs}$ are Abelian, i.e.

$$[Z_{rs}, Z_{tk}] = [\tilde{Z}_{rs}, \tilde{Z}_{tk}] = 0. \tag{4}$$

It appears that if we introduce the crossed anticommutator, completing the superalgebra relations

$$\{Q_r, S_s\} \;=\; L_{rs}, \tag{5}$$

we get from the Jacobi identity that the 1024 generators L_{rs} form the $GL(32;\mathbf{R})$ algebra [15]. Summarizing, the resulting superalgebra admits the following five-grading

$$\begin{array}{ccccc} I_{-2} & I_{-1} & I_0 & I_1 & I_2 \\ \tilde{Z}_{rs} & S_r & L_{rs} & Q_r & Z_{rs}. \end{array} \tag{6}$$

The set of generators Z_{rs}, $\tilde{Z}_{rs}$, L_{rs} describe the generalized $D = 11$ conformal algebra $Sp(64)$ (conformal M-algebra) and all the generators from (6) form the superalgebra $OSp(1|64)$[16, 17, 19, 20] , known as generalized $D = 11$ superconformal algebra (conformal M-superalgebra).

The aim of this paper is to propose an analogous construction for the sequence of **F**-valued (**F**=**R**, **C**, **H**, **O**) generalized superalgebras, with the real superalgebras describing generalized supersymmetries in $D = 4$. We shall describe these superalgebras in some detail in Sect. **2** for $D = 4$ (**F**=**R**), 5 (**F**=**C**) and 7 (**F**=**H**). We obtain the generalized Poincaré superalgebras with 10 real bosonic generators for $D = 4$, 16 real bosonic generators for $D = 5$ and 28 real bosonic generators for $D = 7$ and the corresponding $D = 4$, 5 and 7 generalized conformal superalgebras $U_\alpha U(8;1|\mathbf{F})$.

In Sect. **3** we shall consider the relation of our proposal to other ways of introducing generalized supersymmetries, in particular based on Lorentz spin and conformal spin algebras [21, 22, 23]. It appears that our scheme for $D = 7$ can be identified with the one following from the minimal conformal spin algebra, but this is not the case for $D = 4, 5$. On the other hand our generalized superalgebras can be called minimal in another sense since the symmetrized product of supercharges (i.e. the anticommutators) is spanned by the fundamental representation of the respective Clifford algebras ($\mathbf{R}^4 \times \mathbf{R}^4$ for $D = 4$, $\mathbf{C}^4 \times \mathbf{C}^4$ for $D = 5$, $\mathbf{H}^4 \times \mathbf{H}^4$ for $D = 7$)[4]. The proposal is linked with the generalized conformal and superconformal algebra description in terms of **F**-valued Jordan (super)algebras [26, 27, 11].

In Sect. **4** we shall conjecture that one can use the proposed superalgebras with the division algebra **F** given by the octonionic algebra **O**[5]. In particular we obtain in place of the "standard" M-algebra (2) an algebra which we call the *octonionic M-algebra* with 52 real bosonic generators described by a 4×4 octonionic Hermitian matrix. We provide

[4]The fundamental representation of the Clifford algebra is its faithful representation with minimal dimension [24, 25].

[5]For the extension of $U_\alpha U(n;m|\mathbf{F})$ algebra to octonions see also [28, 29].

two alternative descriptions of the octonionic M-algebra: the first one linear and bilinear in the octonionic Γ-matrices and the second with their five-linear products only. We shall also introduce an octonionic conformal M-superalgebra with 232 real bosonic generators.

In Sect. **5** we present the final remarks. In particular we list some aspects of our framework which are postponed to further consideration.

2 The generalized $D = 4, 5$ and 7 supersymmetries described by F-valued superalgebras (F = R, C, H)

i) Generalized Poincaré superalgebras.

The standard $N = 1$ $D = 4$ Poincaré superalgebra has the following complex Hermitian form ($A, B = 1, 2$):

$$\begin{aligned} \{Q_A, \overline{Q}_{\dot{B}}\} &= (\sigma_\mu)_{A\dot{B}} P^\mu, \\ \{Q_A, Q_B\} = 0 \qquad & \{\overline{Q}_{\dot{A}}, \overline{Q}_{\dot{B}}\} = 0, \end{aligned} \tag{7}$$

where $\sigma_\mu = (\mathbf{1}_2, \sigma_i)$ describes the linear basis of Hermitian 2×2 matrices and $Q_A \mapsto {Q_A}^\dagger = \overline{Q}^{\dot{A}}$ is the complex Hermitian conjugation. One can however also introduce complex holomorphic $D = 4$ algebra as follows

$$\begin{aligned} \{Q_A, \overline{Q}_{\dot{B}}\} &= 0, \\ \{Q_A, Q_B\} = Z_{AB}, \qquad & \{\overline{Q}_{\dot{A}}, \overline{Q}_{\dot{B}}\} = \overline{Z}_{\dot{A}\dot{B}}. \end{aligned} \tag{8}$$

Such types of superalgebras are the standard ones in the description of $D = 4$ Euclidean supersymmetry (see e.g. [30, 31]). If we introduce the antisymmetric real two-tensor field

$$Z_{[\mu\nu]} = \frac{1}{2i}({\sigma_{[\mu\nu]}}^{AB} Z_{AB} - \tilde{\sigma}^{\dot{A}\dot{B}}_{[\mu\nu]} \overline{Z}_{\dot{A}\dot{B}}), \tag{9}$$

one can incorporate both Abelian charges P^μ, $Z^{[\mu\nu]}$ in the Majorana form of $D = 4$ superPoincaré algebra ($a, b = 1, \ldots, 4$, $\mu, \nu = 0, 1, 2, 3$)

$$\{Q_a, Q_b\} = X_{ab} \equiv (C{\Gamma^{(4)}}_\mu)_{ab} P^\mu + (C{\Gamma^{(4)}}_{[\mu\nu]})_{ab} Z^{[\mu\nu]}. \tag{10}$$

We see that replacing in $D = 4$ the complex-Hermitian structure by a real one we find a place for six new Abelian tensorial charges. Such superalgebra was recently considered [32, 18, 33] as the $D = 4$ counterpart of the M-algebra (2) and describes the supersymmetric theories with domain walls and four-dimensional supermembranes. One can state that the physical background for the extension (10) of the standard Poincaré superalgebra (7) is now established. It is the starting point of our construction. In order to describe the

$D=5$ and $D=7$ extended Poincaré superalgebras we generalize (10) to the $\mathbf{F}$-valued[6] Hermitian superalgebras

$$\{Q_a, {Q^\dagger}_b\} = Z_{ab}, \qquad Z_{ab} = {Z_{ba}}^\dagger, \\ \{Q_a, Q_b\} = 0, \tag{11}$$

where $\dagger$ denotes the principal conjugation in the $\mathbf{F}$-algebra, namely

$$\begin{array}{lll} \mathbf{C}: & Q_a = {Q_a}^0 + i{Q_a}^1, & {Q_a}^\dagger = {Q_a}^0 - i{Q_a}^1, \\ & Z_{ab} = {X_{ab}}^0 + i{Y_{ab}}^1, & {Z_{ab}}^\dagger = {X_{ab}}^0 - i{Y_{ab}}^1, \\ \mathbf{H}: & Q_a = {Q_a}^0 + e_r{Q_a}^{(r)}, & {Q_a}^\dagger = {Q_a}^0 - e_r{Q_a}^{(r)}, \\ & Z_{ab} = {X_{ab}}^0 + e_r{Y_{ab}}^{(r)}, & {Z_{ab}}^\dagger = {X_{ab}}^0 - e_r{Y_{ab}}^{(r)}. \end{array} \tag{12}$$

In order to write the superalgebra (11) in a Dirac matrices basis we shall at first introduce 2×2 complex Dirac matrices for the $O(3)\simeq SU(2)$ algebra (i.e. the three Pauli matrices σ_r, $r=1,2,3$) and 2×2 quaternionic matrices for the $O(5)\simeq SU(2;\mathbf{H})$ algebra

$$O(5): \qquad \Sigma_r = \begin{pmatrix} 0 & e_r \\ -e_r & 0 \end{pmatrix}, \quad \Sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Sigma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{13}$$

where e_r are the three quaternionic units. Then we shall consider $O(4,1)$ and $O(6,1)$ as $D=3$ and respectively $D=5$ Euclidean conformal algebras and follow the rules (see e.g. [34]) in order to introduce Dirac's γ matrices for $O(p+1,q+1)$ by doubling the dimension of the $O(p,q)$ representations. We obtain

i1) $D=5$.

$${\Gamma_r}^{(5)} = \begin{pmatrix} 0 & \sigma_r \\ \sigma_r & 0 \end{pmatrix}, \quad {\Gamma_4}^{(5)} = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix}, \quad {\Gamma_0}^{(5)} = \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix}. \tag{14}$$

One can describe the complex Hermitian 4×4 matrices as linear combination of 16 Hermitian-symmetric matrices ${\Gamma_\mu}^{(5)}C^{(5)}$, ${\Gamma_{\mu\nu}}^{(5)}C^{(5)}$ and $iC^{(5)}$. One sets

$$\{Q_a, {Q_b}^\dagger\} = Z_{ab} = ({\Gamma_\mu}^{(5)}C^{(5)})_{ab}P^\mu + ({\Gamma_{[\mu\nu]}}^{(5)}C^{(5)})_{ab}Z^{[\mu\nu]} + i{C^{(5)}}_{ab}Z, \tag{15}$$

where $C^{(5)}$ is the $O(4,1)$ complex charge conjugation matrix satisfying the relations

$$\begin{aligned} {{\Gamma_\mu}^{(5)}}^\dagger C^{(5)} &= -C^{(5)}{\Gamma_\mu}^{(5)}, \\ {C^{(5)}}^\dagger &= -C^{(5)}. \end{aligned} \tag{16}$$

In the representation with ${\Gamma_a}^{(5)} = {{\Gamma_a}^{(5)}}^\dagger$ $(a=1,2,3,4)$ (see e.g. (14)) and ${\Gamma_0}^{(5)} = -{{\Gamma_0}^{(5)}}^\dagger$, we should put $C^{(5)} = {\Gamma_0}^{(5)}$. The maximal covariance algebra of the supercharges is given

[6] $\mathbf{F}=\mathbf{C}$ and $\mathbf{H}$; see however Sect. 4 where we consider $\mathbf{F}=\mathbf{O}$. We add that an abstract algebra $\{A,B\}=C$ of matrices having entries (a_{ij}, b_{ij}, c_{ij} respectively) valued in a division algebra $\mathbf{F}$ (i.e. $a_{ij}=\sum_\alpha {a_{ij}}_\alpha \tau_\alpha$, with τ_0 the identity, and similarly for B, C), implies the following relations on the real components ${c_{ik}}_\gamma = \sum_{j,\alpha,\beta}\{{a_{ij}}_\alpha, {b_{jk}}_\beta\}C_{\alpha\beta\gamma}$, where $C_{\alpha\beta\gamma}$ are the structure constants of $\mathbf{F}$.

by the group $GL(4, \mathbf{C})$, however distinguished role is played by its subgroup $U_\alpha(4; \mathbf{C}) = U(2,2)$, because sixteen generators Z_{ab} from (15) belong to the adjoint representation of $U(2,2)$.

i2) $D = 7$.

The Hermitian quaternionic representation of the $O(6,1)$ Clifford algebra can be obtained from (14) as follows ($p = 1, \ldots, 5$, $\mu = 0, 1, \ldots, 6$)

$$\Gamma_p{}^{(7)} = \begin{pmatrix} 0 & \Sigma_p \\ \Sigma_p & 0 \end{pmatrix}, \quad \Gamma_6{}^{(7)} = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix}, \quad \Gamma_0{}^{(7)} = \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix}. \tag{17}$$

The $O(6,1)$ quaternionic charge conjugation matrix $C^{(7)}$ satisfies relations analogous to (16) with quaternionic-Hermitian conjugation. In the representation with $\Gamma_k{}^{(7)} = \Gamma_k{}^{(7)\dagger}$ ($k = 1, 2, 3, 4, 5, 6$) and $\Gamma_0{}^{(7)} = -\Gamma_0{}^{(7)\dagger}$ (see e.g. (17)) we obtain again $C^{(7)} = \Gamma_0{}^{(7)}$.

If we consider the symmetry properties of the products $C\Gamma_{[\mu_1,\ldots,\mu_k]}$ ($k = 1, \ldots, 7$) under quaternionic conjugation we obtain that (11) for $\mathbf{F} = \mathbf{H}$ can be decomposed as follows ($\mu, \nu = 0, 1, \ldots, 6$):

$$\{Q_a, Q_b{}^\dagger\} = Z_{ab} = (C^{(7)}\Gamma_\mu{}^{(7)})_{ab} P^\mu + (C^{(7)}\Gamma_{[\mu\nu]}{}^{(7)})_{ab} Z^{[\mu\nu]}. \tag{18}$$

The most general covariance group of quaternionic Poincaré algebra (18) is $GL(4, \mathbf{H})$, and its distinguished subgroup is $U_\alpha(4, \mathbf{H}) \simeq SO^*(8) \simeq SO(6,2)$. The 28 bosonic real generators spanning Z_{ab} in (18) are described by the adjoint representation of $U_\alpha(4, \mathbf{H})$.

ii) Generalized conformal superalgebras.

Following the procedure of obtaining $OSp(1;64)$ from the M-algebra (2) one can add a second copy of the $\mathbf{F}$-valued superalgebra (11)

$$\{S_a, S_b\} = \tilde{Z}_{ab} \tag{19}$$

and impose the Jacobi identities which imply that the mixed anticommutator $\{Q_a, S_b\} = L_{ab}$ describes the $GL(4|\mathbf{F})$ Lie algebra generators. One obtains the following five-fold graded structure

$$\begin{array}{ccccc} I_{-2} & I_{-1} & I_0 & I_1 & I_2 \\ \tilde{Z}_{ab} & S_a & L_{ab} & Q_a & Z_{ab}. \end{array} \tag{20}$$

If $\mathbf{F} = \mathbf{R}$ the set of generators (20) describe the $D = 4$ generalized conformal superalgebra $OSp(1|8)$ with its bosonic sector describing the $D = 4$ generalized conformal algebra.

The construction for $D = 5$ and $D = 7$ corresponds respectively to $\mathbf{F} = \mathbf{C}$ and $\mathbf{F} = \mathbf{H}$.

ii1) $D = 5$

In such a case the generalized conformal superalgebra is complex. The complex generators Z_{ab}, $\tilde{Z}_{ab}$ in (20) describe complex Hermitian algebras (see (15)), and L_{ab} span the $GL(4|\mathbf{C})$ algebra. It can be checked that the complex bosonic algebra with three-grading

$$\begin{array}{ccc} I_{-2} & I_0 & I_2 \\ \tilde{X}_{ab} & L_{ab} & X_{ab}. \end{array} \tag{21}$$

describes the $U_\alpha(8,\mathbf{C}) = U(4,4)$ algebra which is our $D=5$ generalized conformal algebra. The five-grading (20) provides $SU(4,4;1)$ as $D=5$ generalized conformal superalgebra.

ii2) D=7

This case corresponds to inserting in (20) into the sectors I_2 and I_{-2} two copies of the $D=7$ Poincaré superalgebra given by (18). The sector I_0 is then described by $GL(4;\mathbf{H}) \simeq SU^*(8)$ algebra, and the quaternionic three-graded algebra (21) provides $U_\alpha(8|\mathbf{H}) \simeq O^*(16)$ as the $D=7$ generalized conformal algebra. The supersymmetric extension can be obtained by imposing the five-grading (20) and it leads to the $D=7$ generalized conformal superalgebra $U_\alpha U(8;1|\mathbf{H})$.

Summarizing we see that the $D=4$, $D=5$ and $D=7$ generalized conformal algebras and generalized conformal superalgebras are given respectively by $U_\alpha(8|\mathbf{F})$ and $U_\alpha U(8;1|\mathbf{F})$. We obtain the following numbers of additional bosonic generators, which are present in generalized conformal algebras and conformal superalgebras:

	$\frac{U_\alpha(8;\mathbf{F})}{O(D,2)}$	$\frac{U_\alpha(8;\mathbf{F})\times U(1;\mathbf{F})}{O(D,2)}$
$D=4 \quad (\mathbf{F}=\mathbf{R})$	21 (10)	21 (10)
$D=5 \quad (\mathbf{F}=\mathbf{C})$	42 (16)	43 (16)
$D=7 \quad (\mathbf{F}=\mathbf{H})$	84 (28)	87 (28)

(22)

where in brackets we provided the number of bosonic generators which appear if we pass from the standard to the generalized Poincaré superalgebra. These generators can also be treated as introducing an extended D-dimensional space-time (see e.g. [18, 19, 20]) with additional tensorial coordinates besides the Minkowski space-time variables. We obtain

	standard space-time	extended space-time
D=4	4	$4 + 6 = 10$
D=5	5	$5 + 11 = 16$
D=7	7	$7 + 21 = 28$

(23)

The field realizations on space-time with additional coordinates can be related with the representations of infinite-dimensional spin algebras with infinite spin or helicity spectra [20].

3 Relations with spinor algebras, representations of Clifford algebras and Jordan algebras

The existence of standard conformal supersymmetries at $D=3,4$ and 6 described by the set of superalgebras $U_\alpha U(4;n,\mathbf{F})$ follows from the property that the spinorial coverings of the conformal algebra $O(D,2)$ are described for $D=3,4,6$ by $U_\alpha(4|\mathbf{F})$, i.e.

$$\begin{aligned} Spin(3,2) &= \overline{O(3,2)} = Sp(4;\mathbf{R}), \\ Spin(4,2) &= \overline{O(4,2)} = \overline{SU(2,2)}, \\ Spin(6,2) &= \overline{O(6,2)} = U_\alpha(4|\mathbf{H}) = O^*(8,\mathbf{C}). \end{aligned} \tag{24}$$

For $D=5$ and $D>6$ the spinorial covering of $\overline{O(D,2)}$ is not described by a classical Lie group.

Recently the notion of spin algebra has been introduced [21, 22, 23] in any dimension generalizing the notion of standard spin covering (24). Let us firstly introduce for the orthogonal group $O(n,m)$ its fundamental spinorial representation described by an N-dimensional vector space $\mathbf{F}_{(n,m)}$ with the choice of $\mathbf{F}$ ($\mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$) depending on the pair of numbers (n,m)[7]. The spin group $Spin(n,m)$ is the $\mathbf{F}$-valued $N\times N$ matrix Lie group of endomorphisms of $\mathbf{F}_{(n,m)}$ which contains the spinorial covering of $\overline{O(n,m)}$[8]

$$\overline{O(n,m)}\subset Spin(n,m), \quad D=n+m. \tag{25}$$

In particular one can distinguish the minimal spin group $Spin_{min}(n,m)$, with minimal number of real generators. In the standard case $D=3,4$ or 6 we have $Spin_{min}(D,2)=\overline{O(D,2)}$, but for $D=5$ and $D\geq 7$ we obtain that $dim\ Spin_{min}(D,2) > dim\ O(D,2) = \frac{1}{2}(D+1)(D+2)$. If we supersymmetrize the $Spin_{min}(D,2)$ algebra we obtain the minimal conformal spin superalgebra $\widetilde{Spin}_{min}(D,2)$. In $D=3,4$ and 6 one gets $\widetilde{Spin}_{min}(D,2)=U_\alpha U(4;n|\mathbf{H})$.

i) $D=5$.

It can be shown that for $D=5$ the fundamental conformal spinors $\mathbf{F}_{(4,1)}\equiv\mathbf{H}^4$ and (see [21, 22, 23])

$$Spin_{min}(5,2) = U_\alpha(4,\mathbf{H})\simeq O^*(8)\simeq O(6,2), \tag{26}$$

$$\widetilde{Spin}_{min}(5,2) = U_\alpha U(4;n|\mathbf{H}). \tag{27}$$

The formula (26) describes the $D=6$ conformal algebra and the relation (27) assigns as minimal $D=5$ spin superalgebra the standard $D=6$ conformal superalgebra (see (2)) with bosonic sector $O(6,2)\times O(3)$. In order to interpret $U_\alpha(4)$ as $D=5$ conformal spin algebra with $D=5$ tensor structure we should perform the dimensional reduction $D=6\mapsto D=5$. The seven generators spanning the coset $\frac{O(6,2)}{O(5,2)}$ are described by $O(5,2)$ seven-vector, and after dimensional reduction they will form an $O(4,1)$ five-vector and two $D=5$ scalars. These seven generators will extend the $D=5$ conformal algebra $O(5,2)$. In the supersymmetric case $U_\alpha U(4;n|\mathbf{H})$ is used as $D=5$ conformal spin superalgebra and will contain, besides the seven generators from $\frac{O(6,2)}{O(5,2)}$, also three scalar $O(3)$ generators ($U(1|\mathbf{H})\simeq SU(2)\simeq O(3)$) describing the internal symmetry sector[9].

¿From the considerations in Sect. 2 follows that our $D=5$ generalized conformal algebra $SU(4,4)\supset U_\alpha(4;\mathbf{H})$. Subsequently, for our $D=5$ generalized conformal superalgebra we obtain

$$SU(4,4;2) \quad\supset\quad U_\alpha U(4;1|\mathbf{H}), \tag{28}$$

[7]The fundamental spinor representation is determined by the minimal faithful Clifford algebra representation of $O(n,m)$ with generators $I_{\mu\nu}=\frac{1}{2}[\Gamma_\mu,\Gamma_\nu]$.

[8]In [21, 22, 23] $Spin(n,m)$ are real algebras; we assume that $Spin(n,m)$ are $\mathbf{F}_{(n,m)}$-valued matrices. Both descriptions are equivalent.

[9]This internal sector is usually referred to as describing R-symmetries.

but $U_\alpha U(4;1|\mathbf{H})$ is not contained in $SU(4,4;1)$ in analogy with the relations between $D=4$ standard and generalized conformal superalgebras [35], where $OSp(8;2) \supset SU(2,2;1)$, but one cannot embed $SU(2,2;1)$ into $OSp(8;1)$.

It is easy to see that $SU(4,4)$ in comparison with minimal $D=5$ conformal spin algebra $SU(4,4) \supset U_\alpha(4;\mathbf{H})$ contains more additional bosonic generators. In fact the principle of constructing our generalized supersymmetries in $D=4$ and $D=5$ are analogous. In $D=4$ we relaxed the restrictions on superalgebra by replacing the complex structure by a real one, and in $D=5$ the quaternionic structure is replaced by the complex one.

ii) $D=7$.

In $D=7$ the situation is different. The fundamental $D=7$ conformal spinors are given by $\mathbf{H}^8$ and our generalized conformal superalgebra $U_\alpha U(8;1,\mathbf{H})$ is identical with the minimal conformal spin superalgebra

$$Spin_{min}(7,2) \quad = \quad U_\alpha U(8;1|\mathbf{H}) \qquad (29)$$

with bosonic sector containing 123 bosonic generators (36 $O(7,2)$ generators + 84 additional tensorial generators + 3 generators describing $U(1|\mathbf{H}) = SU(2)$ R-symmetry).

In order to compare the minimal conformal fundamental spin algebras [21, 22, 23] with our generalized conformal algebras let us write for $D=4,5$ and 7 the fundamental Lorentz spin representations and minimal Clifford algebra modules permitting to represent faithfully $O(D-1,1)$ Γ-matrices:

	$min.\ \ spin\ \ F_{(D-1,1)}$	$min.\ \ Clifford\ \ mod.\ \ C_{(D-1,1)}$
$D=4$	$\mathbf{C}^2$	$\mathbf{R}^4$
$D=5$	$\mathbf{H}^2$	$\mathbf{C}^4$
$D=7$	$\mathbf{H}^4$	$\mathbf{H}^4$

(30)

We see that our generalized Poincaré supercharges Q_a are described by minimal Clifford algebra modules $C_{(D-1,1)}$. We also see from (30) why there is a difference between the conformal spin superalgebra approach with supercharges described by $F_{(D-1,1)}$ and our proposal for $D=4$ and $D=5$.

We would like to mention here that the sequence of superalgebras (12) as well as the corresponding conformal algebras (23) and conformal superalgebras (22) can be put in the framework of Jordanian algebras and Jordanian superalgebras [33,11]. Our goal here was to assign concrete generalized supersymmetries to the particular $\mathbf{F}$-valued chains of superalgebras. It should be added, that in the framework of Jordanian (super)algebras one can also include 3×3 octonionic-Hermitian algebra of matrices $J_3(\mathbf{O})$, but the extesion of the generators Z_{ab} to octonionic-valued 4×4 Hermitian matrices with 52 real generators is not included into Jordan superalgebras sequence. Such a superalgebra, with 32 real supercharges due to the imposed octonionic-Hermitian structure will be called *octonionic M-algebra*. In the next section we shall discuss our octonionic-valued superalgebras.

4 Octonionic M-superalgebras and Octonionic M-theory

One of the features of the proposed sequence of generalized supersymmetries is the possibility of extending the **F**-valued superalgebra structures to octonions. Octonions are described by eight real numbers ($k = 1, \ldots, 7$)

$$X \in \mathbf{O}: \qquad X = X_0 + X_k t_k, \tag{31}$$

where the seven octonionic units t_k satisfy the nonassociative algebra

$$t_k t_l \;=\; -\delta_{kl} + \frac{1}{2} f_{kl}{}^m t_m. \tag{32}$$

The octonions are endowed with the principal involution $\overline{t}_k = -t_k$, and unit octonions describe the unit sphere S^7 through $X\overline{X} = 1$[10].

i) The octonionic Poincaré M-superalgebra (octonionic M-algebra).

¿From extending (11) to octonions [6] it follows that in $D = 11$ ($a = 1, \ldots, 4$)

$$\begin{aligned}\{Q_a, {Q_b}^\dagger\} &= Z_{ab},\\ \{Q_a, Q_b\} = \{{Q_a}^\dagger, {Q_b}^\dagger\} &= 0,\end{aligned} \tag{33}$$

where

$$\begin{aligned}Q_a &= {Q_a}^0 + {Q_a}^{(k)} t_k,\\ Z_{ab} &= {Z_{ab}}^0 + {Z_{ab}}^{(k)} t_k\end{aligned} \tag{34}$$

and $Z_{ab} = {Z_{ab}}^\dagger =$ implies that ${Z_{ab}}^0 = {Z_{ba}}^0$, ${Z_{ab}}^{(k)} = -{Z_{ba}}^{(k)}$, i.e. the algebra (33) is described by 52 bosonic generators. Following (13) one can introduce the octonionic 2×2 gamma matrices ($k = 1, \ldots, 7$) realizing the $(9,0)$ signature[11]

$$\Sigma_k = \begin{pmatrix} 0 & t_k \\ -t_k & 0 \end{pmatrix}, \quad \Sigma_8 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Sigma_9 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{35}$$

and further the following 4×4 $D = 11$ octonionic Γ_μ matrices ($R = 1, \ldots, 9$)

$${\Gamma_R}^{(11)} = \begin{pmatrix} 0 & \Sigma_R \\ \Sigma_R & 0 \end{pmatrix}, \quad {\Gamma_{10}}^{(11)} = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix}, \quad {\Gamma_0}^{(11)} = \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix}, \tag{36}$$

[10] One can also say that S^7 describes the octonionic extension $U(1|\mathbf{O})$ of $U(1)$; S^7 is however not a Lie group, but rather the so-called soft Lie group [36].

[11] Equivalently, to construct $D = 11$ octonionic gamma matrices we could start from the octonionic realization of Clifford algebra with $(1,8)$ signature. It is worth mentioning that octonionic realizations of Clifford algebras only exist in specific signatures, such as $(0,7)$, $(9,0)$, $(1,8)$, $(10,1)$, $(2,9)$ etc. They are related to the nonassociative realizations of $D = 1$ N-extended supersymmetries (see [37]), which are beyond the classification of the representations of associative $D = 1$ N-extended supersymmetries [38], based on the Clifford algebras over **R**, **C** and **H**.

with the $D=11$ matrix $C^{(11)}$ given again by the matrix $\Gamma_0{}^{(11)}$ and satisfying relations analogous to (16) with octonionic principal conjugation. Taking into consideration that $\Gamma_\mu{}^{(11)}$ for $\mu=1,\ldots,10$ are octonionic-Hermitian and $\Gamma_0{}^{(11)}$ is antihermitian, one can write (33) as follows

$$\{Q_a, Q_b{}^\dagger\} \quad = \quad (C^{(11)}\Gamma_\mu{}^{(11)})_{ab}P^\mu + (C^{(11)}\Gamma_{\mu\nu}{}^{(11)})_{ab}Z^{\mu\nu}. \tag{37}$$

¿From the multiplication table of the octonions follows that $(k,l=1,\ldots,7)$

$$\Gamma_{kl}{}^{(1)} = \Gamma_{[k}{}^{(11)}\Gamma_{l]}{}^{(11)} = f_{kl}{}^m\Gamma_m{}^{(11)}\Gamma_8{}^{(11)}\Gamma_9{}^{(11)}\Gamma_{10}{}^{(11)}\Gamma_{11}{}^{(11)}. \tag{38}$$

We see that out of the 21 bilinear products of the first seven matrices (36) only 7 are independent and they correspond to the generators of $\frac{O(7)}{G_2}$. The remaining 4×4 octonionic Γ-matrices in the antisymmetric products $\Gamma_{\mu\nu}{}^{(11)} = \Gamma_{[\mu}{}^{(11)}\Gamma_{\nu]}{}^{(11)}$ are linearly independent, i.e. we get $55-14=41$ generators describing the coset $\frac{O(10,1)}{G_2}$, which plays also the role of octonionic $D=11$ Lorentz algebra. It is easy to see that the maximal number of real generators on the r.h.s. of (33) is $52=11+41$, i.e. the relation (37) indeed saturates the octonionic-valued anticommutator $\{Q_a, Q_b{}^\dagger\}$ $(a,b=1,2,3,4)$.

It should also be stressed that in the definition of $n>2$ antisymmetric products of the octonionic Γ-matrices (36) one should provide the order of multiplication, because the Dirac algebra with the basis (36) is non-associative. To be explicit, the antisymmetrized product of n octonionic matrices A_i $(i=1,2,\ldots,n)$ is given by

$$[A_1\cdot A_2\cdot\ldots\cdot A_n] \quad \equiv \quad \frac{1}{n!}\sum_{perm.}(-1)^{\epsilon_{i_1\ldots i_n}}(A_{i_1}\cdot A_{i_2}\ldots\cdot A_{i_n}), \tag{39}$$

where $(A_1\cdot A_2\ldots\cdot A_n)$ denotes the symmetric product

$$(A_1\cdot A_2\cdot\ldots\cdot A_n) \quad \equiv \quad \frac{1}{2}(.((A_1A_2)A_3\ldots)A_n) + \frac{1}{2}(A_1(A_2(\ldots A_n)).). \tag{40}$$

In such a case one can show that the three-fold product of octonionic Γ-matrices $C[\Gamma_i\cdot\Gamma_j\cdot\Gamma_k]$ provides 75 antihermitian matrices, describing together with C an arbitrary 4×4 octonionic antihermitian matrix. The definition (39), applied to the five-fold products of octonionic Γ-matrices provides their octonionic hermiticity. Further, by explicit calculation one can show that there are 52 independent real tensorial charges describing the five-tensor sector of the octonionic M-algebra, i.e. they span arbitrary 4×4 octonionic-hermitian matrices. We thus see that equivalently one can write the octonionic M-algebra (37) as follows

$$\{Q_a, Q_b{}^\dagger\} \quad = \quad C_{ac}[\Gamma_{\mu_1}\cdot\ldots\cdot\Gamma_{\mu_5}]_{cb}Z^{\mu_1\cdots\mu_5}, \tag{41}$$

where out of the 462 real antisymmetric 5-tensorial charges of the standard M-algebra only 52 are linearly independent, due to the relation

$$[\Gamma_{\mu_1\ldots\mu_5}] \quad = \quad A_{[\mu_1\ldots\mu_5]}{}^\nu\Gamma_\nu + A_{[\mu_1\ldots\mu_5]}{}^{[\nu_1\nu_2]}\Gamma_{[\nu_1}\Gamma_{\nu_2]}, \tag{42}$$

with constant c-number coefficients $A_{[\mu_1\ldots\mu_5]}{}^{\nu}$, $A_{[\mu_1\ldots\mu_5]}{}^{[\nu_1\nu_2]}$ [12].

The relation (42) implies that in $D = 11$ octonionic M-algebra there is an equivalence of the octonionic five-superbrane and the octonionic two-superbrane (supermembrane) sectors. We would like to stress again that[13]

a) The octonionic supermembrane is characterized by constrained number of two-tensorial charges - from 55 to 41. The remaining 14 generators of G_2 describe inner automorphisms of the algebra (32) of octonionic units.

b) If we keep the 11 degrees of freedom corresponding to the momentum sector, the five-superbrane is also characterized by 41 independent degrees of freedom i.e. 462 degrees of freedom of standard five-tensor charges in M-theory are restricted very much indeed. It is interesting to find a geometric interpretation of such a huge reduction of degrees of freedom.

c) The $D = 11$ Lorentz covariance algebra is described also by 41 generators of $\frac{O(10,1)}{G_2}$.

d) The anti-de-Sitter extension of octonionic M-algebra is described by the octonionic extension of superalgebra $OSp(1;4)$, which we denote by $U_\alpha U(4;1|\mathbf{O})$ (see also [29]). The octonionic M-algebra is a contraction of $U_\alpha U(4;1|\mathbf{O})$, when the anti-de-Sitter radius $R \to \infty$.

It should be pointed out that the octonionic M-algebra (33), whose superalgebraic structure for real generators we computed with the prescription in footnote 6, is a Lie superalgebra.

ii) *The octonionic conformal M-algebra.*

Let us recall that in the cases $\mathbf{F} = \mathbf{R}, \mathbf{C}$ and $\mathbf{H}$ the $\mathbf{F}$-valued supercharges Q_a as well as the set of bosonic charges Z_{ab} were carrying the representation of the superalgebra $U_\alpha(4;\mathbf{F})$, describing respectively $D = 4$, $D = 5$ and $D = 7$ standard anti-de-Sitter algebras. If we introduce the algebra $U_\alpha(4;\mathbf{O})$, by means of 4×4 octonionic-valued matrices K_{ab}, satisfying the relation

$$K^{\#}\, A = -A\, K \tag{43}$$

where $A = -A^T$ (or in general $A^{\#} = -A$) describes the antisymmetric (or in general case octonionic-antiHermitian) metric, we obtain the $D = 11$ de-Sitter-like octonionic algebra introduced in [27].

The octonionic conformal M-algebra $U_\alpha(8|\mathbf{O})$ which we propose as defined by extending to octonions the scheme described in (21) is graded as follows:

$$\begin{array}{ccc} I_{-2} & I_0 & I_2 \\ \tilde{Z}_{ab} & L_{ab} \subset gl(4|\mathbf{O}) & Z_{ab} \end{array} \tag{44}$$

It contains 232 real generators and we conjecture it has the following properties:

[12] The relation (38) is a particular case of the formula (42).

[13] We note that in order to express the octonionic structure as constraints on the 528 real Abelian tensorial charges describing the generalized supersymmetry of standard M-theory we use the definition of octonionic-valued anticommutator from footnote 6.

a) We postulate that using the generators of $U_\alpha(8|\mathbf{O})$ one can obtain the realization of $\frac{O(11,2)}{G_2}$, replacing standard $D=11$ conformal algebra.

b) The 128 real generators of $Gl(4|\mathbf{O})$ describe the 4×4 octonionic matrix in the place of the general real $Gl(32|\mathbf{R})$ covariance group of standard M-theory with 1024 real generators.

iii) The octonionic conformal M-superalgebra.

The superextension of $U_\alpha U(8;1|\mathbf{F})$ to $\mathbf{F}=\mathbf{O}$ describes the octonionic conformal M-super-algebra, with bosonic sector described by $U_\alpha(8|\mathbf{O})\times U(1|\mathbf{O})$ (232+7=239 real generators), where the internal sector $U(1|\mathbf{O})\simeq S^7$ describes a parallelizable manifold which only can be described by an extension of the notion of standard Lie algebra - the so called soft Lie algebras [36]. Some indications suggest that the structure of the octonionic supergroups $U_\alpha U(4;1|\mathbf{O})$ and $U_\alpha U(8;1|\mathbf{O})$ in the real basis is that of a (graded) Malcev (super)algebra [39, 40]. We leave this investigation for future work.

5 Concluding remarks

Our proposal is the extension of Kugo and Townsend [5] relation between division algebras and standard sequence $D=3,4,6,10$ of supersymmetries within the HLS scheme to the case of generalized supersymmetries in the dimensions $4,5,7$ and 11 . The idea that d-dimensional Minkowski space-time should be extended by additional dimensions, describing tensorial central charges coordinates, has been proposed already some time ago (see e.g. [41, 42, 43, 44]). Our framework provides a concrete way of extending standard space-time framework to dimensions 5 and 7 and ultimately to $D=11$.

In this paper we did not develop various aspects of the proposed scheme. Let us only present a list of them, as problems for possible further considerations.

i) One can ask if the choice of our sequence and its space-time supersymmetry interpretation is unique. Indeed, because $U_\alpha U(8;1|\mathbf{C})\equiv SU(4,4;1)$ include $D=6$ conformal symmetries $O(6,2)$, one could also assign our sequences of superalgebras to $D=4,6$ and 7. The argument for using the sequence $D=4,5,7$ comes from the link with minimal Clifford algebra realizations (see (33)). The other choice of division superalgebra sequence can be obtained if we replace the quaternionic structure of $D=6$ Poincaré superalgebra by the real one[14]. In such a case one obtains the sequence $U_\alpha U(16;1|\mathbf{F})$ as describing generalized conformal superalgebras in $D=6$ ($OSp(1;16|\mathbf{R})$), $D=7$ ($SU(8,8:1)$) and $D=9$ ($U_\alpha U(16;1|\mathbf{H})$).

ii) For simplicity we do not consider here more explicitly the extended generalized symmetries, but such a generalization is obvious. In particular the extended generalized conformal supersymmetry with N copies of of $\mathbf{F}$-valued supercharges ${Q_a}^i$, ${S_a}^i$ is given by the superalgebra $U_\alpha U(8;N|\mathbf{F})$, with the internal sector (R–symmetries) $U(\mathbf{F})$ ($O(N)$ for $\mathbf{F}=\mathbf{R}$, $U(N)$ for $\mathbf{F}=\mathbf{C}$ and $U(N;\mathbf{H})\equiv USp(2N)$ for $\mathbf{F}=\mathbf{H}$).

[14] In our case it is replaced by the complex one.

iii) Our considerations are on purely algebraic level. One should also consider the representation theory of the generalized (super)symmetry algebras, e.g. express the generators in terms of oscillators[15] and consider the complete set of Casimir's.

iv) We did not mention here generalized de-Sitter supersymmetries and mentioned only for $\mathbf{F} = \mathbf{O}$ anti-de-Sitter symmetries and supersymmetries. The generalized anti-de-Sitter superalgebras in dimension D should be identified with the generalized conformal superalgebras in dimension $D-1$, i.e. our set of superalgebras $U_\alpha U(8;1|\mathbf{F})$ describes the generalized anti-de-Sitter algebras in $D = 5, 6, 8$ and possibly $D = 12$ (the last case for $\mathbf{F} = \mathbf{O}$). The discussion of generalized D-dimensional de-Sitter superalgebras, equivalent to generalized Lorentz superalgebras in dimension $D + 1$ is similar in principle, however with differences in technical details (see also [14]).

v) We did not discuss here an important issue of supersymmetric dynamics, covariant under generalized Poincaré and conformal supersymmetries. We would like only to mention that the preliminary results in such a direction has been already presented [18, 45, 19] for massless $D0$-superbranes (supersymmetric particles) mainly in $D = 4$ with $OSp(1;8)$ as generalized conformal algebra.

Acknowledgments

The authors would like to thank Dmitri Sorokin for valuable comments. One of the authors (J.L.) would like to thank José Helayel-Neto for his hospitality at CBPF in Rio de Janeiro, where the main part of this paper was prepared. F.T. is grateful for the hospitality at the Institute of Theoretical Physics of the University of Wrocław, where this work has been completed.

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